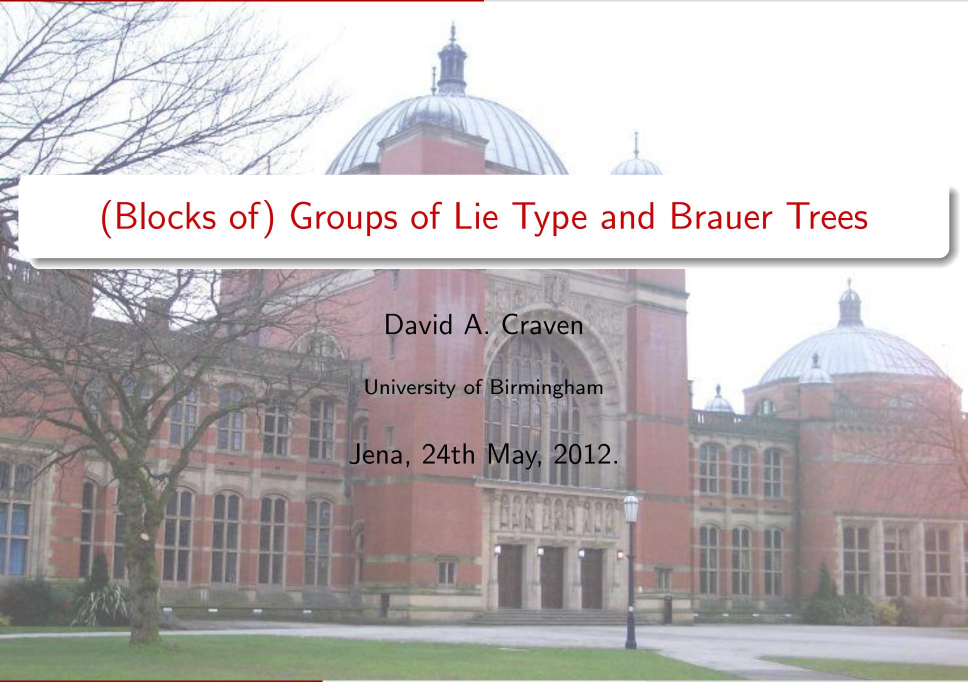

(Blocks of) Groups of Lie Type and Brauer Trees

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Doing Modular Representation Theory

Let G be a finite group and let ℓ be a prime dividing $|G|$. Write k for a field of characteristic ℓ.

In Charles's talk he introduced blocks, and mentioned defect groups, which are ℓ-subgroups of G. If this defect group D is cyclic, then the block B can be described very nicely in a combinatorial way, using the Brauer tree.

Derived Equivalences

Broué's abelian defect group conjecture states that any block with abelian defect groups is derived equivalent to a block with 'nice' structure, in fact the group algebra $P \rtimes E$ for some ℓ'-group E. In the cyclic defect group case, this has Brauer tree a star with the exceptional node in the middle.

In 1989, Rickard proved that any two Brauer tree algebras – e.g., blocks with cyclic defect groups – with the same number of vertices and same exceptionality are derived equivalent, by describing a derived equivalence with the appropriate star.

This derived equivalence is defined combinatorially, with no reference made to the exceptionality.

Groups of Lie Type

Let $G = G(q)$ be a group of Lie type: the order of G is

$$|G| = q^N \prod_{d \in I} \Phi_d(q)^{a_d}.$$

If $\ell \mid |G|$ then either $\ell \mid q$, which leads to one theory, or $\ell \nmid q$, in which case $\ell \mid \Phi_d(q)$ for some d. We are mostly interested in the case where there is no other d' such that $\ell \mid \Phi_{d'}(q)$; in this case, the Sylow ℓ-subgroup P is abelian, homocyclic, of rank a_d. In particular, if $a_d = 1$ then P is cyclic.

We will always assume that ℓ divides exactly one $\Phi_d(q)$ from now on.

Unipotent Blocks

Obviously, if we fix $G(-)$ and vary q we get different numbers of irreducible characters. However, they split into two collections: unipotent and non-unipotent. (Very!) roughly speaking, unipotent is like the non-exceptional characters in the Brauer tree case, and non-unipotent equals exceptional characters.

