On the Unit Conjecture

Joint with Peter Pappas

David A. Craven

University of Oxford

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What the colo(u)rs mean

Green: a technical consideration, which may be ignored.

Blue: denotes a mathematical term.

The Unit Conjecture in a quotation

"The true test of a first-rate mind is the ability to hold two contradictory ideas at the same time and still function."

F. Scott Fitzgerald

The Unit Conjecture is sometimes clearly true and sometimes clearly false, it alternates between the two continually. The community has no idea whether it is true or not, and so we need to attack the problem from both directions.

"Do I believe the Unit Conjecture? What day of the week is it?"

Peter Pappas

An example

Does there exist a polynomial $p(x)$, with real coefficients, such that $(\pi x^3 + 3x + 1)p(x) = 1$?

The answer is **NO**, because the highest power in p, times πx^3, is the unique term of the product of highest power.

What if we allow negative powers of x?

The answer is still **NO**, because there is still no cancellation among the highest powers of x.

Invertible polynomials

So which polynomials **are** invertible, even if we allow negative powers of x?

Clearly polynomials of the form αx^n for $n \in \mathbb{Z}$ are invertible if α is invertible, and the same argument as before proves that these are the only ones.

This applies even if we have more than one variable, say polynomials in $x_1, x_2, \ldots, x_n$, and even if the coefficients come from any field, say the reals, rationals, complexes, or finite fields.

An invertible element will be called a unit. All units here will be two-sided. It is not known (for general group rings) whether all units are two-sided.

Towards the Unit Conjecture

- Write $G = \langle x \rangle = \{\dots x^{-2}, x^{-1}, 1, x, x^2, \dots\}$. Write $K[x^{\pm 1}]$ for all polynomials in x and x^{-1} with coefficients in K; the units of $K[x^{\pm 1}]$ are αx^n for $n \in \mathbb{Z}$ and $\alpha \in K \setminus \{0\}$.
- Every element of $K[x^{\pm 1}]$ can be written as a linear combination of elements of G in a unique way. When we can do this, we write $K[x^{\pm 1}] = KG$.
- The units of KG are simply scalar multiples of the elements of G. Such elements of KG are called trivial units. There are no non-trivial units of KG.
- A typical element of KG is

$$\begin{aligned}\alpha &= c_{g_1} g_1 + c_{g_2} g_2 + \dots + c_{g_n} g_n \\ &= \sum_{g \in G} c_g g \text{ (finitely many } c_g \text{ non-zero)}\end{aligned}$$

What is the Unit Conjecture?

- We now want to change from $\langle x \rangle$ to torsion-free groups G.
- A torsion-free group G is a multiplicative system, in which all elements have inverses (like $G = \langle x \rangle$ on the previous slide), and like polynomials, no power of a non-trivial element is trivial.
  WARNING: these need not be abelian; i.e., we need not have $x \cdot y = y \cdot x$.
- Take all linear combinations of the elements of G, just as before. Call it KG; it is called the group ring, and the elements are

$$\sum_{g \in G} c_g g \text{ (finitely many } c_g \text{ non-zero)}$$

- The trivial units are elements αg, where $\alpha \in K \setminus \{0\}$ and $g \in G$.

Conjecture (Kaplansky, 1969)

All units of KG are trivial.

When is the Unit Conjecture true?

- When G is abelian; i.e., when $x \cdot y = y \cdot x$. In this case KG is essentially a polynomial ring, and we saw this case before.
- When G is a unique-product group. A group is a UP group if, whenever X and Y are finite subsets, there is an element z such that z is expressible **uniquely** as a product $x \cdot y$, where $x \in X$ and $y \in Y$.
- Are all torsion-free groups unique-product groups? **NO**. It was proved by Rips and Segev that there are torsion-free, non-UP groups. An easier example, Γ, was considered by Promislow.
- Using a computer, Promislow searched randomly in Γ, and found a subset X (with $|X| = 14$) such that $X \cdot X$ had no unique product.
- This was the first real use of the computer in this field.

