On the Unit Conjecture for Supersoluble Groups

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A Toy Example

Let K be a field of characteristic $p \geq 0$ and let $R = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Laurent polynomial ring with n variables.
What are the zero divisors of R?

Clearly there are no zero divisors in R.

Thinking of R as KG, where G is the group $\mathbb{Z}^n$, we see that there are no zero divisors in the group rings of torsion-free abelian groups.

The Zero Divisor Conjecture

If G is a group and x is an element of order n in G, then $x^n = 1$, and so the element $x - 1$ is a zero divisor. Hence if we want that there are no zero divisors in KG, as in the case of the abelian group $\mathbb{Z}^n$, then we need G to be torsion free (i.e., have no non-trivial elements of finite order).

Conjecture

If G is a torsion-free group and K is a field, then KG has no zero divisors.

Zero Divisors in Group Rings

The zero divisor conjecture has been solved for increasingly large classes.

Theorem (Bovdi, 1960)

Let G be a poly-$\mathbb{Z}$ group. Then KG has no zero divisors.

Theorem (Formanek, 1973)

Let G be a torsion-free supersoluble group and K be a field. Then KG has no zero divisors.

Theorem (Farkas–Snider, 1976, and Cliff, 1980)

Let G be a torsion-free, virtually polycyclic group and K be a field. Then KG has no zero divisors.

Theorem (Kropholler–Linnell–Moody, 1988)

Let G be a torsion-free, virtually soluble group and K be a field. Then KG has no zero divisors.

A Toy Example Again

Let K be a field of characteristic $p \geq 0$ and let $R = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Laurent polynomial ring with n variables.
What are the invertible elements of R?

These are simply the monomials $\lambda x_{i_1} x_{i_2} \ldots x_{i_m}$, with $\lambda \in K \setminus \{0\}$.

Thinking of R as KG, where G is the group $\mathbb{Z}^n$, we see that the units of $R = KG$ are simply λg, where $\lambda \in K$ and $g \in G$.

The Unit Conjecture

Let G be a finite 2-group, and let K be the field $\mathbb{F}_2$. If ζ lies in the Jacobson radical of the group algebra KG, then $1+\zeta$ is a unit. Since the Jacobson radical has codimension 1, this means that half of the elements of KG are units.

More generally, if G is a group and x has order p in G, then $\hat{x}^p - p\hat{x} = 0$ (where $\hat{x} = 1 + x + \cdots + x^{p-1}$) and so often $(\hat{x}-a)(\hat{x}-1/a) = 1$ for some a. (For the other cases, there are similar constructions.) Hence if we want

$$U(KG) = \{\lambda g : \lambda \in K,\ g \in G\},$$

as in the case of the abelian group $\mathbb{Z}^n$, then we need G to be torsion free (i.e., have no non-trivial elements of finite order).

We always have that λg is a unit: these are called **trivial units**.

Conjecture (Kaplansky, 1969)

If G is a torsion-free group and K is a field, then all units of KG are trivial.

Facts about the Unit Conjecture

- If there is a zero divisor in KG then there is a non-trivial unit in KG. Hence the unit conjecture for G implies the zero divisor conjecture for G.
- If G is a unique-product group then G satisfies the unit conjecture for all fields K. A group is a UP group if, whenever X and Y are finite subsets, there is an element z such that z is expressible **uniquely** as a product $x \cdot y$, where $x \in X$ and $y \in Y$. (Strojnowski proved that every UP group is a 2-UP group.)
- Are all torsion-free groups unique-product groups? **NO**. It was proved by Rips and Segev that there are torsion-free, non-UP groups. An easier example, Γ, was considered by Promislow.
- Using a computer, Promislow searched randomly in Γ, and found a subset X (with $|X| = 14$) such that $X \cdot X$ had no unique product.
- This was the first real use of the computer in this field.

The Passman group Γ

This group Γ is given by the presentation

$$\Gamma = \langle x, y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2}\rangle.$$

Write $z = xy$, $a = x^2$, $b = y^2$, $c = z^2$.
Idea 1: $H = \langle a, b, c\rangle$ is an abelian normal subgroup, and G/H is the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Idea 2: $N = \langle a, b\rangle$ is an abelian normal subgroup, and G/N is the infinite dihedral group D_∞. This second quotient gives a length function on the elements of the group.

