The Core of the Partition Function

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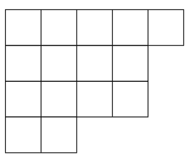
In this talk we will look at the partition function and symmetric group representation theory. Firstly we will talk about the modular representation theory of symmetric groups, and how it differs from the ordinary representation theory. After introducing the core of a partition and giving a few theorems about it, we discuss a conjecture of Stanton's, and prove some of it.

1 Partitions

The number of partitions of n is a very natural object to study: as a rather apposite example, the number of conjugacy classes of S_n is equal to the number of partitions of n. Immediately we see that $p(n)$, the number of partitions of n, is intimately connected with symmetric groups.

As we all know, in representation theory, the number of conjugacy classes of a finite group is equal to the number of irreducible (ordinary) characters of that group. At the turn of the twentieth century, Frobenius constructed the irreducible characters of the symmetric groups. We will firstly recall how their degrees are given, via the concept of Young diagrams.

Let $\lambda \vdash n$ be a partition of n, and write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$. Then the Young diagram (or tableau, or Ferrers diagram) of λ is a grid of boxes, with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. For example, if $\lambda = (5, 4, 4, 2)$, then the Young diagram of λ is given below.



We will identify λ with its Young diagram, and talk of, for instance, a box of λ.

For each box x of λ, we can associate a hook, H_x, which is the set of boxes below x, those to the right of x, and x itself. The hook length, h_x, is the number of boxes in H_x. In our example, these hook lengths are

$$\begin{array}{|c|c|c|c|c|}\hline 8 & 7 & 5 & 4 & 1 \\ \hline 6 & 5 & 3 & 2 \\ \cline{1-4} 5 & 4 & 2 & 1 \\ \cline{1-4} 2 & 1 \\ \cline{1-2} \end{array}.$$

For each partition λ of n, define

$$\chi^\lambda(1) = \frac{n!}{\prod_{x\in\lambda} h_x}.$$

Then the $\chi^\lambda(1)$, as λ ranges over all partitions of n, are the irreducible character degrees of S_n. The simple $\mathbb{C}S_n$-modules can be constructed explicitly, for example by going to C2.2 next year. We will not pursue this here.

2 Modular Representations

After Frobenius and Burnside came Brauer, who started from the position that representations are just homomorphisms $G \to \mathrm{GL}(V)$, so why should V be a vector space over a field of characteristic 0? His modular representation theory, where the field involved is finite, is (a lot) more complicated than ordinary representation theory.

The analogue of ordinary characters—modular characters—are similary defined, as the trace of the corresponding matrix of the representation, with slight modifications. There is an obstacle to taking the trace of a representation directly: for example, if you have a representation of dimension p, over a finite field of order p^n, then the character will take value 0 on the identity. We get round this by noting that the trace of the matrix is the sum of roots of 1, and so we choose a primitive $(p^n - 1)$th root of unity, and write these roots of 1 as if they were in the complex numbers. This solves the problem of losing too much information, but introduces a problem of its own: since the value of a character on an element of order m is a sum of mth roots of 1, if m is a multiple of p, this is going to be impossible in K, a finite field of characteristic p. To overcome this, we only define modular characters on elements whose order is prime to p.

The irreducible modular characters form a basis over the vector space of all class functions on the p-regular elements of a group, and so the number of irreducible modular characters is equal to the number of conjugacy classes of p-regular elements. We can also write an ordinary character as a sum of irreducible modular characters, at least for p-regular elements, in a unique way.

Doing this for the irreducible ordinary characters gives the decomposition numbers, and we can place these in a matrix, understandably called the decomposition matrix. An example is S_3 for the prime 2, which is

$$\begin{array}{c|c|c|} & \psi_1 & \psi_2 \\ \hline \chi_1 & 1 & 0 \\ \hline \chi_2 & 1 & 0 \\ \hline \chi_3 & 0 & 1 \\ \hline \end{array}$$

These numbers are all non-negative integers. A much more interesting example is M_{12} in characteristic 3, which is given on the accompanying slide. The first thing that one who is looking for such things would notice is that this matrix can be split into blocks, which are called blocks. These are important.

