Algebraic Modules for Finite Groups

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For the purposes of this talk, G is a finite group, K is an algebraically closed field of characteristic p, where $p \mid |G|$, and all modules are finite-dimensional.

1 Algebraic Modules

The *Green ring* of KG-modules is defined to be the free abelian group on the basis set of all indecomposable KG-modules, with $M + N$ defined to be equal to $M \oplus N$, and the product of two modules defined as $M \otimes N$. Notice that not all elements of the Green ring can be thought of as modules, since they could have negative multiplicities attached; they are *virtual modules*.

The structure of the Green ring, while a commutative ring with a 1, is far from that of traditional commutative rings. For example, it is not an integral domain: in general, it has nilpotent elements. It is also in general infinite-dimensional. We can still, however, carry over some notions from algebraic number theory. One of those is algebraic modules.

A module is said to be *algebraic* if it satisfies some polynomial equation in the Green ring, with co-efficients in $\mathbb{Z}$.

Proposition 1.1 Let M be a KG-module. Then the following are equivalent:

(i) M is algebraic;

(ii) M satisfies a *monic* polynomial equation in the Green ring with co-efficients in $\mathbb{Z}$; and

(iii) there are only finitely many different indecomposable summands of the (infinite-dimensional) module
$$T(M) = M \oplus M^{\otimes 2} \oplus M^{\otimes 3} \oplus \cdots .$$

The third equivalent condition is often the easiest to use in actually deciding if modules are algebraic or not. In particular, it is very easy to use this condition to prove the following.

Lemma 1.2 Let M and N be KG-modules.

(i) M and N are algebraic if and only if $M \oplus N$ is algebraic.

(ii) If M and N are algebraic, then so is $M \otimes N$.

Hence the algebraic modules form a subring $\mathrm{Alg}(G)$ of the Green ring.

We pause to briefly give some examples of algebraic modules.

Example 1.3 All permutation modules are algebraic, as are all projective modules.

A module is called *endo-trivial* if $M \otimes M^*$ is the direct sum of a projective module and the trivial module K. These crop up all over representation theory, and have recently been classified, in a sequence of long papers. The most complicated case is proving the general non-existence of torsion endo-trivial module, by which we mean $M^{\otimes n} = K \oplus P$ for some projective module P and some $n > 0$. It is easy to see that an endo-trivial is torsion if and only if it is algebraic, and so the methods of algebraic modules might be applicable to shorten this proof somewhat, or even offer an alternative proof.

On a related note to algebraic modules, a module is said to be *simply generated* if it is a summand of some tensor product of simple modules.

Lemma 1.4 The sum and tensor product of two simply generated modules are simply generated. Also, summands of simply generated modules are simply generated.

Again, the simply generated modules form a subring $\mathrm{SG}(G)$ of the Green ring. One natural object to study in the context of these two subrings is their intersection, $\mathrm{Alg}(G) \cap \mathrm{SG}(G)$. We will come back to this later.

Let H be a subgroup of G, and let M be a KG-module. We can simply think of M as a KH-module, and get the *restriction* of M to H, denoted by $M\downarrow_H$. Note that $\dim M = \dim M\downarrow_H$. Dual to this is *induction*, which takes a KH-module N, and produces a KG-module, namely the module

$$N \otimes_{KH} KG.$$

This module has dimension $|G : H| \dim N$.

Lemma 1.5 Suppose that M is an algebraic KG-module, and let H be a subgroup of G. Then $M\downarrow_H$ is an algebraic KH-module. Conversely, suppose that N is an algebraic KH-module. Then $N\uparrow^G$ is an algebraic KG-module.

Let M denote an indecomposable KG-module. It turns out that there is a certain p-subgroup P (determined only up to conjugacy) and a certain indecomposable KP-module

S such that M is a summand of $S \uparrow^G$, and that this property does not hold for any smaller subgroups Q of P. The subgroup P is called a *vertex*, and the module S is called a *source*.

Theorem 1.6 Let M be an indecomposable KG-module, and let P and S be its vertex and source. Then M is algebraic if and only if S is.

Notice that if P is cyclic, we have mentioned previously that there are only finitely many indecomposable KP-modules, and so all KP-modules are algebraic.

2 Algebraic Modules and p-Soluble Groups

The first theorem in this direction came fairly soon after Alperin defined algebraic modules, in 1976.

Theorem 2.1 (Berger (1976)) Let G be a soluble group. Then every simply generated module is algebraic. In particular, all simple modules are algebraic. Hence $\mathrm{SG}(G) \leqslant \mathrm{Alg}(G)$.

In 1980, Walter Feit extended this to p-soluble groups. His proof is five pages long (which is a substantial reduction on Berger's proof) but requires the classification of the finite simple groups. In his proof he noted that it was curious that while the result was only about p-soluble groups, the proof required knowledge about all finite simple groups.

Our first result is that Feit's theorem can be extended, simplified, and have its dependence on the classification removed. The first obvious statement that a group G has algebraic simple modules if and only if $G/\mathrm{O}_p(G)$ has, since $\mathrm{O}_p(G)$ acts trivially on all simple modules. One can similarly consider passing from G to $\mathrm{O}^p(G)$ and G to $\mathrm{O}^{p'}(G)$, and this proves the follwoing result.