- The elements of N (of the form $a^i b^j$) are defined to be length 0.
- Length 1 elements are αx or αy, with $\alpha \in N$.
- Length 2 elements are αxy or αyx, with $\alpha \in N$.
- And so on.

The group ring $K\Gamma$

- We now want to consider the group ring $K\Gamma$, where K is any field.
- We extend the length function from Γ to $K\Gamma$: the length of a sum of elements of G is the maximum of the lengths of the elements.
- We want to rewrite the elements of $K\Gamma$, using the subgroup $H = \langle a, b, c \rangle$ this time. Any element may be written as $Ax + By + C + Dz$, where $A, B, C, D \in KH$.
- This rewriting allows us to construct a representation as matrices over $K\langle a, b, c \rangle$.

$$\begin{pmatrix} C & A & B & D \\ A^x a & C^x & D^x a & B^x \\ B^y b & D^y a^{-1} c^{-1} & C^y & A^y a^{-1} b c^{-1} \\ D^z c & B^z b^{-1} & A^z b^{-1} c & C^z \end{pmatrix}$$

(Here, A^x indicates the conjugate of A by x, and so on.)

Theorems on $K\Gamma$

Using a splitting theorem for units in $K\Gamma$, we can produce two important theorems.

Theorem

The length of a unit in $K\Gamma$ is equal to the length of its inverse.

Theorem

An element of $K\Gamma$ is a unit if and only if its determinant is in K.

Thus it must be really easy to check if an element of $K\Gamma$ is invertible, simply by checking its determinant. A length-3 element looks like the following:

$$\alpha_1 x + (\alpha_2 + \alpha_3 c)y + \alpha_4 + (\alpha_5 + \alpha_6 c^{-1})z.$$

(Here, $\alpha_i \in N$.)