Formally, a unipotent character of $G = \mathbf{G}^F$ is a constituent of the Deligne–Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(1)$. (This probably isn't much help if you didn't know what unipotent characters were in the first place.)

In particular, the set of unipotent characters do not depend on q, and in fact their distribution into ℓ-blocks (thought of as an equivalence relation) does not depend on ℓ or q, and only on d. A unipotent block is a block containing a unipotent character; we therefore get, for any q and (appropriate) ℓ with $\ell \mid \Phi_d(q)$ 'the same' set of unipotent blocks.

Genericity

Let $G = G(q)$ be a Lie type, let q and q' be prime powers, $\ell \nmid q$ and $\ell' \nmid q'$ be primes, and suppose that $\ell \mid \Phi_d(q)$, $\ell' \mid \Phi_d(q')$. Let B be a unipotent ℓ-block of $G(q)$, and let B' be the same unipotent ℓ'-block of $G(q')$.

If B has cyclic defect groups (and G is not of type E_8, at least for now) then the Brauer trees of B and B' are identical, even including planar embedding, except for the exceptionality. Hence B and B' are 'the same', and so the behaviour of G is generic for the block B, or for Φ_d if this is true for all unipotent Φ_d-blocks.

What is 'the same' for blocks with non-cyclic defect groups? (It isn't Morita or Puig equivalence here, since the primes are different, the defect groups are of different exponents, etc.)

Genericity

We first recast the cyclic case. Rickard's derived equivalence depends only on the shape of the Brauer tree, so B and B' are the same if the derived equivalence defined by Rickard works for both algebras.

Rickard's equivalence is defined by a set of combinatorial data. What would allow us to define genericity is a derived equivalence defined purely combinatorially, and we would state that two blocks are the same if the same combinatorial data can be used to define a derived equivalence with $k(P \rtimes E)$. So all we need is a general definition of a type of derived equivalence, defined combinatorially, that extends Rickard's definition to all unipotent blocks of group of Lie type possessing abelian defect groups.

This need is satisfied by perverse equivalences.

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras, $\mathcal{A} = \textit{mod-}A$, $\mathcal{B} = \textit{mod-}B$

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, T_1, $T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \ldots, S_i$, and $D^b_i(\mathcal{A})$ denotes the subcategory of $D^b(\mathcal{A})$ with support modules in $\mathcal{A}_i$, then

- F induces equivalences $D^b_i(\mathcal{A}) \to D^b_i(\mathcal{B})$, and
- $F[\pi(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{B}_i/\mathcal{B}_{i-1}$.

Note that mod-B is determined, up to equivalence, by A, π, and the ordering of the S_i.

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras, $\mathcal{A} = mod\text{-}A$, $\mathcal{B} = mod\text{-}B$.

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, T_1, $T_2, \ldots$, T_r, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, for all i, the cohomology of $F(S_i)$ has only one copy of T_i in degree $-\pi(i)$, and any other T_j can only appear in lower degrees than $-\pi(j)$.

Geometric Broué

Broué's conjecture has a special version for unipotent blocks of groups of Lie type, called the geometric form.

Conjecture

Let $G = G(q)$ be a finite group of Lie type, and let D be an abelian defect group of a unipotent block B of G. We may embed D inside a Φ_d-torus T, and there is a Deligne–Lusztig variety Y, carrying an action of G on the one side and T on the other, whose complex of cohomology Γ has the following properties:

1. *the action of T can be extended to an action of $\mathrm{N}_G(T) = \mathrm{N}_G(D)$;*
2. *the complex induces a derived equivalence between B and its Brauer correspondent.*

Geometric Broué

In fact, if ζ is a primitive dth root of unity, then there should be a Deligne–Lusztig variety Y_ζ associated naturally to ζ, and whose complex of cohomology produces the desired equivalence.

While this is (a lot) more specific than the abstract version of Broué's conjecture, it still needs to be more specific, as the variety Y_ζ can be hideously complicated.

This equivalence should be perverse. If the associated data can be extracted without analyzing the variety Y_ζ, then the derived equivalence should be able to be constructed without the variety at all, purely combinatorially.