Theorem 2.2 (C. (2006)) Let G be a finite group, let H be a normal subgroup, and suppose that G/H is p-soluble. Let M be a simple KG-module. Then M is algebraic if and only if $M \downarrow_H$ is algebraic. In particular, if all simple KH-modules are algebraic then all simple KG-modules are algebraic.

We will briefly discuss the proof of this theorem, since it is entirely elementary, modulo basic representation theory. Firstly, notice that we may assume that $\mathrm{O}^{p'}(G) = G$, since if M is a simple KG-module, and $H = \mathrm{O}^{p'}(G)$, then M is a summand of $(M \downarrow_H) \uparrow^G$ (every simple module is relatively H-projective if $p \nmid |G : H|$). Hence our only problem comes from a normal subgroup H of index p. In this case, either $M \downarrow_H$ is simple or it is not. If it is not, then again

$$M|(M \downarrow_H) \uparrow^G,$$

and in the simple case, the KG-module M is simple the extension of the simple KH-module $M \downarrow_H$ by a trivial G/H-action. In all cases, then M is algebraic if and only if $M \downarrow_H$ is algebraic, and induction proves the result.

We can go in the opposite direction to this theorem.

Proposition 2.3 Let G be a finite group with algebraic simple modules, and let H be a normal subgroup of G. Then both H and G/H have algebraic simple modules. Consequently, if G has algebraic simple modules then so do all composition factors.

The converse to this, as a natural extension to the p-soluble case, is unfortunately still conjectural; this conjecture appears to strictly control how complicated simple KG-modules can get, based on the composition factors for G.

Conjecture 2.4 Let G be a finite group. The G has the SMA property if and only if all composition factors of G have the SMA property.

Theorem 2.2 gives us some headway in this conjecture.

Corollary 2.5 Let G be a finite group all of whose composition factors have simple algebraic modules. Suppose that $E(G)$ is centreless, and that $G/F^*(G)$ is p-soluble. Then G has algebraic simple modules.

The requirement on $E(G)$ comes from the fact that we do not yet know whether a quasisimple group G has algebraic simple modules whenever $G/\mathrm{Z}(G)$ does. An equivalent condition is to require that for all simple composition factors of $E(G)$, the corresponding universal central extension has algebraic simple modules. This is true in all cases tested so far.

3 Simple Groups for $p = 2$

We will concentrate on the case where $p = 2$, although there are results for odd primes as well. The first result suggests that having algebraic simple modules signifies that the simple group is 'small'. The positive case is due to Alperin, and the negative case is considered 'well-known' by Berger.

Theorem 3.1 Let $G = \mathrm{GL}_n(2^a)$, and let K be a field of characteristic 2. Then G has algebraic simple modules for $n = 2$, and the natural module is not algebraic for $n \geqslant 3$.

One would think, therefore, that most finite simple groups have non-algebraic modules, because $\mathrm{PSL}_3(2)$ is quite a small group, and appears inside many larger simple groups.

Theorem 3.2 (C. (2006)) Let $G = \mathrm{SL}_2(q)$, where q is odd. Then G has algebraic simple modules if and only if $q \not\equiv 7 \mod 8$.

In particular, this offers an infinite sequence of simple groups

$$\mathrm{PSL}_2(7) < \mathrm{PSL}_2(7^2) < \mathrm{PSL}_2(7^3) < \cdots$$

such that the terms alternate between having algebraic simple modules and not.

In fact, the groups $\mathrm{PSL}_2(q)$ almost exhaust those simple groups with dihedral Sylow 2-subgroup, with the remaining group being A_7.

Theorem 3.3 Let G be a simple group with dihedral Sylow 2-subgroups. Then G is algebraic unless $G \cong \mathrm{PSL}_2(8n + 7)$ for some n.

In fact, if we assume that whenever $G/\mathrm{O}_{p'}(G)$ has algebraic simple modules then G has algebraic simple modules, we can get a stronger result.

Theorem 3.4 (C. (2006)) Let G be a finite group with dihedral Sylow 2-subgroups. Then $G/\mathrm{O}_{2'}(G)$ has algebraic simple modules unless $\mathrm{PSL}_2(8n + 7)$ is a composition factor of G.

In simple groups with abelian Sylow 2-subgroups we get a similar result, which has to be modified to reflect the fact that we do not know whether J_1 has algebraic simple modules. (The three largest simple modules in the principal block are of unknown algebraicity.)

Theorem 3.5 (C. (2006)) Let G be a finite group with abelian Sylow 2-subgroups, and assume that J_1 is not a composition factor of G. Then $G/\mathrm{O}_{2'}(G)$ has algebraic simple modules.

In particular this implies that the groups ${}^2G_2(3^{2n+1})$ have algebraic simple modules.

For simple groups with semidihedral Sylow 2-subgroup, the picture is more difficult. The following appears to be true.

