Blocks with Abelian Defect Groups

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Notation and Conventions

Throughout this talk,

- G is a finite group,
- ℓ is a prime,
- K is a field of characteristic ℓ,
- P is a Sylow ℓ-subgroup of G, and
- Q is a general ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

This talk is (in places) joint work with Charles Eaton, Radha Kessar, Markus Linckelmann, Hyohe Miyachi and Raphaël Rouquier.

Representation Theory is Local

The deepest and most difficult conjectures in representation theory tend to relate the representation theory of G in characteristic ℓ with that of $(\ell\text{-})$local subgroups $N_G(Q)$, where Q is an ℓ-subgroup of G.

Recall that a block B of KG is an indecomposable, 2-sided ideal summand of KG. To every block is attached a defect group D, which 'controls' the representation theory of B. The local conjectures are localized further to relate B with a block b of $KN_G(D)$, called the Brauer correspondent.

Alperin's weight conjecture gives a precise conjecture about the number of simple B-modules, $\ell(B)$, in terms of local information. If D is abelian, the conjecture reduces to

$$\ell(B) = \ell(b).$$

Broué's Conjecture

[B is a block of KG, defect group D, b its Brauer correspondent in $N_G(D)$.]

If D is abelian, Alperin's weight conjecture states that

$$\ell(B) = \ell(b);$$

is there a structural/geometric reason for B and b having the same number of simple modules?

Conjecture (Broué, 1990)

Let G be a finite group, and let B be a ℓ-block of G with abelian defect group D. Let b be the Brauer correspondent in $N_G(D)$. Then B and b are derived equivalent.

When Is Broué's Conjecture Known?

Broué's conjecture is known for quite a few groups:

- A_n, S_n (Chuang–Rouquier, Marcus);
- $GL_n(q)$ (Chuang–Rouquier);
- D cyclic, $C_2 \times C_2$ (Rouquier, Erdmann, Rickard);
- G finite, $\ell = 2$, B principal;
- G finite, $\ell = 3$, $|P| = 9$ B principal (Koshitani, Kunugi, Miyachi, Okuyama, Waki);
- $SL_2(q)$, $\ell \mid q$ (Chuang, Kessar, Okuyama)
- various low-rank Lie type groups $L(q)$ with $\ell \nmid q$.

The Principal Block

If $B_1, \ldots, B_r$ are the blocks of KG, then the simple KG-modules are exactly the union of the simple B_i-modules.

The block contributing the trivial module is called the principal block, and denoted by $B_0(KG)$. Its defect group is always the Sylow ℓ-subgroup P, so its Brauer correspondent is a block of $KN_G(P)$.

Theorem (Brauer's third main theorem)

The Brauer correspondent of $B_0(KG)$ is $B_0(KN_G(P))$.

Thus if we are considering principal blocks, we need to relate the principal block of KG with the principal block of $KN_G(P)$.

Principal Blocks Are Good

In representation theory, one standard method of proof is to reduce a conjecture to finite simple groups and then use the Classification of the Finite Simple Groups.

In general, there is no (known) reduction of Broué's conjecture to simple groups, but for principal blocks there is.

Theorem

Let G be a finite group, and suppose that P is abelian. Then there are normal subgroups $H \leq L$ such that

- *$\ell \nmid |H|$,*
- *$\ell \nmid |G : L|$, and*
- *L/H is a direct product of simple groups and an abelian ℓ-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L passes up to G. Thus if Broué's conjecture for principal blocks holds for all simple groups, it holds for all groups.

How Do You Find Derived Equivalences?

There are four main methods to prove that B and b are derived equivalent.

1. Okuyama deformations: using many steps, deform the Green correspondents of the simple modules for B into those for b. This works well for small groups.

2. Rickard's Theorem: randomly find complexes in the derived category of b related to the Green correspondents of the simple modules for B, and if they 'look' like simple modules (i.e., Homs and Exts behave nicely) then there is a derived equivalence $B \to b$.

3. More structure: if B and b are more closely related (say Morita or Puig equivalent) then they are derived equivalent. More generally, find another block B' for some other group, an equivalence $B \to B'$, and a (previously known) equivalence $B' \to b$.

4. Perverse equivalence: build a derived equivalence up step by step in an algorithmic way.

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras.

An equivalence $F : D^b(\text{mod-}A) \to D^b(\text{mod-}B)$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r, T_1, T_2, \ldots, T_r$, and
- a function $p : \{1, \ldots, r\} \to \mathbb{Z}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \ldots, S_i$, then

- F induces equivalences $D^b(\mathcal{A}_i) \to D^b(\mathcal{B}_i)$, and
- $F[p(i)]$ induces an equivalence $\mathcal{A}_i / \mathcal{A}_{i-1} \to \mathcal{B}_i / \mathcal{B}_{i-1}$.