Cyclotomic Hecke Algebras

Let's stick with the cyclic case for now. The (specialized) cyclotomic Hecke algebra $\mathcal{H}(Z_e, \mathbf{u})$ is an algebra over $R = \mathbb{C}(q^{1/2})$, and is given by

$$R[T]/(T - u_1)(T - u_2)\dots(T - u_e),$$

where $u_i = \omega_i q^{v_i}$ for roots of unity ω_i and semi-integers v_i. (For GL_n we have $\omega_i = 1$, for classical groups $\omega_i = \pm 1$, for exceptional untwisted it goes up to sixth roots, and can go up twelfth roots for Ree and Suzuki groups.)

This is invariant under global multiplication by a root of unity or power of q. Arrange them so that $u_1 = q^{v_1}$, $v_1 \geq v_i$ for all i.

The generic degree associated to the parameter u_i is

$$\frac{u_1}{u_i} \prod_{\substack{j=1 \\ j \neq i}}^{e} \frac{(u_1 - u_j)}{(u_i - u_j)}.$$

Cyclotomic Hecke Algebras

To every unipotent block there is associated a cyclotomic Hecke algebra, and the generic degrees are actually the degrees (as polynomials in q) of the unipotent characters (up to a fixed polynomial, which is 1 for the principal block).

Let ζ be a primitive dth root of unity, B be a unipotent block with cyclic defect groups, and let $\mathcal{H} = \mathcal{H}(Z_e, \mathbf{u})$ be its associated cyclotomic Hecke algebra. The specialization $q \mapsto \zeta$ turns $\mathcal{H}$ into the group algebra $\mathbb{C}Z_e$. This gives us a natural bijection between the parameters of the cyclotomic Hecke algebra and the irreducible characters of Z_e.

Since k has characteristic ℓ, and P is an ℓ-group, the simple kH-modules for $H = Z_{\ell^a} \rtimes Z_e$ are in natural bijection with the irreducible characters of Z_e.

generic degrees $\leftrightarrow$ parameters $\leftrightarrow$ characters of Z_e $\leftrightarrow$ kH-modules

Combinatorial Broué

In order to get a perverse equivalence, we need a bijection between the simple modules for a block B and its Brauer correspondent b.

- The simple b-modules are in **non-canonical** bijection with the simple modules for $Z_{\ell^a} \rtimes Z_e$;
- the simple $Z_{\ell^a} \rtimes Z_e$-modules are in bijection with the parameters of the cyclotomic Hecke algebra;
- the parameters of the cyclotomic Hecke algebra are (at least for cyclic defect groups) in natural bijection with the unipotent characters in B, which are in natural bijection with the simple B-modules.

The perversity function $\pi(-)$ can be calculated from the generic degrees (C., 2011).

Theorem (C. (2012))

This bijection and perversity function yield a perverse equivalence whenever the Brauer tree is known.

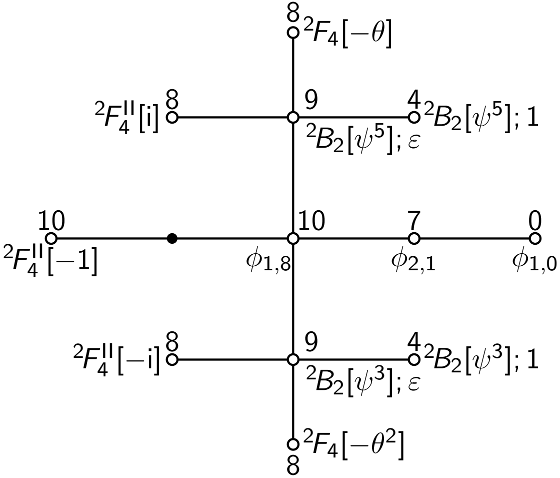
Combinatorial Broué: Example

$G = {}^2F_4(q^2)$, $\ell \mid \Phi'_{24}(q)$. (By Φ'_{24} we mean the polynomial factor of Φ_{24} with ζ_{24} as a root.)