Conjecture 3.6 Let G be a simple group with semidihedral Sylow 2-subgroups. If G is isomorphic with M_{11} or $\mathrm{PSU}_3(q)$ for $q \equiv 1 \mod 4$, then G has algebraic simple modules, and if G is isomorphic with $\mathrm{PSL}_3(q)$ for $q \equiv 3 \mod 4$, then G has algebraic simple modules if and only if $q \equiv 3 \mod 8$.

This result is true for all of the modules in the principal block by work of Karin Erdmann in the late 1970s.

Moving from positive results to negative results, we have the following result, most of which follows easily from the fact that $\mathrm{GL}_3(2)$ has non-algebraic simple modules, and facts about modules for V_4, which we will talk about later.

Theorem 3.7 (C. (2006)) let G be a finite simple group, and let K be a field of characteristic 2. Then G has non-algebraic simple modules in all of the following cases:

(i) $\mathrm{PSL}_n(2^a)$, $n \geqslant 3$;

(ii) $\mathrm{PSp}_{2n}(2^a)$, $n \geqslant 3$;

(iii) $\mathrm{P}\Omega_{2n}^{+}(2^a)$, $n \geqslant 4$;

(iv) $\mathrm{P}\Omega_{2n}^{-}(2^a)$, $n \geqslant 4$;

(v) $\mathrm{PSU}_n(2^a)$, $n \geqslant 6$;

(vi) $E_6(2^a)$;

(vii) $E_7(2^a)$;

(viii) $F_4(2^a)$;

(ix) $G_2(2^a)$;

(x) ${}^2E_6(2^a)$;

(xi) ${}^3D_4(2^a)$; and

(xii) M_{12}, M_{22}, M_{23}, M_{24}, J_2, J_3, J_4, Suz, He, HN, Fi_{22}, and Ru.

The missing sporadics are: M_{11}, which does have algebraic simple modules, J_1, which is not known, Ly, which has smallest simple module 2480, ON, whose simple modules are huge—10000 and above—B and M, whose simple modules are obviously big, B having 4370, Fi_{24} with 3774-dim, Fi_{23} with 782-dim, Th with 248-dim, but has smallest perm rep of degree 143 127 000, the Conway groups and McL, whose 24-dimish rep might actually be algebraic.

One notable result is that in characteristic 3, M_{22} has algebraic simple modules.

4 Blocks with V_4 Defect Group

The easiest case in modular representation theory is blocks with cyclic defect group. The next easiest is when the defect group is V_4. In the case of V_4-modules, the following result is due to Conlon.

Theorem 4.1 (Conlon) Let M be an indecomposable KV_4-module. Then M is algebraic if and only if M is even-dimensional or trivial.

It is true that if M is a simple module lying in a block B, then a vertex of M is contained within a defect group of B. In the case where the defect group is abelian, all vertices of simple modules are equal to the defect group, and so we will focus on simple modules with V_4 vertex.

Conjecture 4.2 Let M be a simple KG-module with V_4 vertex. Then M is algebraic.

This more general conjecture is much more amenable to reduction than the weaker conjecture that if B is a block with V_4 defect group, then all simple B-modules are algebraic. (The result is not true for general abelian defect groups, at least in the case where p is odd, and is also not true for blocks with dihedral defect group.)

Proposition 4.3 (C. (2007)) Let G be a minimal counterexample to the conjecture above. Then G is perfect and M is faithful.

The reason for interest in this is the Puig conjecture.

Theorem 4.4 (C. (2006)) Suppose that Conjecture 4.2 is true. Then Puig's conjecture holds for blocks with V_4 defect group.

5 The Heller Translate

Recall that if M is a non-projective indecomposable module then $\Omega(M)$ is the kernel of the projection from the projective cover of M onto M. This is defined inductively for $\Omega^i(M)$, and $\Omega^{-i}(M) = \Omega^i(M^*)^*$. If $\Omega^i(M) = M$ for some $i > 0$, then M is referred to as *periodic*. A periodic module for a group of p-rank at least 2 has dimension divisible by p.

Theorem 5.1 (C. (2006)) Assume that the p-rank of G is at least 2. Let $A(n)$ denote the number of isomorphism classes of indecomposable KG-module of dimension prime to p and at most n, and let $B(n)$ denote the number of algebraic modules of dimension prime to p and at most n. Then

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = 0.$$

Theorem 5.2 (C. (2006)) Let G be a group of 2-rank 2. Let $A(n)$ denote the number of isomorphism classes of indecomposable non-periodic KG-modules of dimension at most n, and let $B(n)$ denote the number of algebraic indecomposable non-periodic modules of dimension at most n. Then

$$\lim_{n\to\infty} \frac{B(n)}{A(n)} = 0.$$

Proposition 5.3 Let G be a finite group of p-rank at least 2, and let M be a self-dual non-periodic indecomposable module. Then $\Omega^i(M)$ is not algebraic for $i \neq 0$.

Proposition 5.4 Let G be a finite group, and let M be a periodic indecomposable module. Then M is algebraic if and only if $\Omega^i(M)$ is algebraic for all i.