Perverse Equivalences and Broué's Conjecture

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Notation and Conventions

Throughout this talk,

- G is a finite group,
- ℓ is a prime,
- k is a field of characteristic ℓ,
- B is a block of kG, with defect group D and Brauer correspondent b;
- P is a Sylow ℓ-subgroup of G,
- Q is a general ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

This talk is joint work with Raphaël Rouquier.

Representation Theory is Local

Some of the deepest and most difficult conjectures in modular representation theory relate the structure of a block B of kG to the structure of its Brauer correspondent b, a block of $k\mathrm{N}_G(D)$. Write $\ell(B)$ for the number of simple B-modules.

Alperin's weight conjecture gives a precise conjecture about the number of simple B-modules, $\ell(B)$, in terms of local information. If D is abelian, the conjecture reduces to

$$\ell(B) = \ell(b).$$

Is there a structural/geometric reason for B and b having the same number of simple modules?

Conjecture (Broué, 1990)

Let G be a finite group, and let B be a ℓ-block of G with abelian defect group D. If b is the Brauer correspondent of B in $N_G(D)$, then B and b are derived equivalent.

When Is Broué's Conjecture Known?

Broué's conjecture is known for quite a few groups:

- A_n, S_n (Chuang–Rouquier, Marcus);
- $\mathrm{GL}_n(q)$, $\ell \nmid q$ (Chuang–Rouquier);
- D cyclic, $C_2 \times C_2$ (Rouquier, Erdmann, Rickard);
- G finite, $\ell = 2$, B principal;
- G finite, $\ell = 3$, $|P| = 9$, B principal (Koshitani, Kunugi, Miyachi, Okuyama, Waki);
- $\mathrm{SL}_2(q)$, $\ell \mid q$ (Chuang, Kessar, Okuyama);
- various low-rank Lie type groups $L(q)$ with $\ell \nmid q$ and sporadic groups. (Okuyama, Holloway, etc.)

Principal Blocks Are Good

In representation theory, one standard method of proof is to reduce a conjecture to finite simple groups and then use their classification.

In general, there is no (known) reduction of Broué's conjecture to simple groups, but for principal blocks there is.

Theorem

Let G be a finite group. If P is abelian, then there are normal subgroups $H \leq L$ such that

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- L/H *is a direct product of simple groups and an abelian* ℓ*-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L (compatible with automorphisms of the simple components) passes up to G. Thus if Broué's conjecture for principal blocks holds for all simple groups (with automorphisms), it holds for all groups.

How Do You Find Derived Equivalences?

There are four main methods to prove that B and b are derived equivalent.

1. Okuyama deformations: using many steps, deform the Green correspondents of the simple modules for B into the simple modules for b. This works well for small groups.
2. Rickard's Theorem: randomly find complexes in the derived category of b related to the Green correspondents of the simple modules for B, and if they 'look' like simple modules (i.e., Homs and Exts behave nicely) then there is a derived equivalence $B \to b$.
3. More structure: if B and b are more closely related (say Morita or Puig equivalent) then they are derived equivalent. More generally, find another block B' for some other group, an equivalence $B \to B'$, and a (previously known) equivalence $B' \to b$.
4. Perverse equivalence: build a derived equivalence up step by step in an algorithmic way.

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras, $\mathcal{A} = mod\text{-}A$, $\mathcal{B} = mod\text{-}B$.

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, T_1, $T_2, \ldots$, T_r, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \ldots, S_i$, and $D_i^b(\mathcal{A})$ denotes the subcategory of $D^b(\mathcal{A})$ with support modules in $\mathcal{A}_i$, then

- F induces equivalences $D_i^b(\mathcal{A}) \to D_i^b(\mathcal{B})$, and
- $F[\pi(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{B}_i/\mathcal{B}_{i-1}$.

Note that mod-B is determined, up to equivalence, by A, π, and the ordering of the S_i.

What is a Perverse Equivalence?

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An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, T_1, $T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, for all i, the cohomology of $F(S_i)$ only involves T_j for $j < i$, except for one copy of T_i in degree $-\pi(i)$, and T_j can only appear in degrees less than $-\pi(j)$.

Benefits of a Perverse Equivalence

The perverse equivalence is 'better' than a general derived equivalence.

