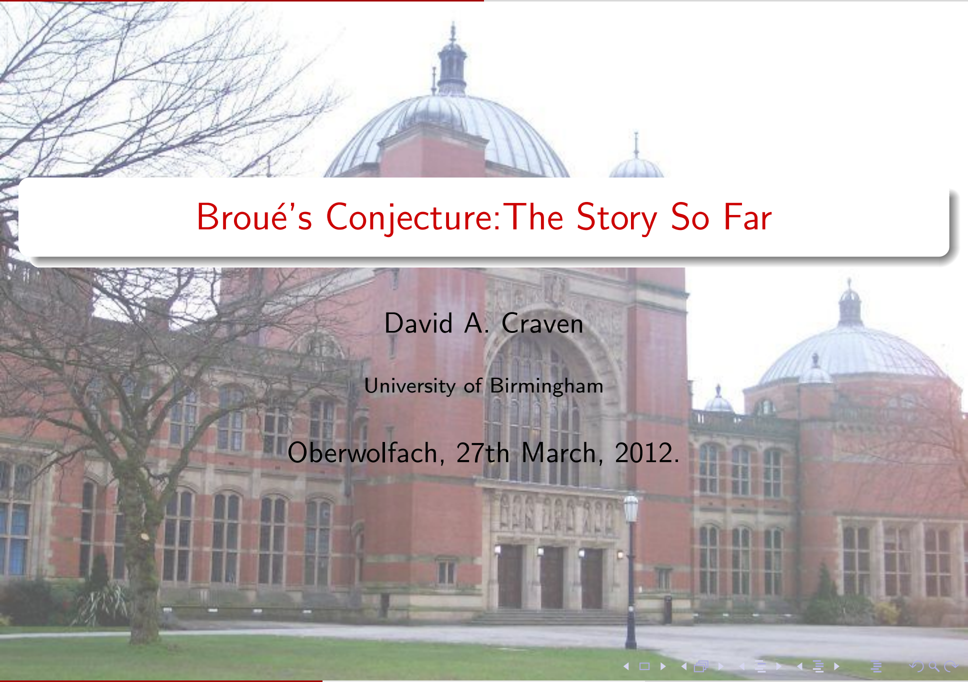

Broué's Conjecture:The Story So Far

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Notation and Conventions

Throughout this talk,

- G is a finite group,
- ℓ is a prime,
- k is a field of characteristic ℓ,
- B is a block of kG, with defect group D and Brauer correspondent b;
- P is a Sylow ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

Some of this talk is joint work with Raphaël Rouquier.

Reduction to Simple Groups

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However, there are other conjectures that so far resist a reduction to simple groups, such as Broué's conjecture for all blocks of finite groups.

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Theorem

Let G be a finite group. If P is abelian, then there are normal subgroups $H \leq L$ of G such that

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- L/H *is a direct product of simple groups and an abelian ℓ-group.*

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- L/H *is a direct product of simple groups and an abelian ℓ-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L (compatible with automorphisms of the simple components) passes up to G. Thus if Broué's conjecture for principal blocks holds for all simple groups (with automorphisms), it holds for all groups.

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4. $G = G(q)$ is a Lie-type group and $\ell \nmid q$.

Hence in order to prove Broué's conjecture for principal blocks, we need to understand groups of Lie type in non-defining characteristic.

Groups of Lie Type

Let $G = G(q)$ be a group of Lie type: the order of G is

$$|G| = q^N \prod_{i \in I} \Phi_i(q).$$

Suppose that $\ell \nmid q$ divides exactly one of the cyclotomic polynomials $\Phi_d(q)$ in the product. Then the Sylow ℓ-subgroup is abelian, and contained in a Φ_d-torus.

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The unipotent characters of G are certain irreducible characters of G, not depending on q. A unipotent block of G is one containing a unipotent character, such as the principal block, which contains the trivial character.

