Normal Subsystems of Fusion Systems

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1 Fusion in Groups

Let G be a finite group. In the classification of the finite simple groups, results like the following are fundamental.

Theorem 1.1 (Glauberman's Z^*-theorem) Let G be a finite group such that $\mathrm{O}_{2'}(G) = 1$ with a Sylow 2-subgroup P, and let x be an involution. If $x^G \cap P = \{x\}$ then $x \in \mathrm{Z}(G)$.

This result is also true for odd primes, but requires CFSG. Much earlier than this, we had results such as the following.

Theorem 1.2 (Frobenius's normal p-complement theorem) Let G be a finite group and let P be a Sylow p-subgroup of G. If, for any two subgroups A and B of P, and any conjugation map $c_g : A \to B$ for some $g \in G$, there is some $x \in P$ such that $c_g|_A = c_x|_A$, then G possesses a normal Hall p'-subgroup.

If we look at Frobenius's theorem, this suggests that we should focus on the subgroups of a fixed Sylow p-subgroup P, and all of the conjugation maps c_g for the various $g \in G$ between the subgroups of P. Often, if $g \in G$ and $Q \leqslant P$, $Q \cdot c_g = Q^g$ does not lie inside P, and so we simply ignore those maps, and only keep the ones where both the domain and the image are inside P. This leads us to define the fusion system of a finite group to be just this.

Definition 1.3 Let G be a finite group and let P be a Sylow p-subgroup of G. The *fusion system* of G, denoted by $\mathcal{F}_P(G)$, is a category, with objects all subgroups of P, and as morphism sets

$$\mathrm{Hom}_{\mathcal{F}_P(G)}(A, B) = \{c_g|_A \mid g \in G,\ A^g \leqslant B\}.$$

(Notice that we allow all injective maps, not just the bijections induced by conjugation by $g \in G$. This is just a technical fact that makes stating many things easier.) Frobenius's theorem becomes the statement that if $\mathcal{F}_P(G) = \mathcal{F}_P(P)$ then G has a normal Hall p'-subgroup, and Glauberman's Z^*-theorem becomes the statement that if an involution is not $\mathcal{F}_P(G)$-conjugate to any other involution then it lies in the centre of $G/\operatorname{O}_{2'}(G)$.

2 From Fusion to Fusion

It's all well and good studying the fusion system of a finite group (and it is good) but we want to have an abstract definition of a fusion system. The reason for this is that, as it stands, the fusion system is an object attached to a finite group, and so working with it is equivalent to working with the group. If we had an axiomatic definition of a fusion system, we could work from the axioms directly, and this might make results that otherwise would not be clear appear so.

In addition, an axiomatic framework might let in other fusion systems than fusion systems of groups. This is a double-edged sword: we are no longer making statements about just groups, so we can no longer use *any* statement from group theory directly. However, we gain new fusion systems, both as interesting objects in their own right, and as objects that would be 'obstacles' to proving theorems about groups using fusion arguments.

We will give the definition of a fusion system now.

Definition 2.1 Let P be a finite p-group. A *fusion system* on P is a category $\mathcal{F}$, whose objects are all subgroups of P and whose morphisms $\operatorname{Hom}_{\mathcal{F}}(A, B)$ are sets of injective homomorphisms $A \to B$ satisfying three axioms:

(i) $\mathcal{F}_P(P) \subseteq \mathcal{F}$;

(ii) if $\phi : A \to B$ is a map in $\mathcal{F}$ then so is the induced isomorphism $A \to A\phi$; and

(iii) if $\phi : A \to B$ is an isomorphism in $\mathcal{F}$, then $\phi^{-1} : B \to A$ lies in $\mathcal{F}$.

This definition is very loose, and we need to make a restriction on which fusion systems we consider. For motivation, we go back to groups. Let Q be a subgroup of P. It is not always true that $\operatorname{N}_P(Q)$ is a Sylow p-subgroup of $\operatorname{N}_G(Q)$, but it is not difficult to show that there is *some* $g \in G$ such that $\operatorname{N}_P(Q^g)$ is a Sylow p-subgroup of $\operatorname{N}_G(Q^g)$ (and Q^g is also contained in P); i.e., Q is $\mathcal{F}_P(G)$-conjugate to a subgroup whose P-normalizer is a Sylow p-subgroup of its G-normalizer. In fact, this subgroup Q^g is simply a subgroup R such that $|\operatorname{N}_P(R)| \geqslant |\operatorname{N}_P(S)|$ whenever S is $\mathcal{F}_P(G)$-conjugate to R.

Suppose that $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$. Notice that $\mathrm{Aut}_G(Q) = \mathrm{N}_G(Q)/\,\mathrm{C}_G(Q)$. The quotient of a Sylow p-subgroup is a Sylow p-subgroup, and so $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_G(Q) = \mathrm{Aut}_{\mathcal{F}_P(G)}(Q)$. We have found our first condition.

Definition 2.2 Let $\mathcal{F}$ be a fusion system on P. If Q is a subgroup of P, we say that Q is *fully automized* if $\mathrm{Aut}_P(Q)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(Q)$.

If $\mathcal{F} = \mathcal{F}_P(G)$ then we know that every $\mathcal{F}$-conjugacy class of subgroups of P contains a fully automized member.

The other condition we want to understand concerns extensions of isomorphisms. Let $\phi : A \to B$ be an isomorphism in $\mathcal{F}$. We want to know whether we can extend the domain of ϕ to some larger subgroup C, so that there is a map $\psi : C \to P$ in $\mathcal{F}$ such that $A \leqslant C$ and $\psi|_A = \phi$. If there is such a $C > A$, then $\mathrm{N}_C(A) > A$, so assume $A \trianglelefteqslant C$.

