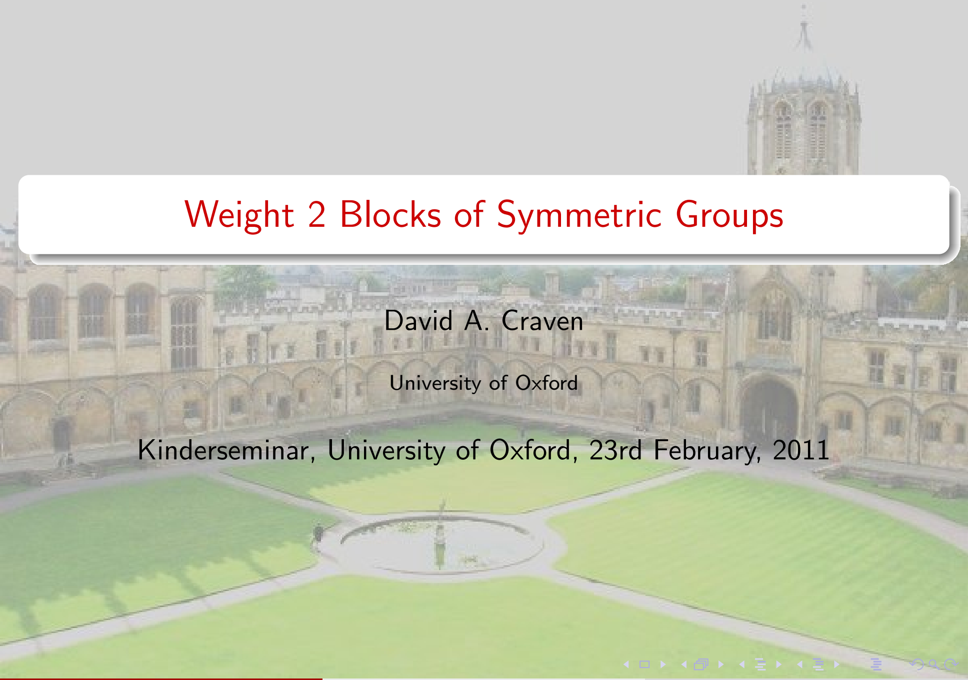

Weight 2 Blocks of Symmetric Groups

David A. Craven

University of Oxford

Kinderseminar, University of Oxford, 23rd February, 2011

Symmetric Groups

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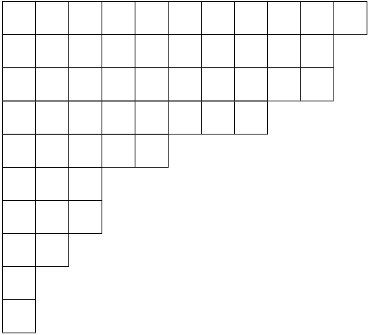
The construction actually defines these modules as $\mathbb{Z}S_n$-modules, and so for any field K we can get Specht modules S^λ: in general, if K has positive characteristic, these modules will **not** be irreducible.

Tableaux and Hooks

The standard device is to represent partitions as tableaux. So (11,10,10,8,5,3,3,2,1,1) becomes



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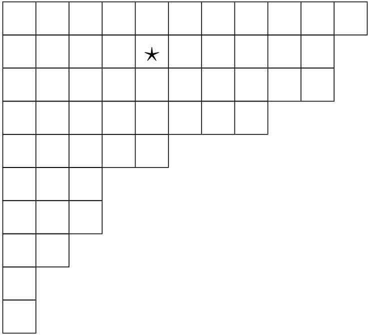

Choose a box.

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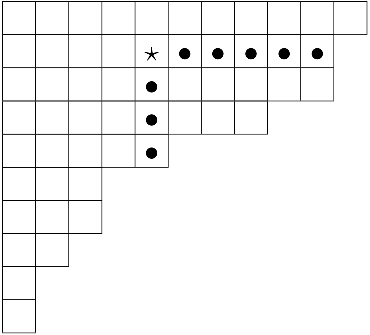

Find all of the boxes to the right and below it. Count the size of the hook. It is 9. (The arm has length 6 and the leg has length 4.)