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Conjecture

Let $G = G(q)$ be a finite group of Lie type, and let D be an abelian defect group of a unipotent block B of G. We may embed D inside a Φ_d-torus T, and there is a Deligne–Lusztig variety Y, carrying an action of G on the one side and T on the other, whose complex of cohomology Γ has the following properties:

1. *the action of T can be extended to an action of $\mathrm{N}_G(T) = \mathrm{N}_G(D)$;*
2. *the complex induces a derived equivalence between B and its Brauer correspondent.*

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While this is (a lot) more specific than the abstract version of Broué's conjecture, it still needs to be more specific, as the variety Y_ζ can be hideously complicated.

This equivalence should be a perverse equivalence, which requires some combinatorial data. If these data can be extracted without analyzing the variety Y_ζ, then the derived equivalence should be able to be constructed without the variety at all, purely combinatorially.

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- orderings on the simple modules $S_1, S_2, \dots, S_r$, $T_1, T_2, \dots, T_r$, and
- a function $\pi : \{1, \dots, r\} \to \mathbb{Z}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \dots, S_i$, and $D^b_i(\mathcal{A})$ denotes the subcategory of $D^b(\mathcal{A})$ with support modules in $\mathcal{A}_i$, then

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- F induces equivalences $D_i^b(\mathcal{A}) \to D_i^b(\mathcal{B})$, and
- $F[\pi(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{B}_i/\mathcal{B}_{i-1}$.

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Note that mod-B is determined, up to equivalence, by A, π, and the ordering of the S_i.

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such that, for all i, the cohomology of $F(S_i)$ has only one copy of T_i in degree $-\pi(i)$, and any other T_j can only appear in lower degrees than $-\pi(j)$.

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2. There is an algorithm that gives us a perverse equivalence from any block B for a group G to **some** algebra, and we need to check that the target is the Brauer correspondent b. (This is simply checking that the Green correspondents are the last terms in the complexes.) This algorithm is very useful!

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3. The alternating sum of the cohomology of the complex X_i corresponding to S_i constructed by this algorithm gives a row of the decomposition matrix, with only S_j appearing for $\pi(S_j) < \pi(S_i)$, except for a single copy of S_i. When placed in ascending order of $\pi(-)$, this yields a lower unitriangular decomposition matrix.

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5. The ordinary characters in the upper square part are the unipotent characters when B is a unipotent block, and hence this gives a natural bijection between the unipotent characters and the simple modules of B (of course, this bijection is the same as given by the triangularity of the decomposition matrix by the a-function).

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6. The algorithm's output should be 'generic' in ℓ. (This is ongoing research of Rouquier and me.) This would imply that, assuming this version of the geometric version of Broué's conjecture, the decomposition numbers of unipotent blocks are independent of ℓ, for all sufficiently large ℓ. It would also suggest an answer to 'sufficiently large'.

The Perversity Function

The cohomology of the variety Y_ζ over $\bar{\mathbb{Q}}_\ell$ should have the property that each unipotent character χ should appear in exactly one degree $\pi_\zeta(\chi)$. (This degree will depend on ζ.) These degrees should be the perversity function for the perverse equivalence from B to b. (Actually, this is the cohomology with non-compact support. For compact support, take $-\pi_\zeta(\chi)$ and shift by twice the length of the variety.)

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If $\ell \mid \Phi_d$, where d is the largest integer such that $\Phi_d \mid |G|$, i.e., d is the Coxeter number, then Lusztig calculated the cohomology for $\zeta = \exp(2\pi i/d)$. Let $f(q)$ be a polynomial. If $a(f)$ denotes the multiplicity of q in a factorization of f, and $A(f) = \deg(f)$, then the degree of the cohomology that χ is in is $(a(\chi(1)) + A(\chi(1)))/d$ plus half the power of $(q-1)$ in $\chi(1)$.

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If $d = 1$ or $d = 2$ then Digne–Michel–Rouquier conjectured, for the principal block, that the degree of cohomology in which χ lies is $2A(\chi(1))/d$.

The Perversity Function

Define $\zeta = e^{2k\pi i/d}$, and for f a polynomial in q write $\phi_\zeta(f)$ for the number of non-zero zeroes of f (with multiplicity) of argument at most that of ζ (with argument in $[0, 2\pi)$), and $\bar{\phi}_\zeta(f)$ for the number of positive real zeroes. Write $a(f)$ for the multiplicity of the zero at 0.