The obvious question to ask is can you find the blocks of a finite group without having to calculate the decomposition matrix? Since the ordinary characters are assigned to blocks, there might be a way of telling which ordinary characters are in which blocks without having to calculate the modular character table (which is in general quite hard). One can determine whether two ordinary characters lie in the same block by evaluating the function

$$f(\chi, x^G) = \frac{\chi(x)}{\chi(1)}|x^G|,$$

for a class of p-regular elements x^G. Two characters χ_1 and χ_2 lie in the same block if and only if $f(\chi_1, \mathscr{C}) = f(\chi_2, \mathscr{C})$ for all conjugacy classes $\mathscr{C}$ of p-regular elements.

Another question we might ask is can we find the blocks of a finite group without even knowing the ordinary characters? Brauer's main theorems give some information on this, but yield nothing on, for example, the blocks of defect 0. For the symmetric groups, however, the blocks can be found unsing just the partitions.

Theorem 2.1 (Nakayama Conjecture) Two irreducible characters χ^λ and χ^μ lie in the same p-block if and only if λ and μ have tha same p-core.

3 Blocks and Cores

To understand the statement of the Nakayama conjecture, we need to know what the (p-)core of a partition is. This is defined using rim p-hooks. The rim of a partition is all boxes on the lower-right edge of a partition. For example, if $\lambda = (5, 4, 4, 2)$, then the rim is

			$\star$	$\star$
			$\star$	
	$\star$	$\star$	$\star$	
$\star$	$\star$			

.

A rim p-hook is a hook H_x, where x lies on the rim, and $h_x = p$. Note that if λ' is the partition got by removing a rim hook from λ, then λ' is the Young diagram of a partition again.

The p-core of a partition λ is the partition λ' got by removing as many rim p-hooks as one can. For example, the 3-core of the partition above is

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & \ & \ \\ \hline \ & \ & \ & 1 \\ \cline{1-4} \ & 2 & 1 & 1 \\ \cline{1-4} 2 & 2 \\ \cline{1-2}\end{array}\ .$$

It can be shown that the core of a partition is independent of the order in which rim hooks are removed. Let $c_p(n)$ denote the number of p-cores with n boxes in them.

By the Nakayama conjecture, it can be seen that the number of p-blocks of S_n is given by

$$\sum_{i=0}^{\lfloor n/p \rfloor} c_p(n - ip),$$

and the number of p-blocks of defect 0 in S_n is simply $c_p(n)$. Thus the function $c_p(n)$ is interesting in its own right.

In our consideration of p-cores and rim p-hooks, we made no use of the fact that p is a prime. Thus we can extend our definitions, and consider $c_t(n)$ for all t, $n > 0$. If $t = 1$, then no partition is a 1-core except for the empty partition. If $t = 2$, then the only 2-core partitions are the triangular partitions, for example

$$\begin{array}{|c|c|c|c|}\hline \ & \ & \ & \ \\ \hline \ & \ & \ \\ \cline{1-3} \ & \ \\ \cline{1-2} \ \\ \cline{1-1}\end{array}\ .$$

If $t = 3$, then there are infinitely many n for which $c_3(n) = 0$. (A good exercise is to calculate $c_3(n)$ for arbitrary n.)

Theorem 3.1 (Granville, Ono) If $t \geqslant 4$, then $c_t(n) > 0$.

The proof of this theorem is complicated, and uses the theory of modular forms. Is there a combinatorial proof?

4 Stanton's Conjecture

A graph containing $c_t(40)$ for all $1 \leqslant t \leqslant 40$ is on an accompanying slide. This suggests the following conjecture of Dennis Stanton.

Conjecture 4.1 (Monotonicity Conjecture) If $4 \leqslant t < n - 1$, then

$$c_t(n) \leqslant c_{t+1}(n).$$

Since $c_4(n)$ is known to be positive for all n independently of the Granville–Ono theorem, this conjecture implies their theorem.