The determinant of a length-3 element

$$\begin{gathered}
\alpha_1\alpha^x_1\alpha^y_1\alpha^z_1 - \alpha_1\alpha^x_1\alpha^y_4\alpha^z_4 a - \alpha_1\alpha^x_2\alpha^y_3\alpha^z_1 + \alpha_1\alpha^x_2\alpha^y_4\alpha^z_6 - \alpha_1\alpha^x_3\alpha^y_2\alpha^z_1 + \alpha_1\alpha^x_3\alpha^y_4\alpha^z_5 - \alpha_1\alpha^x_5\alpha^y_1\alpha^z_5 b + \alpha_1\alpha^x_5\alpha^y_2\alpha^z_4 ab \\
-\alpha_1\alpha^x_6\alpha^y_1\alpha^z_6 b + \alpha_1\alpha^x_6\alpha^y_3\alpha^z_4 ab - \alpha_2\alpha^x_1\alpha^y_1\alpha^z_3 + \alpha_2\alpha^x_1\alpha^y_6\alpha^z_4 + \alpha_2\alpha^x_2\alpha^y_2\alpha^z_2 + \alpha_2\alpha^x_2\alpha^y_3\alpha^z_3 - \alpha_2\alpha^x_2\alpha^y_5\alpha^z_5 a^{-1} \\
-\alpha_2\alpha^x_2\alpha^y_6\alpha^z_6 a^{-1} + \alpha_2\alpha^x_3\alpha^y_2\alpha^z_3 - \alpha_2\alpha^x_3\alpha^y_6\alpha^z_5 a^{-1} + \alpha_2\alpha^x_4\alpha^y_1\alpha^z_5 ba^{-1} - \alpha_2\alpha^x_4\alpha^y_2\alpha^z_4 b - \alpha_3\alpha^x_1\alpha^y_1\alpha^z_2 + \alpha_3\alpha^x_1\alpha^y_5\alpha^z_4 \\
+\alpha_3\alpha^x_2\alpha^y_3\alpha^z_2 - \alpha_3\alpha^x_2\alpha^y_5\alpha^z_6 a^{-1} + \alpha_3\alpha^x_3\alpha^y_2\alpha^z_2 + \alpha_3\alpha^x_3\alpha^y_3\alpha^z_3 - \alpha_3\alpha^x_3\alpha^y_5\alpha^z_5 a^{-1} - \alpha_3\alpha^x_3\alpha^y_6\alpha^z_6 a^{-1} + \alpha_3\alpha^x_4\alpha^y_1\alpha^z_6 ba^{-1} \\
-\alpha_3\alpha^x_4\alpha^y_3\alpha^z_4 b - \alpha_4\alpha^x_2\alpha^y_4\alpha^z_2 b^{-1} + \alpha_4\alpha^x_2\alpha^y_5\alpha^z_1 a^{-1}b^{-1} - \alpha_4\alpha^x_3\alpha^y_4\alpha^z_3 b^{-1} + \alpha_4\alpha^x_3\alpha^y_6\alpha^z_1 a^{-1}b^{-1} - \alpha_4\alpha^x_4\alpha^y_1\alpha^z_1 a^{-1} \\
+\alpha_4\alpha^x_4\alpha^y_4\alpha^z_4 + \alpha_4\alpha^x_5\alpha^y_1\alpha^z_3 - \alpha_4\alpha^x_5\alpha^y_6\alpha^z_4 + \alpha_4\alpha^x_6\alpha^y_1\alpha^z_2 - \alpha_4\alpha^x_6\alpha^y_5\alpha^z_4 + \alpha_5\alpha^x_1\alpha^y_4\alpha^z_2 ab^{-1} - \alpha_5\alpha^x_1\alpha^y_5\alpha^z_1 b^{-1} \\
+\alpha_5\alpha^x_4\alpha^y_3\alpha^z_1 - \alpha_5\alpha^x_4\alpha^y_4\alpha^z_6 - \alpha_5\alpha^x_5\alpha^y_2\alpha^z_2 a - \alpha_5\alpha^x_5\alpha^y_3\alpha^z_3 a + \alpha_5\alpha^x_5\alpha^y_5\alpha^z_5 + \alpha_5\alpha^x_5\alpha^y_6\alpha^z_6 - \alpha_5\alpha^x_6\alpha^y_3\alpha^z_2 a + \alpha_5\alpha^x_6\alpha^y_5\alpha^z_6 \\
+\alpha_6\alpha^x_1\alpha^y_4\alpha^z_3 ab^{-1} - \alpha_6\alpha^x_1\alpha^y_6\alpha^z_1 b^{-1} + \alpha_6\alpha^x_4\alpha^y_2\alpha^z_1 - \alpha_6\alpha^x_4\alpha^y_4\alpha^z_5 - \alpha_6\alpha^x_5\alpha^y_2\alpha^z_3 a + \alpha_6\alpha^x_5\alpha^y_6\alpha^z_5 - \alpha_6\alpha^x_6\alpha^y_2\alpha^z_2 a \\
-\alpha_6\alpha^x_6\alpha^y_3\alpha^z_3 a + \alpha_6\alpha^x_6\alpha^y_5\alpha^z_5 + \alpha_6\alpha^x_6\alpha^y_6\alpha^z_6 + c(-\alpha_1\alpha^x_2\alpha^y_2\alpha^z_1 + \alpha_1\alpha^x_2\alpha^y_4\alpha^z_5 - \alpha_1\alpha^x_6\alpha^y_1\alpha^z_5 b + \alpha_1\alpha^x_6\alpha^y_2\alpha^z_4 ab \\