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$$\pi_\zeta(f) = k(\deg f + a(f))/d + \phi_\zeta(f) - \bar{\phi}_\zeta(f).$$

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Conjecture

If f denotes the relative degree of χ, then $\pi_\zeta(f)$ is the unique degree in the cohomology with non-compact support of Y_ζ in which χ appears, and the $\pi_\zeta(\chi)$ form the perversity function for a perverse equivalence from B to b.

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In particular, when $d=1$, for $k=0$ we get $\pi_\zeta \equiv 0$ and for $k=1$ we get $\pi_\zeta \equiv 2A(-)$. The first is Puig's Morita equivalence, and the second is the predicted equivalence of Digne–Michel–Rouquier.

Pulling π Downstairs

The quantity $\pi_\zeta(\chi)$ is defined for unipotent characters in B, but for the algorithm computing perverse equivalences it needs to be defined on the simple b-modules, which can be thought of as the ordinary characters of the automizer $E = \mathrm{N}_G(D, b)/\mathrm{C}_G(D)$.

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The object we should need for this is the cyclotomic Hecke algebra, which in one specialization gives the characters of B and in another gives the characters of b.

The Cyclic Case, I

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Theorem

Suppose that G is of Lie type, B is a unipotent block, and D is cyclic. If G does not have type E_7 or E_8 (and even then in many cases) the 'combinatorial form' of Broué's conjecture is true, with $\pi(-) = \pi_\zeta(-)$ and bijection given by mapping χ to $\omega_\chi \zeta^{(a(\chi)+A(\chi))/\ell(b)}$ (ω_χ is a root of unity, normally ± 1), with the Brauer tree of b (a star) being represented on the complex plane.

In order for this theorem to make sense, for non-principal blocks anyway, we have to fix a rotation of the Brauer tree of b, to decide which non-exceptional b-character is placed at the position 1 (in $\mathbb{C}$). This can be done by taking Green correspondents of a simple B-module with smallest π_ζ-function.

The Cyclic Case, II

The theorem suggests that we should think of a Brauer tree as being embedded in $\mathbb{C}$, not in $\mathbb{R}^2$: one of the important directions this research might take is a generalization of the Brauer tree to (some other) abelian defect groups.

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The method of proof of the theorem is fairly simple: using the π_ζ-function and bijection, we construct the Brauer tree of the block, and compare it to the known one (when it is known, i.e., not for some blocks of E_7 and E_8). Combinatorial Broué's conjecture holds if and only if the Brauer tree is correct.

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Notice that this allows us to make conjectures as to the shape of the Brauer tree in the remaining cases, and this has led to some outstanding cases being resolved.

Examples: Known Tree

$G = {}^2F_4(q^2)$, $\ell \mid \Phi'_{24}(q)$. (By Φ'_{24} we mean the polynomial factor of Φ_{24} with ζ_{24} as a root.)

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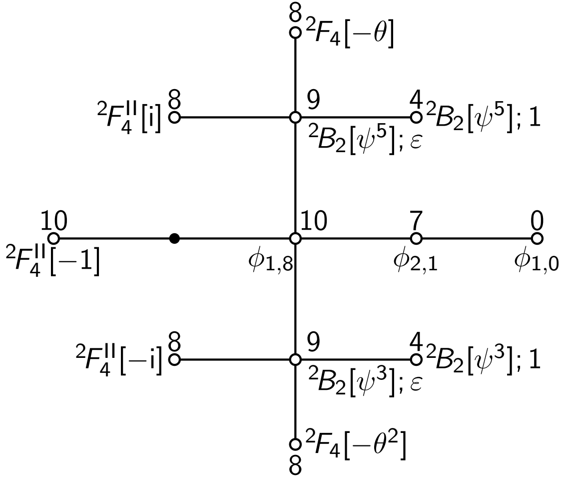
Character	$\omega_i q^{aA/e}$	$k=5$	$k=11$	$k=13$	$k=19$
$\phi_{1,0}$	1	0	0	0	0
${}^2B_2[\psi^3];1$	$\psi^7 q$	4	10	12	18
${}^2F_4^{II}[-\mathrm{i}]$	$-\mathrm{i}q^2$	8	18	22	32
${}^2F_4[-\theta^2]$	$-\theta q^2$	8	18	22	32
${}^2B_2[\psi^5];1$	ψq	4	10	12	18
$\phi_{2,1}$	q^2	7	17	21	31
${}^2B_2[\psi^3];\varepsilon$	$\psi^7 q^3$	9	21	25	37
${}^2F_4[-\theta]$	$-\theta^2 q^2$	8	18	22	32
${}^2F_4^{II}[\mathrm{i}]$	$\mathrm{i}q^2$	8	18	22	32
${}^2B_2[\psi^5];\varepsilon$	ψq^3	9	21	25	37
$\phi_{1,8}$	q^4	10	22	26	38
${}^2F_4^{II}[-1]$	$-q^2$	10	20	24	32