Note that mod-B is determined, up to equivalence, by A, p, and the ordering of the S_i.

Benefits of a Perverse Equivalence

The perverse equivalence is 'better' than a general derived equivalence.

- Has an underlying geometric interpretation (for Lie-type groups).
- The p-function comes from Lusztig's a-function (so is known).
- There is an algorithm that gives us a perverse equivalence from $B_0(KN)$ to **some** algebra, so only need to check that the target is $B_0(KG)$. (This is simply checking that the Green correspondents are the last terms in the complexes.)

This algorithm is so useful it will be simply called the algorithm.

An Example

Let $G = M_{11}$, $\ell = 3$.

p	Ord. Char.	S_1	S_3	S_7	S_2	S_4	S_6	S_5
0	1	1						
2	10		1					
3	10			1				
4	16	1	1		1			
5	11	1			1	1		
6	44			1	1	1	1	
7	55	1	1		1	1	1	1
	10							1
	16	1				1		1

The homology of the complexes gives the rows of the decomposition matrix.

Which Groups Have Perverse Equivalences?

- $PSL_3(q)$, $\ell = 3 \mid (q-1)$
- $PSL_4(q)$, $PSL_5(q)$, $\ell = 3 \mid (q+1)$
- $PSU_3(q)$, $\ell = 3 \mid (q+1)$
- $PSU_4(q)$, $PSU_5(q)$, $\ell = 3 \mid (q-1)$
- $PSp_4(q)$, $\ell = 3 \mid (q-1), (q+1)$
- (almost) $PSp_8(q)$, $\ell = 5 \mid (q^2+1)$, $P = C_5 \times C_5$
- (almost) $\Omega_8^+(q)$, $\ell = 5 \mid (q^2+1)$, $P = C_5 \times C_5$
- $G_2(q)$, $\ell = 5 \mid (q+1)$, $P = C_5 \times C_5$
- S_6, A_7, A_8, $\ell = 3$ (A_6 does not)
- M_{11}, $M_{22}.2$, M_{23}, HS, $\ell = 3$ (M_{22} does not)

However...

In most of the cases above, Lusztig's *a*-function gives a perverse equivalence (if G is of Lie type), but in the case $\mathrm{PSL}_4(q)$, something weird goes on.

p	Ord. Char.	S_1	S_2	S_3	S_4	S_5
0	1	1				
3	$q(q^2 + q + 1)$	1	1			
4	$q^2(q^2 + 1)$		1	1		
5	$q^3(q^2 + q + 1)$	1	1	1	1	
6	q^6	1			1	1

Running the algorithm on this yields complexes that **almost** work. They are too close to being correct for it to be a coincidence, but it's not a 'standard' perverse equivalence.

Representation Theory is Finite

If D is an abelian ℓ-group, then Broué's conjecture says that there are only finitely many different blocks B with defect group D, up to derived equivalence.

However, derived equivalences lose a lot of structure. Donovan's conjecture states that there should be only finitely many different blocks B with defect group D, up to Morita equivalence (the module categories are equivalent). In fact, there should be only finitely many different blocks with defect group D, up to Puig equivalence (Puig's finiteness conjecture).

Theorem (Linckelmann, 1996)

If D is cyclic, then Puig's finiteness conjecture holds.

Puig's conjecture is known for various classes of group G, but this was the only result on D.

$D = C_2 \times C_2$

Using results of Erdmann (1982) and Linckelmann (1994), we proved the following result, last year.

Theorem (C.–Eaton–Kessar–Linckelmann)

Puig's finiteness conjecture holds for $D = C_2 \times C_2$. (The Puig equivalence types are KD, KA_4 and $B_0(KA_5)$.

The proof is annoying; i.e., it relies on the Classification of the Finite Simple Groups.

Puig's finiteness conjecture is too difficult at the moment for all blocks, although in general there is a reduction to simple groups. If we want to generalize this result to other blocks, we need a different form of it.

D an Abelian 2-group

One equivalent form of the previous theorem is the following.

Theorem
Let B be a block with defect group $C_2 \times C_2$. Then all simple B-modules are algebraic, in the sense that they satisfy polynomials with integer coefficients in $\oplus$ and $\otimes$.

Theorem (C., 2008)
Let G be a finite group, $\ell = 2$, and P be abelian. Then all simple modules are algebraic.

Conjecture
Let B be a block with an abelian defect group, and suppose that $\ell = 2$. Then all simple B-modules are algebraic.

This is known to be false for $\ell = 3, 5$, and very likely all odd primes (even if $P = C_\ell \times C_\ell$). What makes $\ell = 2$ special?