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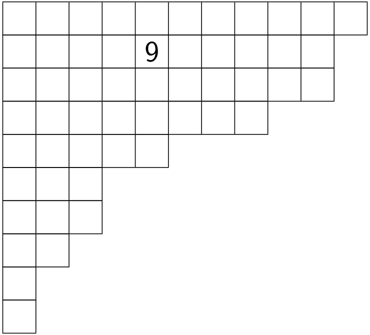

So 9 goes in the box. It's the hook length.

Tableaux and Hooks

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20	17	15	12	11	9	8	7	5	4	1
18	15	13	10	9	7	6	5	3	2	
17	14	12	9	8	6	5	4	2	1	
14	11	9	6	5	3	2	1			
10	7	5	2	1						
7	4	2								
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2										
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Repeat until bored.

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- This gives a method of calculating the dimension of S^λ. Indeed, if $H(\lambda)$ is the set of hook lengths, and $|\lambda| = n$, then

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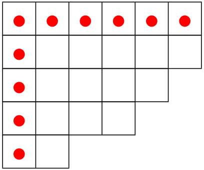
  The possible d-cores classify the d-blocks of the symmetric group. (More later!)

Removing hooks

Removing a hook is equivalent to removing a rim hook, which involves peeling off part of the rim, start at the arm of the hook, and ending at the leg.



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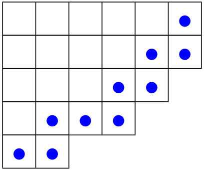

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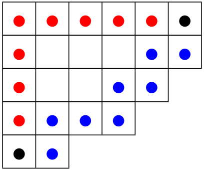

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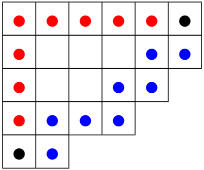

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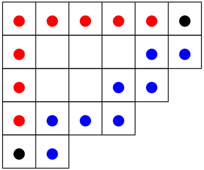

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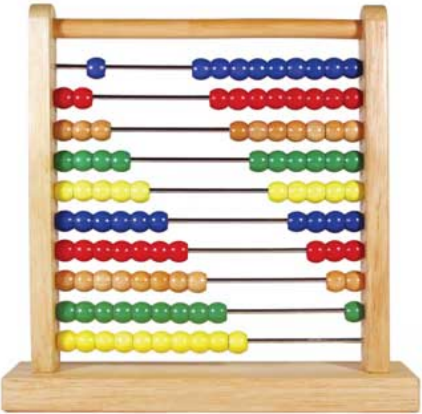
What does this do to the hook numbers? To see this, we use the abacus, and the hook lengths of the first column.

The Abacus



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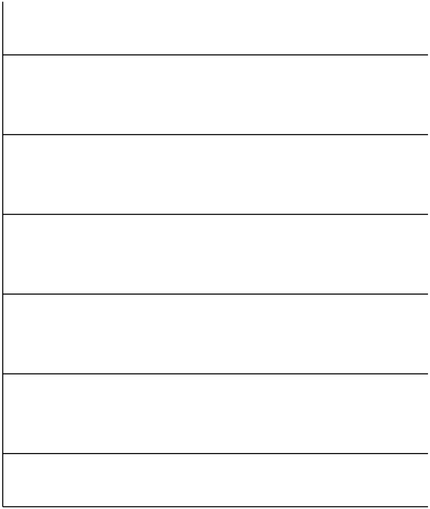