Examples: Known Tree

$G = {}^2F_4(q^2)$, $\ell \mid \Phi'_{24}(q)$. (By Φ'_{24} we mean the polynomial factor of Φ_{24} with ζ_{24} as a root.)



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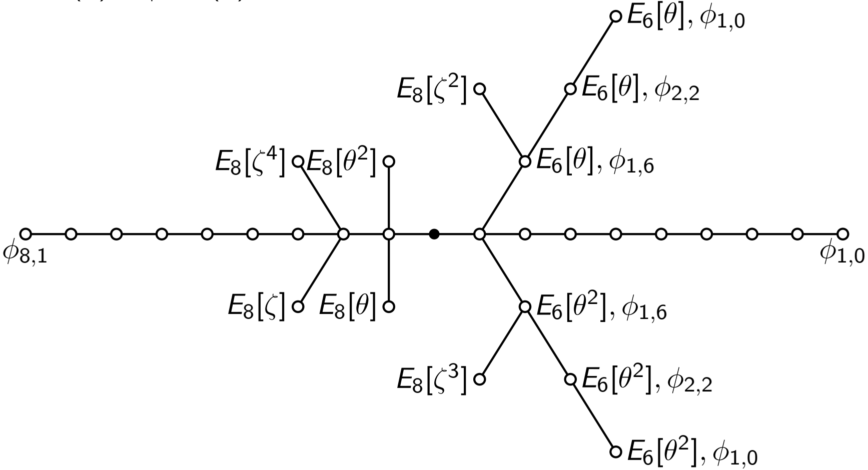

Examples: Made-up Stuff

$G = H_3(q)$, $\ell \mid \Phi_5'(q)$. (By Φ_5' we mean the polynomial $(q - \zeta_5^2)(q - \zeta_5^3)$.)

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Character	$\omega_i q^{aA/e}$	$k=2$	$k=3$
$\phi_{1,0}$	q^0	0	0
$\phi_{4,3}$	$-q^{3/2}$	9	15
$I_2(5)[1,3];\varepsilon$	$\zeta^2 q^2$	11	17
$I_2(5)[1,3];1$	$-\zeta^2 q$	7	11
$I_2(5)[1,2];\varepsilon$	$\zeta^3 q^2$	11	17
$I_2(5)[1,2];1$	$-\zeta^3 q$	7	11
$\phi_{4,4}$	$q^{3/2}$	9	15
$\phi_{1,15}$	$-q^3$	12	18
$\phi_{3,6}$	q^2	12	16
$\phi_{3,1}$	$-q$	8	10

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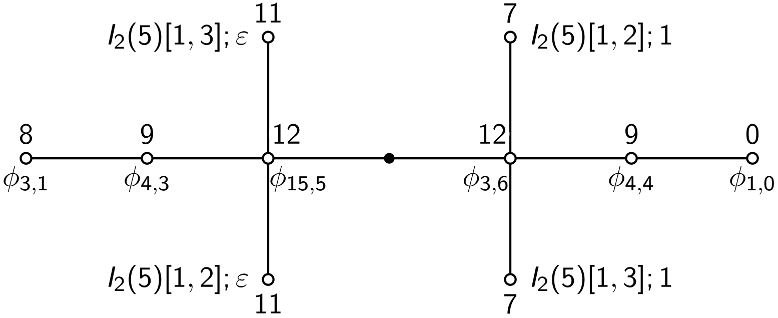

Examples: Even-more-made-up Stuff

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This slide would have another example on it, but Olivier and Gerhard (mathematically) distracted me last night.