- Has an underlying geometric interpretation (for Lie-type groups).
- The π-function 'comes from' Lusztig's A-function. For $\ell \mid \Phi_d(q)$, if $d = 1$ or $d = 2$, π is the A-function, but for $d \geq 3$ it is not clear (yet) what it is, exactly.
- There is an algorithm that gives us a perverse equivalence from $B_0(kN)$ to **some** algebra, so only need to check that the target is $B_0(kG)$. (This is simply checking that the Green correspondents are the last terms in the complexes.)

This algorithm is very useful!

The Algorithm

Label the simple modules $k = T_1, \ldots, T_n$, with perversity function $\pi : \{1, \ldots, n\} \to \mathbb{Z}$. Assume that $\pi(i) \geq \pi(i-1)$ and $\pi(1) = 0$. We will describe the complex corresponding to the ith simple.

The complex X_i for T_i is $0 \to P_{i,\pi(i)} \to P_{i,\pi(i-1)} \to \cdots \to P_{i,1} \to 0$.

1. Let $P_{i,\pi(i)} = \mathcal{P}(i)$. Let $H^{-\pi(i)}(X_i)$ be the largest submodule of $P_{i,\pi(i)}$ with composition factors T_j, where $\pi(j) < \pi(i)$. Write $A_{i,\pi(i)}$ for the quotient $P_{i,\pi(i)}/H^{-\pi(i)}(X_i)$.
2. At the mth stage, let $P_{i,m} = \mathcal{P}(A_{i,m+1})$, and let $H^{-m}(X_i)$ be the largest submodule of $P_{i,m}$ with composition factors T_j, where $\pi(j) < m$. Let A_m be the quotient of $P_{i,m}$ by $A_{i,m+1}$ and $H^{-m}(X_i)$.
3. Let $H^{-1}(X_i) = P_{i,1}/A_{i,2}$. Let Y_i be the largest submodule of $H^{-1}(X_i)$ with composition factors T_j, where $\pi(j) = 0$. $H^{-1}(X_i)/Y_i$ is the Green correspondent of a simple B-module, if we have the right $\pi(-)$.

If induction and restriction do not yield the right stable equivalence, we need to replace $P_{i,m}$ with relatively projective modules, for some m.

An Example

Let $G = M_{11}$, $\ell = 3$.

π	Ord. Char.	S_1	S_3	S_7	S_2	S_4	S_6	S_5
0	1	1						
2	10		1					
3	10			1				
4	16	1	1		1			
5	11	1			1	1		
6	44			1	1	1	1	
7	55	1	1		1	1	1	1
	10							1
	16	1				1		1

The cohomology of the complexes gives the rows of the decomposition matrix.

Which Groups Have Perverse Equivalences?

- All groups, D cyclic or $C_2 \times C_2$
- $\mathrm{PSL}_3(q)$, $\ell = 3 \mid (q-1)$, P abelian
- $\mathrm{PSL}_4(q)$, $\mathrm{PSL}_5(q)$, $\ell = 3 \mid (q+1)$, $P = C_3 \times C_3$
- $\mathrm{PSU}_3(q)$, $\ell = 3 \mid (q+1)$, P abelian
- $\mathrm{PSU}_4(q)$, $\mathrm{PSU}_5(q)$, $\ell = 3 \mid (q-1)$
- b a block of $\mathrm{PSU}_n(q)$, $\ell = 5 \mid (q+1)$, b has defect group $C_5 \times C_5$
- $\mathrm{PSp}_4(q)$, $\ell = 3 \mid (q-1)$ or $(q+1)$, $P = C_3 \times C_3$
- (almost) $\Omega_8^+(q)$, $\ell = 5 \mid (q^2+1)$, $P = C_5 \times C_5$
- $G_2(q)$, $\ell = 5 \mid (q+1)$, $P = C_5 \times C_5$
- S_6, A_7, A_8, $\ell = 3$ (A_6 does not)
- M_{11}, $M_{22}.2$, M_{23}, HS, $\ell = 3$ (M_{22} does not)
- $\mathrm{SL}_2(8)$, J_1, ${}^2G_2(q)$, $\ell = 2$ in two steps
- S_n, A_n, $\mathrm{GL}_n(q)$ in multiple steps

An Example: $PSL_3(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char	S_1	S_5	S_2	S_3	S_4
0	1	1				
2	$q(q+1)$	1	1			
3	$(q+1)(q^2+q+1)/3$	1	1	1		
3	$(q+1)(q^2+q+1)/3$	1	1		1	
3	$(q+1)(q^2+q+1)/3$	1	1			1