If g lies in C, then $g\psi$ lies in $\mathrm{N}_P(B)$, since g normalizes a. Furthermore, we actually know the automorphism that $g\psi$ induces on B, namely c_g^ϕ, since any isomorphism $A \to B$ induces an isomorphism $\mathrm{Aut}(A) \to \mathrm{Aut}(B)$.

If we think just about $\mathrm{Aut}(A)$ then, in order for ϕ to extend to C, the image of $\mathrm{Aut}_C(A)$ under ϕ must lie inside $\mathrm{Aut}_P(B)$. This is the second condition.

Definition 2.3 Let $\mathcal{F}$ be a fusion system on P. If Q is a subgroup of P, we say that Q is *receptive* if, whenever $\phi : R \to Q$ is an isomorphism in $\mathcal{F}$, and S is a subgroup of $\mathrm{N}_P(R)$ containing Q such that $\mathrm{Aut}_S(Q)^\phi \leqslant \mathrm{Aut}_P(Q)$, there is an extension ψ of ϕ to S.

Exercise: prove that, for finite groups, if Q is a subgroup of P such that $\mathrm{N}_P(Q)$ is a Sylow p-subgroup of $\mathrm{N}_G(Q)$, then Q is receptive.

We can now state the definition of saturation.

Definition 2.4 A fusion system $\mathcal{F}$ is *saturated* if every $\mathcal{F}$-conjugacy class of subgroups contains a fully automized, receptive member.

3 Normal Subsystems

Let $\mathcal{F}$ and $\mathcal{E}$ be two fusion systems, on P and Q respectively. If $\phi : P \to Q$ is a group homomorphism, then any morphism of $\mathcal{F}$ passes through ϕ and induces a morphism between subgroups of Q. If this morphism lies in $\mathcal{E}$ for any morphism in $\mathcal{F}$ then ϕ is said to induce a *morphism* of fusion systems. If $\Phi : \mathcal{F} \to \mathcal{E}$ is a morphism of fusion systems, then the kernel K of the underlying morphism is *strongly $\mathcal{F}$-closed*: that is, any morphism in $\mathcal{F}$ with domain a subgroup of K has image a subgroup of K. It turns out that, for every strongly $\mathcal{F}$-closed subgroup of P, there is a surjective morphism of fusion systems with kernel that subgroup, and the morphism is determined by the underlying group homomorphism.

With morphisms come normal subsystems, but a slight defect of the theory is that with the definition of a normal subsystem we have (and it seems like the right one), there is not always a normal subsystem on a given strongly closed subgroup.

Definition 3.1 Let $\mathcal{F}$ be a saturated fusion system on P. A saturated subsystem $\mathcal{E}$ of $\mathcal{F}$, on a strongly $\mathcal{F}$-closed subgroup Q of P is *weakly normal* if every element of $\mathrm{Aut}_{\mathcal{F}}(Q)$ induces an automorphism of $\mathcal{E}$, and for $A, B \leqslant Q$, every $\phi \in \mathrm{Hom}_{\mathcal{F}}(A, B)$ may be written as $\phi = \alpha\beta$, where $\alpha \in \mathrm{Aut}_{\mathcal{F}}(Q)$ and $\beta \in \mathrm{Hom}_{\mathcal{E}}(A\alpha, B)$.

If, in addition, every $\alpha \in \mathrm{Aut}_{\mathcal{E}}(Q)$ extends to an automorphism $\beta \in \mathrm{Aut}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))$ such that β acts trivially on $Q\,\mathrm{C}_P(Q)/Q$.

The definition of a weakly normal subsystem given here is not the standard one, and is an analogue of the Frattini argument.

Example 3.2 Let G be a finite group, and let H be a normal subgroup of G. Let P be a Sylow p-subgroup of G, and let $Q = P \cap H$. If $\mathcal{F} = \mathcal{F}_P(G)$ and $\mathcal{E} = \mathcal{F}_Q(H)$, then identifying $\mathcal{E}$ with the corresponding subsystem of $\mathcal{F}$, we have that $\mathcal{E}$ is a normal subsystem of $\mathcal{F}$.

Proposition 3.3 Let $\mathcal{F}$ be a saturated fusion system on P, and let $\mathcal{E}$ be a weakly normal subsystem on T. If $Q \leqslant T$ then $\mathrm{Aut}_{\mathcal{E}}(Q) \trianglelefteqslant \mathrm{Aut}_{\mathcal{F}}(Q)$.

In particular, if $\mathcal{E}$ is a weakly normal subsystem on P itself, then $\mathrm{Aut}_{\mathcal{E}}(P)$ is a normal subgroup of p'-index in $\mathrm{Aut}_{\mathcal{F}}(P)$.

Proposition 3.4 Let T be a strongly $\mathcal{F}$-closed subgroup of P, and suppose that there are (weakly) normal subsystems on T. Order this set by inclusion. There exists a unique minimal (weakly) normal subsystem and a unique maximal (weakly) normal subsystem, denoted (in the weakly normal case) by $\mathcal{R}_{\mathcal{F}}(T)$ and $\mathcal{R}^{\mathcal{F}}(T)$ respectively.

How are weakly normal and normal subsystems related to one another?

Theorem 3.5 If T has weakly normal subsystems, then $\mathcal{R}_{\mathcal{F}}(T)$ is normal in $\mathcal{F}$.

One pleasing consequence of this is the following.

Corollary 3.6 $\mathcal{F}$ has no proper, non-trivial normal subsystems if and only if it has no proper, non-trivial weakly normal subsystems.