+\alpha_2\alpha^x_2\alpha^y_2\alpha^z_3 - \alpha_2\alpha^x_2\alpha^y_6\alpha^z_5 a^{-1} - \alpha_3\alpha^x_1\alpha^y_1\alpha^z_3 + \alpha_3\alpha^x_1\alpha^y_6\alpha^z_4 + \alpha_3\alpha^x_2\alpha^y_2\alpha^z_2 + \alpha_3\alpha^x_2\alpha^y_3\alpha^z_3 - \alpha_3\alpha^x_2\alpha^y_5\alpha^z_5 a^{-1} \\
-\alpha_3\alpha^x_2\alpha^y_6\alpha^z_6 a^{-1} + \alpha_3\alpha^x_3\alpha^y_2\alpha^z_3 - \alpha_3\alpha^x_3\alpha^y_6\alpha^z_5 a^{-1} + \alpha_3\alpha^x_4\alpha^y_1\alpha^z_5 ba^{-1} - \alpha_3\alpha^x_4\alpha^y_2\alpha^z_4 b - \alpha_4\alpha^x_2\alpha^y_4\alpha^z_3 b^{-1} \\
+\alpha_4\alpha^x_2\alpha^y_6\alpha^z_1 a^{-1}b^{-1} + \alpha_4\alpha^x_6\alpha^y_1\alpha^z_3 - \alpha_4\alpha^x_6\alpha^y_6\alpha^z_4 + \alpha_5\alpha^x_1\alpha^y_4\alpha^z_3 ab^{-1} - \alpha_5\alpha^x_1\alpha^y_6\alpha^z_1 b^{-1} + \alpha_5\alpha^x_4\alpha^y_2\alpha^z_1 - \alpha_5\alpha^x_4\alpha^y_4\alpha^z_5 \\
-\alpha_5\alpha^x_5\alpha^y_2\alpha^z_3 a + \alpha_5\alpha^x_5\alpha^y_6\alpha^z_5 - \alpha_5\alpha^x_6\alpha^y_2\alpha^z_2 a - \alpha_5\alpha^x_6\alpha^y_3\alpha^z_3 a + \alpha_5\alpha^x_6\alpha^y_5\alpha^z_5 + \alpha_5\alpha^x_6\alpha^y_6\alpha^z_6 - \alpha_6\alpha^x_6\alpha^y_2\alpha^z_3 a \\
+\alpha_6\alpha^x_6\alpha^y_6\alpha^z_5) + c^{-1}(-\alpha_1\alpha^x_3\alpha^y_3\alpha^z_1 + \alpha_1\alpha^x_3\alpha^y_4\alpha^z_6 - \alpha_1\alpha^x_5\alpha^y_1\alpha^z_6 b + \alpha_1\alpha^x_5\alpha^y_3\alpha^z_4 ab - \alpha_2\alpha^x_1\alpha^y_1\alpha^z_2 + \alpha_2\alpha^x_1\alpha^y_5\alpha^z_4 \\
+\alpha_2\alpha^x_2\alpha^y_3\alpha^z_2 - \alpha_2\alpha^x_2\alpha^y_5\alpha^z_6 a^{-1} + \alpha_2\alpha^x_3\alpha^y_2\alpha^z_2 + \alpha_2\alpha^x_3\alpha^y_3\alpha^z_3 - \alpha_2\alpha^x_3\alpha^y_5\alpha^z_5 a^{-1} - \alpha_2\alpha^x_3\alpha^y_6\alpha^z_6 a^{-1} + \alpha_2\alpha^x_4\alpha^y_1\alpha^z_6 ba^{-1} \\
-\alpha_2\alpha^x_4\alpha^y_3\alpha^z_4 b + \alpha_3\alpha^x_3\alpha^y_3\alpha^z_2 - \alpha_3\alpha^x_3\alpha^y_5\alpha^z_6 a^{-1} - \alpha_4\alpha^x_3\alpha^y_4\alpha^z_2 b^{-1} + \alpha_4\alpha^x_3\alpha^y_5\alpha^z_1 a^{-1}b^{-1} + \alpha_4\alpha^x_5\alpha^y_1\alpha^z_2 - \alpha_4\alpha^x_5\alpha^y_5\alpha^z_4 \\
-\alpha_5\alpha^x_5\alpha^y_3\alpha^z_2 a + \alpha_5\alpha^x_5\alpha^y_5\alpha^z_6 + \alpha_6\alpha^x_1\alpha^y_4\alpha^z_2 ab^{-1} - \alpha_6\alpha^x_1\alpha^y_5\alpha^z_1 b^{-1} + \alpha_6\alpha^x_4\alpha^y_3\alpha^z_1 - \alpha_6\alpha^x_4\alpha^y_4\alpha^z_6 - \alpha_6\alpha^x_5\alpha^y_2\alpha^z_2 a \\
-\alpha_6\alpha^x_5\alpha^y_3\alpha^z_3 a + \alpha_6\alpha^x_5\alpha^y_5\alpha^z_5 + \alpha_6\alpha^x_5\alpha^y_6\alpha^z_6 - \alpha_6\alpha^x_6\alpha^y_3\alpha^z_2 a + \alpha_6\alpha^x_6\alpha^y_5\alpha^z_6)
\end{gathered}$$