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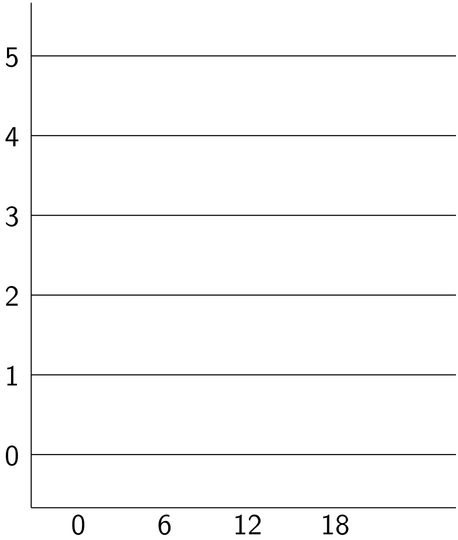

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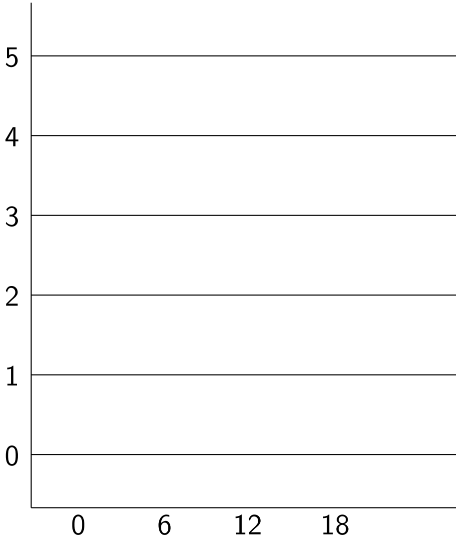

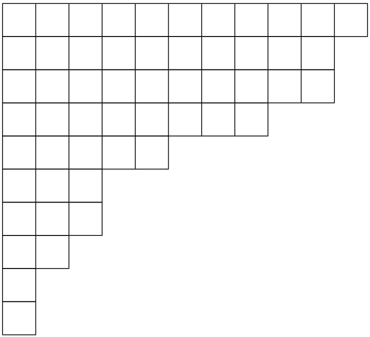

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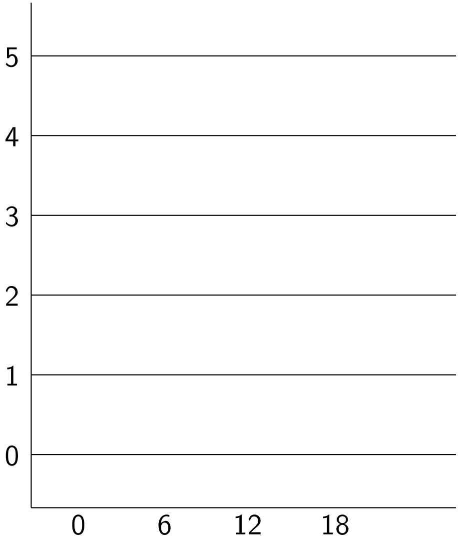

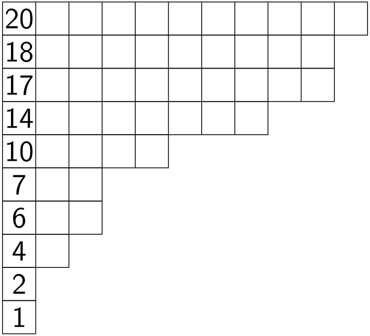

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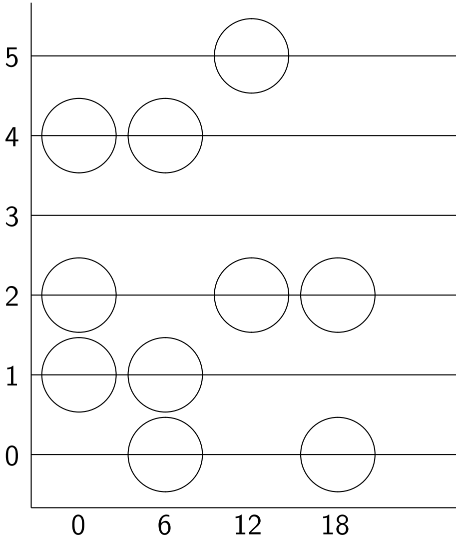