		H^{-3}	H^{-2}	H^{-1}	Total
X_5:	$0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to C_5 \to 0.$		$1/5$	11	$5-1$
X_2:	$0 \to \mathcal{P}(2) \to \mathcal{P}(34) \to \mathcal{P}(5) \to C_2 \to 0.$	$1/5/2$	1		$2-5$
X_3:	$0 \to \mathcal{P}(3) \to \mathcal{P}(24) \to \mathcal{P}(5) \to C_3 \to 0.$	$1/5/3$	1		$3-5$
X_4:	$0 \to \mathcal{P}(4) \to \mathcal{P}(23) \to \mathcal{P}(5) \to C_4 \to 0.$	$1/5/4$	1		$4-5$

An Example: $\mathrm{PSp}_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char	S_1	S_5	S_2	S_3	S_4
0	1	1				
3	$q(q-1)^2/2$		1			
3	$q(q^2+1)/2$	1		1		
3	$q(q^2+1)/2$	1			1	
4	q^4	1	1	1	1	1

$$X_5: \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0.$$
$$X_2: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0.$$
$$X_3: \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(5) \to \mathcal{P}(2) \oplus M_{1,1} \to C_3 \to 0.$$
$$X_4: \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(23) \to \mathcal{P}(5) \to C_4 \to 0.$$

An Example: $PSL_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
3	$q(q^2+q+1)$	1	1			
4	$q^2(q^2+1)$		1	1		
5	$q^3(q^2+q+1)$	1	1	1	1	
6	q^6	1			1	1

$$\begin{aligned}
X_2 &: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0. \\
X_5 &: \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(234) \oplus M_{4,1} \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0. \\
X_3 &: \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(5) \oplus M_{1,1} \to M_{1,1} \oplus M_{1,2} \to C_3 \to 0. \\
X_4 &: \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(4) \to M_{4,2} \to C_4 \to 0.
\end{aligned}$$

Some Remarks

- Since $\pi(-)$, the ordering and the first category determine the perverse equivalence, it is a very compact way of defining a (type of) derived equivalence.
- Computationally, this reduces finding a derived equivalence to finding the Green correspondents of the simple modules for G, a much simpler task.
- For groups of Lie type, it seems as though the complexes above do not really depend on ℓ, and only on d, where $\ell \mid \Phi_d(q)$. It might be possible to use these perverse equivalences to prove real results in this direction.

Complex Reflection Groups

Generically, the automizer H of a Φ_d-torus in a group of Lie type is a complex reflection group with $|H|$ and $\Phi_d(q)$ coprime. The action of H on the torus gives a representation of H over $\mathbb{F}_\ell^n$ for $\ell \mid \Phi_d(q)$.

This representation is (for sufficiently large ℓ) invariant of ℓ and q, and only dependent on d and G.

The principal block of the normalizer is Morita equivalent to the group algebra of $k\mathrm{N}_G(P)/\mathrm{O}_{\ell'}(\mathrm{C}_G(P))$, which is $\mathbb{F}_\ell^n \rtimes H$.

This focuses attention on (in particular complex reflection) groups acting on the 'natural' $\mathbb{F}_\ell$-module M, and $M \rtimes H$.

ℓ-Extended Finite Groups

Let H be a finite group, and let ρ be a faithful complex representation of H. It is well known that there exists an algebraic number field K, with ring of integers $\mathcal{O} = \mathcal{O}_K$, such that $H \leq \mathrm{GL}_n(\mathcal{O})$ and this embedding induces ρ.

Let ℓ be an integer with $\gcd(\ell, |H|) = 1$, such that the map $\mathcal{O} \to \mathbb{Z}/\ell\mathbb{Z}$ induces a faithful representation of H over $\mathbb{Z}/\ell\mathbb{Z}$ via ρ. Write M for the $\mathbb{Z}/\ell\mathbb{Z}H$-module, and $G_\ell = M \rtimes H$.

- $k(G_\ell)$ is a polynomial in ℓ, and $k(G_\ell) \cdot |H|$ is a monic polynomial in ℓ with integer coefficients.
- If H is a reflection group and ρ is its natural representation over $\mathbb{Z}$, then the second coefficient of $k(G_\ell) \cdot |H|$ is $3N$, where N is the number of reflections in H. (A similar formula exists for complex reflection groups.)

