A Survey on Algebraic Modules

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1 Introduction

In this talk, I will survey the theory of algebraic modules. Let M be a finite-dimensional KG-module, where K is a field of characteristic p and G is a finite group. Write $a(KG)$ for the Green ring, which is all formal sums of indecomposable KG-modules, with addition and multiplication given by direct sum and tensor product. An *algebraic* module is an element of $a(KG)$ that is algebraic over K; that is to say, the module M satisfies a polynomial $f(x)$ with integer coefficients.

This definition will only be usseful if it includes some interesting modules, excludes some modules, and has some nice properties. For the first, all trivial source modules are algebraic, and for example we have the following theorem.

Theorem 1.1 (Feit, 1979) Let G be a finite p-soluble group, and let M be a simple KG-module. Then M is algebraic.

Thus there are interesting modules in the class of algebraic modules. In addition, for example an endo-permutation module is algebraic if and only if it has finite order in the Dade group, so there is overlap with other definitions. It also needs to exclude some modules. Here is an example.

Theorem 1.2 (C., 2008) Let M be an indecomposable algebraic module, of complexity at least 3. Then M lies on the end of its Auslander–Reiten component, and no other module on that component is algebraic.

[This is reminiscent of various theorems on the locations of simple modules on Auslander–Reiten components by Kawata, Michler, and Uno.]

There is no example of a non-periodic component of the Auslaner–Reiten quiver that contains at least two algebraic modules, but there is no proof extending to all non-periodic modules.

It also needs to have various properties.

Proposition 1.3 The class of algebraic modules is closed under taking summands, sums, tensor products, induction, restriction, taking sources and Green correspondents (in either direction).

So it forms a nice subring of the Green ring of a finite group.

2 Simple Modules

In the case of simple modules, much of the results are about blocks with abelian defect groups. In 1979, Alperin proved the following theorem.

Theorem 2.1 (Alperin, 1979) Let G be the group $\mathrm{SL}_2(2^n)$ for some n, and let K be a field of characteristic 2. Then all simple KG-modules are algebraic.

Shortly afterwards, Kovacs extended this to $\mathrm{SL}_2(p^n)$, over a field of characteristic p. From here, it might be reasonable to conjecture that groups with abelian Sylow p-subgroups might have algebraic simple modules. Things get even better, because the groups $\mathrm{PSL}_2(q)$ with Klein four Sylow 2-subgroup also have only algebraic simple modules. In total, we can prove the following theorem.

Theorem 2.2 (C., 2009) Let G be a finite group with abelian Sylow 2-subgroup, and let K be a field of characteristic 2. Then all simple KG-modules are algebraic.

Let's stay with the prime 2 for a while: moving from groups to blocks, we look at whether the simple modules in blocks with abelian defect group are algebraic. In 1982, Erdmann determined the sources of the simple modules in blocks with Klein four defect group, up to some iteration of the Heller operator. In particular, the sources are all either odd-dimensional (and so some Heller translate of the trivial module) or periodic. Conlon (1966) proved an indecomposable module for V_4 is algebraic if and only if it is even-dimensional (and hence periodic) or trivial. Using the classification of the finite simple groups, this parameter can be identified, proving the following theorem.

Theorem 2.3 (C., Eaton, Kessar, Lickelmann) Let K be a field of characteristic 2. Let B be a 2-block of a finite group, with Klein four defect group.

(i) B is Puig equivalent to one of KV_4, KA_4, and $B_0(KA_5)$.

(ii) All simple B-modules are algebraic.

(iii) The source of a simple B-module is either trivial or a periodic 2-dimensional simple module, realizable over GF(4).

Given that we have such a result for both abelian Sylow 2-subgroups and Klein four 2-blocks, one might try to combine the two into a conjecture.

Conjecture 2.4 Let B be a 2-block with abelian defect groups. Then all simple B-modules are algebraic.

Another direction is changing the prime. This is less successful.

Theorem 2.5 (C.) Let G be a finite group with Sylow 3-subgroup $C_3 \times C_3$, and let K be a field of characteristic 3. Then the simple module in the principal block are algebraic if and only if neither M_{11} nor M_{23} is a composition factor of G.

So there is not such a satisfactory result moving to odd primes, but with only finitely many exceptions (up to $\mathrm{O}_{3'}(G)$) the theorem holds, at least for this particular Sylow 3-subgroup.

Moving beyond 3, even this result cannot be rescued. Let $G = F_4(2)$, and let $p = 5$. Then the Sylow p-subgroup of G is $C_5 \times C_5$, and there is a simple module of dimension 875823 in the principal 5-block that lies on the second row of the Auslander–Reiten component. Although the theorem I gave earlier does not apply in this case, it can be shown that an algebraic module cannot lie on the second row in this case, and so this simple module is non-algebraic. There should be infinitely many groups $G = F_4(q)$ whose principal 5-block is Puig equivalent to that of $F_4(2)$, providing infinitely many examples of simple groups with abelian Sylow 5-subgroups, and with non-algebraic simple modules in the principal block.

3 Indecomposable Modules

Moving away from simple modules, let us look at the whole module category. The only (non-cyclic) p-group for which we have perfect knowledge is V_4. There are some results for dihedral groups, but we will focus on $C_p \times C_p$ for odd primes.

Conjecture 3.1 Let p be an odd prime, let $P = C_p \times C_p$, and let K be a finite field of characteristic p. Let M be an absolutely indecomposable KP-module. Then M is algebraic if and only if M is periodic.

The presence of the finite field is because it is not clear whether periodic should be equivalent to algebraic, or merely that the dimensions of the summands are bounded; for finite fields, these notions coincide of course. This conjecture is interesting because it relates the tensor structure of the group algebra to the homological structure in a way that does not appear to have been considered before.

4 Non-algebraic Modules

Having had a look at algebraic modules, let's ask similar questions about other modules. Given a module M, let $a_n(M)$ denote the number of non-isomorphic indecomposable summands of

$$K \oplus M \oplus M^{\otimes 2} \oplus \cdots \oplus M^{\otimes n}.$$

Then a module is algebraic if and only if $a_n(M)$ is universally bounded, i.e., eventually constant. (Also notice that if $a_n(M) = a_{n1}(M)$ for some n, then M is algebraic.) Thus if M is non-algebraic, the slowest that the function $a_n(M)$ can grow is linearly.

Proposition 4.1 Let M be an algebraic module. Then $a_n(\Omega^i(M))$ is bounded by a linear function in n.

This still leaves almost all modules, for which very little is known. However, one may attach to each module a growth type, and this stratifies the module category by an invariant dependent on the monoidal structure of the module category. Algebraic objects, and this more general notion, may of course be defined for any symmetric monoidal category, like Hopf algebras for example. Results here are obviously not as impressive as for groups, but something may still be said.