You don't want to see the length-4 determinant.

A Splitting Theorem for Supersoluble Groups

Suppose that G has an infinite dihedral quotient with kernel N, generated by Nx and Ny.

Let σ be an element of KG, and suppose that there is τ such that $\sigma\tau \in KN$. Then

$$\sigma = \eta^{-1}(\alpha_1 + \beta_1\gamma_1)(\alpha_2 + \beta_2\gamma_2)\dots(\alpha_n + \beta_n\gamma_n),$$

with $\gamma_i \in \{x, y\}$, and $\alpha_i, \beta_i, \eta \in KN$.

This splitting is unique in the following sense: if

$$\sigma = \varepsilon^{-1}(\alpha'_1 + \beta'_1\gamma_1)(\alpha'_2 + \beta'_2\gamma_2)\dots(\alpha'_n + \beta'_n\gamma_n)$$

is some other splitting, then (up to units) $\eta = \varepsilon$, $\alpha_i = \alpha'_i$ and $\beta_i = \beta'_i$.

What is this η?

Write s for the split of σ, so that $\sigma = \eta^{-1}s$. For example, if $s = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)$, then

$$s = \alpha_1\alpha_2 + \alpha_1\beta_2 y + \beta_1\alpha_2^x x + \beta_1\beta_2^x xy,$$

In order for $\eta^{-1}s$ to be in KG, we must have that η divides each coefficient in front of the words in x and y.

Proposition

If $\eta = 1$ in the split of σ, then σ is not a unit.

Theorem

In a 'minimal counterexample' to the unit conjecture, there are no non-trivial units of length 1 or 2.

This Doesn't Work for Length 3

The length-3 elements are as follows:

Word	Coefficient
xyx	$\beta_1\beta_2^x\beta_3^{yx}$
yx	$\alpha_1\beta_2\beta_3^y$
xy	$\beta_1\beta_2^x\alpha_3^{yx}$
y	$\alpha_1\beta_2\alpha_3^y$
x	$\alpha_1\alpha_2\beta_3 + \beta_1\alpha_2^x\alpha_3^x$
1	$\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2^x\beta_3^x x^2$

Let $G = \Gamma$. If $\alpha_1 = \alpha_2 = \alpha_3 = \beta_3 = 1$, $\beta_1 = -a$, $\beta_2 = 1 - a$, then $a - 1$ divides each of the coefficients, and so you can have $\eta \neq 1$ for length-3 elements.

At the moment we cannot do length-3 elements in full generality, and so we need another idea.

The $*$-antiautomorphism

Let $*$ denote the antiautomorphism $g \mapsto g^{-1}$. Extend $*$ to a K-linear map on KG. An element of KG is $*$-**symmetric** if $\sigma^* = \sigma$, and $*$-**inverse** if $\sigma\sigma^* \in K$.

Proposition

There are no non-trivial units in KG if and only if there are no non-trivial $*$-symmetric or $*$-inverse units.

Let σ be a non-trivial $*$-inverse unit; write $\sigma = \eta^{-1}s$, so that $\sigma^* = s^*\eta^{*-1}$. We have $\sigma\sigma^* = \eta^{-1}ss^*(\eta^*)^{-1}$, so if $\sigma\sigma^* = k \in K$, then $ss^* = k\eta\eta^*$. In the case where G is the Passman group Γ, the element ss^* is

$$\prod(\alpha_i\alpha_i^* + \beta_i\beta_i^*) := \prod D_i.$$

This makes the left-hand side particularly easy to evaluate, and allows us to prove that there are no length-3, $*$-inverse units in Γ.

Higher Lengths?

This is where things start getting interesting, and unfortunately right now we don't have an answer. It seems as though length 4 can be tackled in a similar way, but with more complications. However, as the length increases it seems possible that one can have that each of the D_i divides either η or η^*.

This is assuming that there are non-trivial units, which is my current opinion.

If this is not the case, then an inductive approach might work. This would prove either that there are non-trivial $*$-inverse units or there are not.

Even if there are not, there still might be $*$-symmetric units. These might need a different approach, since there is no 'nice' expression for the product of the splits in this case.