Being Generic: An Example

Let $G = \mathrm{PSU}_3(q)$, $\ell \mid (q+1)$. There are three simple modules in the principal block, as $\mathrm{N}_G(P)/\mathrm{C}_G(P) \cong S_3$. G has a permutation representation on q^3+1 points, let Q be a Sylow ℓ-subgroup of the point stabilizer, so that $|Q| = \ell$.

π	Ord. Char.	1_1	2_1	1_2
0	1	1		
2	$q(q-1)$		1	
3	q^3	1	2	1

Let M_1 be the trivial source module $23/13/23/\cdots/23$ with vertex Q, $\dim M_1 = 3\ell$. Let M_2 be the relatively projective summand of $(1/1/1)\uparrow_Q^N$ with head 23, $\dim M_2 = 9\ell$. For $\ell = 5, 7, 17$ we have the following:

$$X_3: \quad 0 \to \mathcal{P}(3) \to \Omega(M_1) \to C_3 \to 0.$$
$$X_2: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(22) \to \mathcal{P}(2) \oplus M_2 \to C_2 \to 0.$$

Linear vs Unitary

The philosophy is the following:

$$\mathrm{GU}_n(q) \quad \leftrightsquigarrow \quad \mathrm{GL}_n(-q).$$

More specifically, let $\ell \mid \Phi_d(q)$, so that d denotes the order of q modulo ℓ. Let e denote the order of $-q$ modulo ℓ. We have

$$e = \begin{cases} 2d & d \text{ odd} \\ d/2 & d/2 \text{ odd} \\ d & 4 \mid d \end{cases}.$$

Call ℓ unitary if e is odd, and linear otherwise. Split linear up into linear-1 for d odd, and linear-4 for $4 \mid d$.
The Φ_d-representations of $\mathrm{GU}_n(q)$ should correspond to the Φ_e-representations of $\mathrm{GL}_n(q)$.

What Can Be Done?

Suppose that we want to adapt the Chuang–Rouquier approach to unitary groups.

There are three types of primes, as we have seen, and there are two questions that need to be answered: 'Can the categorification take place?' and 'Is there a "good" block, which is easily seen to satisfy Broué's conjecture?'

The following is our best guess at this time:

Prime	Categorification?	Good block
Linear-4	Yes	Yes
Unitary	Yes	No
Linear-1	Don't know	Don't know

Being Generic: Another Example

Let $G = \mathrm{PSp}_4(q)$, $\ell \mid (q+1)$.

π	Ord. Char.	1_1	2_1	1_2	1_3	1_4
0	1	1				
3	$q(q-1)^2/2$		1			
3	$q(q^2+1)/2$	1		1		
3	$q(q^2+1)/2$	1			1	
4	q^4	1	1	1	1	1

Let $M_{1,1} = 34/5/34/\cdots/34$, $M_{1,2} = 5/12/5\cdots/5$, $M_{2,1} = 24/5/24/\cdots/24$, $M_{2,2} = 5/13/5\cdots/5$, of dimension 2ℓ. For $\ell = 3, 5, 7$ we have the following:

$$
\begin{aligned}
X_5 &: \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to \mathcal{P}(4) \oplus M_{1,1} \oplus M_{2,1} \to C_5 \to 0. \\
X_2 &: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{2,2} \to C_2 \to 0. \\
X_3 &: \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(5) \to \mathcal{P}(2) \oplus M_{1,2} \to C_3 \to 0. \\
X_4 &: \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(44) \to \mathcal{P}(2345) \to \mathcal{P}(5) \oplus M_{1,2} \oplus M_{2,2} \to C_4 \to 0.
\end{aligned}
$$

The Held Group, $\ell = 5$

π	χ	S_1	$S_{2,i}$	S_8	$S_{6,i}$	$S_{4,i}$	S_9	S_{10}	S_3	$S_{5,i}$	S_7
0	1	1									
0	51_i		1								
3	4352	1	1	1							
4	7497_i		1	1	1						
5	153_i					1					
5	6528						1				
7	21504_1	1		1	1			1			
?	17493	1					1	1	1		
?	1029_i								1	1	
?	6272					1				1	1
	21504_2						1	1			1
	23324				1			1	1	1	1