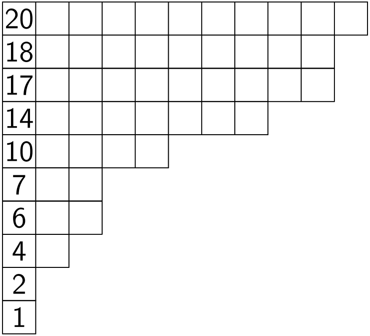

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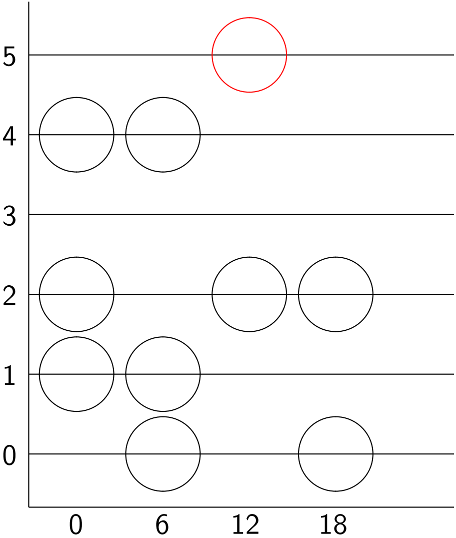

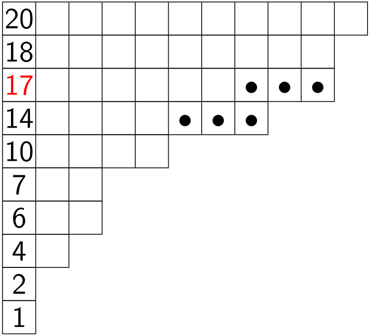

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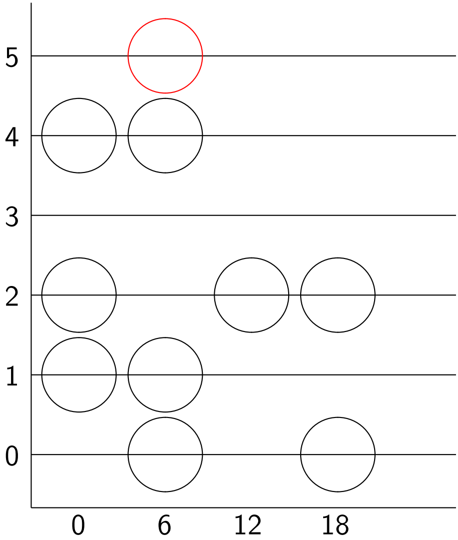

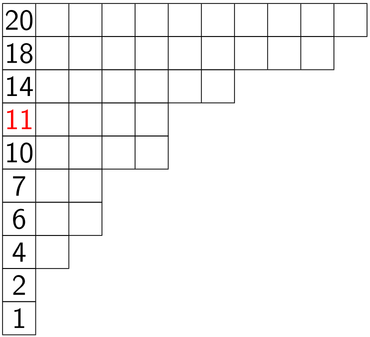

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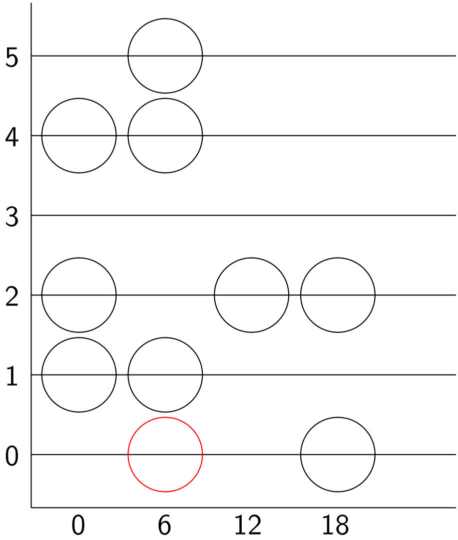