The π_ζ-function cannot go through without modification, because of the following fact:

Proposition

Let f be a real polynomial whose roots are either 0 or roots of unity. If ζ is a root of unity such that $f(\zeta) \neq 0$ then

$$\pi_\zeta(f)/\pi = \arg(f(\zeta)).$$

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For complex polynomials this statement is false, and so the π_ζ-function needs to be changed on non-real cyclotomic polynomials. This can be done, and the structure of combinatorial Broué's conjecture yielding a Brauer tree (i.e., nodes alternate $+$ to $-$ a sign) might well produce consistent choices of the signs in front of the unipotent degrees for spetses.

Disclaimer

We have to make some choices of course with signs on the spetses. It might well be that running the algorithm on *any* set of parameters and bijections should produce some sort of Brauer tree, and so I have to look at the signs in front of the parameters, together with the Coxeter case philosophy of Hiss–Lübeck–Malle, to guess the right answer.

More generally, the combinatorial Broué conjecture for non-cyclic cases relies on ordering the parameters in a certain way to get the bijection. Although the bijection is defined independently of the cyclotomic Hecke algebra in real life, i.e., there is only one choice, by permuting the parameters associated to each root we get 'different' bijections.

An Example: $\mathrm{PSL}_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
3	$q(q^2+q+1)$	1	1			
4	$q^2(q^2+1)$		1	1		
5	$q^3(q^2+q+1)$	1	1	1	1	
6	q^6	1			1	1

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$$
\begin{aligned}
X_2 &: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0.\\
X_5 &: \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(234) \oplus M_{4,1} \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0.\\
X_3 &: \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(5) \oplus M_{1,1} \to M_{1,1} \oplus M_{1,2} \to C_3 \to 0.\\
X_4 &: \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(4) \to M_{4,2} \to C_4 \to 0.
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ℓ-Extended Finite Groups

Let H be a finite group, and let ρ be a faithful complex representation of H. It is well known that there exists an algebraic number field K, with ring of integers $\mathcal{O} = \mathcal{O}_K$, such that $H \leq \mathrm{GL}_n(\mathcal{O})$ and this embedding induces ρ.

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Let ℓ be an integer with $\gcd(\ell, |H|) = 1$, such that the map $\mathcal{O} \to \mathbb{Z}/\ell\mathbb{Z}$ induces a faithful representation of H over $\mathbb{Z}/\ell\mathbb{Z}$ via ρ. Write M for the $\mathbb{Z}/\ell\mathbb{Z}H$-module, and $G_\ell = M \rtimes H$.

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- $k(G_\ell)$ is a polynomial in ℓ, and $k(G_\ell) \cdot |H|$ is a monic polynomial in ℓ with integer coefficients.
- If H is a reflection group and ρ is its natural representation over $\mathbb{Z}$, then the second coefficient of $k(G_\ell) \cdot |H|$ is $3N$, where N is the number of reflections in H. (A similar formula exists for complex reflection groups.)

Being Generic: An Example

Let $G = \mathrm{PSU}_3(q)$, $\ell \mid (q+1)$. There are three simple modules in the principal block, as $\mathrm{N}_G(P)/\mathrm{C}_G(P) \cong S_3$. G has a permutation representation on q^3+1 points, let Q be a Sylow ℓ-subgroup of the point stabilizer, so that $|Q| = \ell$.

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$$\begin{array}{ll} X_3: & 0 \to \mathcal{P}(3) \to \Omega(M_1) \to C_3 \to 0. \\ X_2: & 0 \to \mathcal{P}(2) \to \mathcal{P}(22) \to \mathcal{P}(2) \oplus M_2 \to C_2 \to 0. \end{array}$$