We will prove this conjecture for $t > n/2$, the easy half.

Suppose that $t > n/2$: then a partition $\lambda \vdash n$ is either a t-core or can have a single rim t-hook removed. It is known that, if $\lambda \vdash (n-t)$ is a t-core, then the number of partitions of n with t-core λ' is exactly t. Thus

$$c_t(n) = p(n) - tp(n-t).$$

Hence $c_t(n) \leqslant c_{t+1}(n)$ if and only if

$$tp(n-t) \geqslant (t+1)p(n-n-1).$$

This will hold if

$$\frac{p(n-t)}{p(n-t-1)} \geqslant \frac{t+1}{t},$$

since $t > n/2$.

To prove the growth condition on $p(n)$ given above, we will give a function from the set of all partitions of n to those of $n+1$. This yields an exact expression for $p(n+1)/p(n)$, and by inserting the crudest reasonable bounds for the terms involved, we get $p(n+1)/p(n) > (n+1)/n$.

Let $P(n)$ denote the set of all partitions of n: we will decompose the set $P(n)$ into the disjoint union of three subsets, $A(n)$, $B(n)$ and $C(n)$. The first, $A(n)$, consists of all partitions whose last part is 1. The set $B(n)$ consists of all partitions $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ for which $\lambda_r > 1$ and $\lambda_1 > \lambda_2$, together with the partition (n). The third set, $C(n)$, consists of all partitions $(\lambda_1, \ldots, \lambda_r)$ for which $\lambda_r > 1$ and $\lambda_1 = \lambda_2$. Write $a(n) = |A(n)|$, and so on.

Firstly, notice that $a(n) = p(n-1)$, since the function between $P(n-1)$ and $A(n)$ obtained by adding 1 to the end of each partition is clearly bijective. Thus we have

$$p(n) = p(n-1) + b(n) + c(n).$$

Next, we describe a function from $P(n)$ to $B(n+1)$: this function is surjective, but obviously not injective. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n, with the last k parts of λ equal to 1. Then define $f(\lambda)$ by

$$f(\lambda) = (\lambda_1 + 1 + k, \lambda_2, \ldots, \lambda_{r-k});$$

that is, by removing the parts of size 1 from the bottom of λ, attaching them to the first row of λ, and then increasing it by 1. It is obvious that $f(\lambda) \in B(n+1)$, and that every partition in $B(n+1)$ can be written as $f(\lambda)$ for some λ.

Suppose that $\lambda' \in B(n)$, and that $\lambda_1 - \lambda_2 = k + 1$, where λ_2 is taken to be 0 if $\lambda = (n)$. Then it is reasonably obvious that there are exactly k different partitions of $n-1$ whose image under f is equal to λ'. Write $s(n)$ for the mean value of k, as the partition λ' ranges over all elements of $B(n)$. Finally, write $r(n) = c(n)/p(n-1)$. Then we see that

$$p(n) = p(n-1) + \frac{p(n-1)}{s(n)} + r(n)p(n-1) = p(n-1)\left(1 + \frac{1}{s(n)} + r(n)\right).$$

Now, to get a lower bound on $p(n)/p(n-1)$, we simply need a lower bound on $r(n)$ and an upper bound on $s(n)$. A perfectly good lower bound on $r(n)$ is zero, and from the discussion in the previous paragraph, an upper bound on $s(n)$ is $n-1$. Thus

$$p(n) \geqslant p(n-1)\left(1 + \frac{1}{n-1}\right).$$

If $n \geqslant 8$ then $r(n) > 0$, and so for $n \geqslant 8$, this inequality is strict. In fact, $s(n) < n-1$ if $n \geqslant 4$, and so, for $n \geqslant 4$, we have

$$\frac{p(n)}{p(n-1)} > \frac{n}{n-1},$$

as we needed.

The bounds on $r(n)$ and $s(n)$ given above are extremely crude. With more work, one can get sharper bounds, although this does not seem to generate a proof of smaller caes of the monotonicity conjecture.