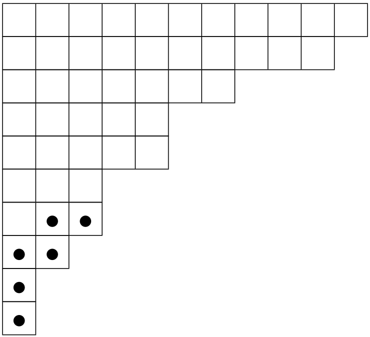

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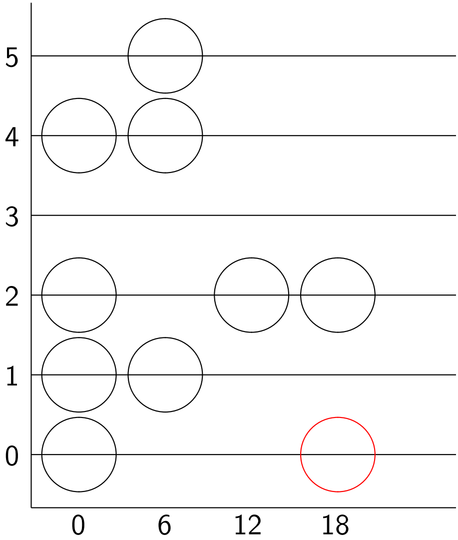

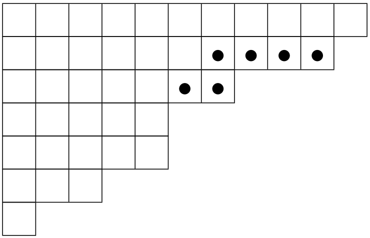

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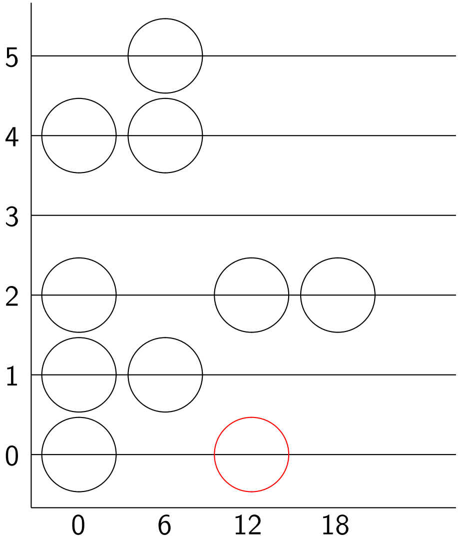

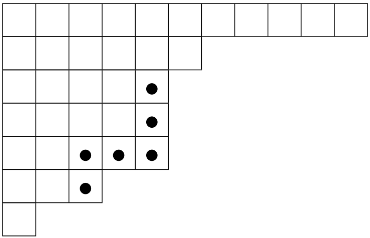

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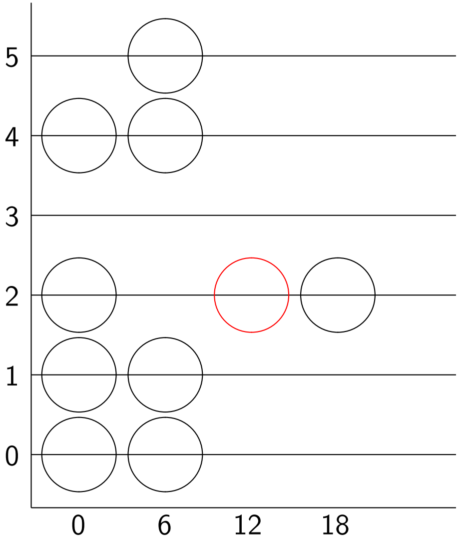