Character	$\omega_i q^{aA/e}$	$k=5$	$k=11$	$k=13$	$k=19$
$\phi_{1,0}$	1	0	0	0	0
${}^2B_2[\psi^3];1$	$\psi^7 q$	4	10	12	18
${}^2F_4^{II}[-\mathrm{i}]$	$-\mathrm{i}q^2$	8	18	22	32
${}^2F_4[-\theta^2]$	$-\theta q^2$	8	18	22	32
${}^2B_2[\psi^5];1$	ψq	4	10	12	18
$\phi_{2,1}$	q^2	7	17	21	31
${}^2B_2[\psi^3];\varepsilon$	$\psi^7 q^3$	9	21	25	37
${}^2F_4[-\theta]$	$-\theta^2 q^2$	8	18	22	32
${}^2F_4^{II}[\mathrm{i}]$	$\mathrm{i}q^2$	8	18	22	32
${}^2B_2[\psi^5];\varepsilon$	ψq^3	9	21	25	37
$\phi_{1,8}$	q^4	10	22	26	38
${}^2F_4^{II}[-1]$	$-q^2$	10	20	24	32

Combinatorial Broué: Example

$G = {}^2F_4(q^2)$, $\ell \mid \Phi'_{24}(q)$. (By Φ'_{24} we mean the polynomial factor of Φ_{24} with ζ_{24} as a root.)



Chasing Brauer Trees

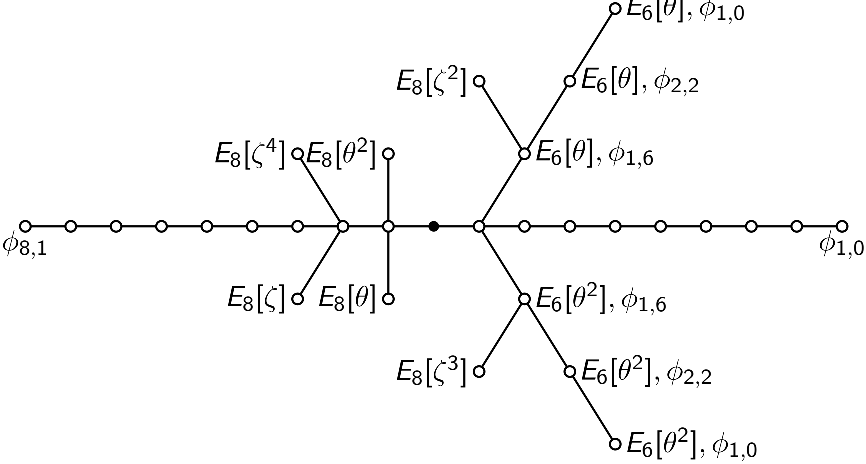
Not all Brauer trees are known. Let's try to fix this.

A perverse equivalence allows you to reconstruct the original block up to Morita equivalence. We can use the cyclotomic Hecke algebra for unipotent blocks for which we do not yet know the Brauer tree to generate a conjecture about its exact shape.

Once you know what you are trying to prove, it becomes a lot easier to do this!

Example: Conjectured Tree

$G = E_8(q)$, $\ell \mid \Phi_{15}(q)$.



So What's New?

The previously unknown Brauer trees were for

- ${}^2G_2$, $d = 12''$
- F_4, $d = 12$
- ${}^2F_4$, $d = 24''$
- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 12$
- E_7, all d
- E_8, all d

Theorem (Dudas (2011))

Known for ${}^2G_2$ and F_4.

Theorem (Dudas, Rouquier (2012))

Known for ${}^2F_4$, E_7 for $d = 18$, E_8 for $d = 30$.

So What's Even Newer?

The previously unknown Brauer trees were for

- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 12$
- E_7, all $d \neq 18$
- E_8, all $d \neq 30$

Proposition (C. (2012))

Many of the trees for E_7 and E_8 are lines, or Morita equivalent to cases solved by Dudas and Dudas–Rouquier.

So What's Ultra-New?

The previously unknown Brauer trees were for

- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 12$
- E_7, $d = 9, 10, 14$
- E_8, $d = 9, 12, 14, 15, 18, 20, 24$

Theorem (C.–Dudas–Rouquier (2012))

The Brauer trees are known for ${}^2E_6$, E_7 *for all* d, *and* E_8 *for* $d = 14$, *two out of three blocks for* $d = 18$, *and* $d = 24$.

So What's Left?

- E_8, $d = 9, 12, 15, 18, 20$.

The same techniques will hopefully work for $d = 9, 12$, and we might be able to sort out $d = 15$. A new idea will probably be needed for the last (non-principal) block of $d = 18$, and for $d = 20$.