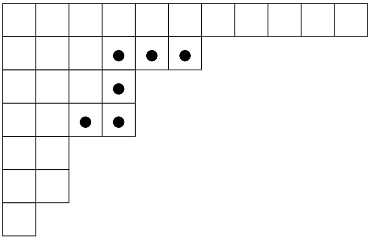

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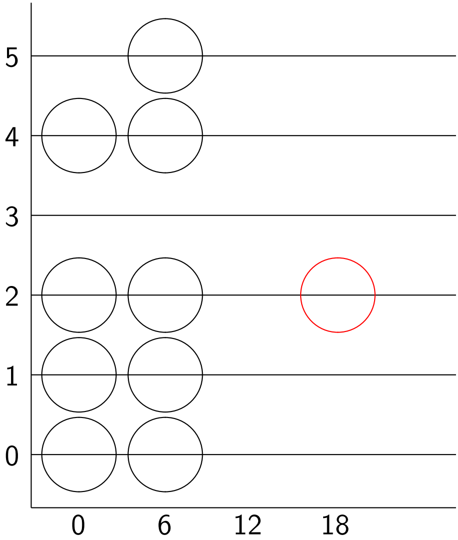

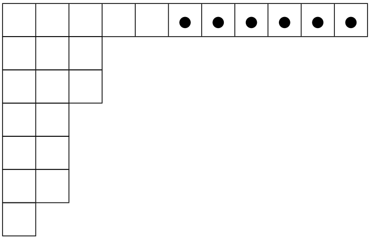

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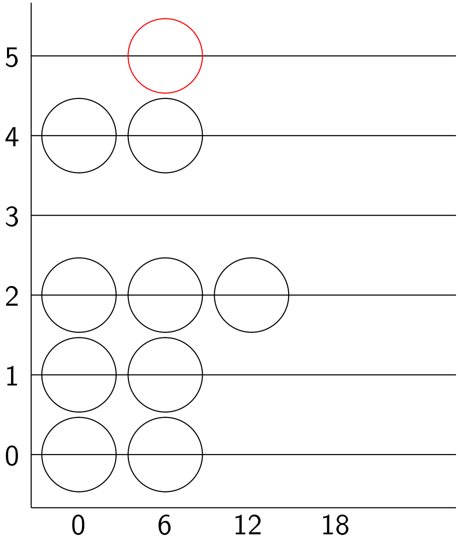

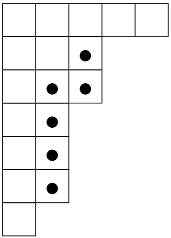

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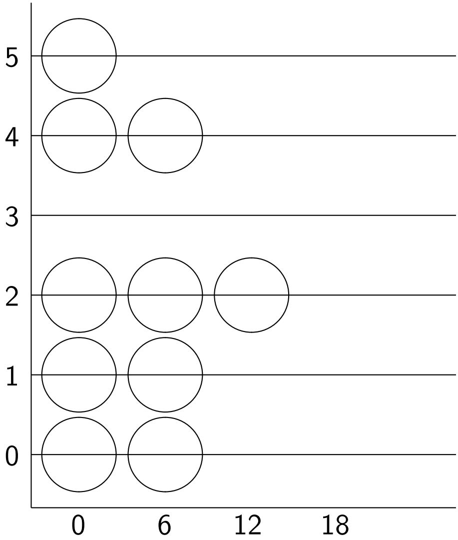

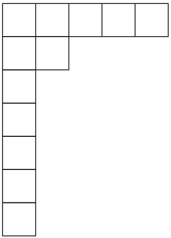

Modular Representations: Blocks

If G is a finite group then the algebra $\mathbb{C}G$ is semisimple, so that it is a direct sum of simple (matrix) algebras. If K is a field of characteristic p, then KG need not be semisimple, and never is if $p \mid |G|$.

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The B_r are called the blocks of KG. Since $1 \in KG$, we can write

$$1 = e_1 + e_2 + \cdots + e_r,$$

where $e_i \in B_i$ is a central idempotent.

Modular Representations: Modules and Blocks

If M is any KG-module, then $M \cdot 1 = M$, so that

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If M is indecomposable then all of the $M \cdot e_i$ except for one, say e_j, are zero. In this case, we say that M belongs to B_j. The simple KG-modules are distributed throughout the blocks in this way, with each block getting at least one.

Modular Representations: Ordinary Representations

There is a way to pass from ordinary to modular representations, and so assign irreducible complex representations to blocks. In general, this bit is quite difficult (think the p-adic rationals $\mathbb{Q}_p$, the p-adic integers $\mathbb{Z}_p$, and the quotient field $\mathbb{F}_p$), but for the symmetric group things are easy.

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How can we tell to which block the module S^λ belongs?

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Let B be a block of S_n, with p-core λ. Since you get from a partition of n to λ by removing p-hooks, we have that $n - |\lambda| = wp$, where w is the number of p-hooks removed. Define the weight of B to be w.

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Having distributed the complex representations into blocks, we now need to understand the simple KS_n-modules, and we have done everything.

Modular Representations: the D^λ

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Define the dominance order, a partial order, on partitions: $(\lambda_1, \ldots, \lambda_r) \trianglelefteq (\mu_1, \ldots, \mu_s)$ if

$$\sum_{i=1}^{k} \lambda_i \leq \sum_{i=1}^{k} \mu_i$$

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for all i. (If $i > r$ or $i > s$, pretend that $\lambda_i = 0$ or $\mu_i = 0$.) It turns out that the composition factors of S^λ modulo p are all present in S^μ for $\lambda \trianglelefteq \mu$, except for exactly one if and only if λ is p-restricted, which we denote D^λ. The D^λ form a complete set of irreducible KS_n-modules.

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In the weight 1 case the defect group is cyclic, and there is an extensive theory of these blocks. Weight 2 is the first time that things get non-trivial.

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Theorem (Branching Rule)

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We are interested in the modular branching rule for weight 2 blocks.

Weight 2 Blocks

Let p be an odd prime. Let B be a weight 2 block, with p-core κ. If λ is a partition of n with p-core κ, then on the abacus, we must move two beads left in order to reach κ. Hence we can parametrize the partitions with p-core κ by symbols $\langle i, j\rangle$, $\langle i, i\rangle$ and $\langle i\rangle$. These mean:

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This is the full list of complex irreducibles S^λ in B, and there are $p(p+3)/2$ of them. For how many of these are p-restricted, notice that being p-restricted means that there are (at least) p consecutive hook numbers, or on the abacus, a complete column of beads (wrapping around the top of the abacus.) Assuming we are in the case of an empty core, this means the symbols $\langle 1, 2 \rangle$ and $\langle i, i \rangle$, so $p+1$. (This is true in general, but with different symbols.)

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It is an important theorem of Scopes that if $k > 1$, then the two blocks B and $\bar{B}$ are "the same" (Morita equivalent). If D^μ has symbol $\langle a, b \rangle$, then $D^\mu \downarrow_B$ has summand D^λ, which has symbol $\langle a\sigma, b\sigma \rangle$.

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This **almost** happens when $k = 1$.

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Each Scopes move has the property that $D^\lambda \uparrow^{\bar{B}}$ is **simple**, except for exactly one λ, the one with symbol $\langle i, i \rangle$.

From Minimal to RoCK

Since there are $p(p-1)/2$ Scopes moves, and each Scopes move "messes up"one simple module, there are at most $p(p-1)/2$ simple modules that are messed up, going from the minimal block to the RoCK block. (The same simple could be messed up twice.)

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(A source is an indecomposable module M for the Sylow p-subgroup of S_{2p} such that the simple module is a summand of $M\uparrow^B$.)

From Minimal to RoCK

Since $p-1$ simple modules share sources with modules from the RoCK block (which has "easy" sources, as it looks like $S_p \wr C_2$), the last thing here is obviously to identify which simple modules from the minimal block have this property.

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The simple modules for the minimal block that share a source with the RoCK block are labelled by $\langle p \rangle$ and $\langle i, i+1 \rangle$ for $2 \leq i \leq p-1$, which are the partitions $(2p)$ and $(i^2, 2^{p-1})$ for $3 \leq i \leq p$.

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Main open problem for weight two blocks: Find the sources for the other simple modules!