The Passman group Γ

This group Γ is given by the presentation

$$\Gamma = \langle x, y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2}\rangle.$$

Write $z = xy$, $a = x^2$, $b = y^2$, $c = z^2$.
Idea 1: $H = \langle a, b, c\rangle$ is an abelian normal subgroup, and G/H is the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Idea 2: $N = \langle a, b\rangle$ is an abelian normal subgroup, and G/N is the infinite dihedral group D_∞. This second quotient gives a length function on the elements of the group.

- The elements of N (of the form $a^i b^j$) are defined to be length 0.
- Length 1 elements are αx or αy, with $\alpha \in N$.
- Length 2 elements are αxy or αyx, with $\alpha \in N$.
- And so on.

The group ring $K\Gamma$

- We now want to consider the group ring $K\Gamma$, where K is any field.
- We extend the length function from Γ to $K\Gamma$: the length of a sum of elements of G is the maximum of the lengths of the elements.
- We want to rewrite the elements of $K\Gamma$, using the subgroup $H = \langle a, b, c\rangle$ this time. Any element may be written as $Ax + By + C + Dz$, where $A, B, C, D \in KH$.
- This rewriting allows us to construct a representation as matrices over $K\langle a, b, c\rangle$.

$$\begin{pmatrix} C & A & B & D \\ A^x a & C^x & D^x a & B^x \\ B^y b & D^y a^{-1} c^{-1} & C^y & A^y a^{-1} b c^{-1} \\ D^z c & B^z b^{-1} & A^z b^{-1} c & C^z \end{pmatrix}$$

(Here, A^x indicates the conjugate of A by x, and so on.)

Theorems on $K\Gamma$

Using a splitting theorem for units in $K\Gamma$, we can produce two important theorems.

Theorem

The length of a unit in $K\Gamma$ is equal to the length of its inverse.

Theorem

An element of $K\Gamma$ is a unit if and only if its determinant is in K.

Thus it must be really easy to check if an element of $K\Gamma$ is invertible, simply by checking its determinant. A length-3 element looks like the following:

$$\alpha_1 x + (\alpha_2 + \alpha_3 c)y + \alpha_4 + (\alpha_5 + \alpha_6 c^{-1})z.$$

(Here, $\alpha_i \in N$.)

The determinant of a length-3 element

$$
\begin{gathered}
\alpha_1\alpha^x_1\alpha^y_1\alpha^z_1 - \alpha_1\alpha^x_1\alpha^y_4\alpha^z_4 a - \alpha_1\alpha^x_2\alpha^y_3\alpha^z_1 + \alpha_1\alpha^x_2\alpha^y_4\alpha^z_6 - \alpha_1\alpha^x_3\alpha^y_2\alpha^z_1 + \alpha_1\alpha^x_3\alpha^y_4\alpha^z_5 - \alpha_1\alpha^x_5\alpha^y_1\alpha^z_5 b + \alpha_1\alpha^x_5\alpha^y_2\alpha^z_4 ab \\
-\alpha_1\alpha^x_6\alpha^y_1\alpha^z_6 b + \alpha_1\alpha^x_6\alpha^y_3\alpha^z_4 ab - \alpha_2\alpha^x_1\alpha^y_1\alpha^z_3 + \alpha_2\alpha^x_1\alpha^y_6\alpha^z_4 + \alpha_2\alpha^x_2\alpha^y_2\alpha^z_2 + \alpha_2\alpha^x_2\alpha^y_3\alpha^z_3 - \alpha_2\alpha^x_2\alpha^y_5\alpha^z_5 a^{-1} \\
-\alpha_2\alpha^x_2\alpha^y_6\alpha^z_6 a^{-1} + \alpha_2\alpha^x_3\alpha^y_2\alpha^z_3 - \alpha_2\alpha^x_3\alpha^y_6\alpha^z_5 a^{-1} + \alpha_2\alpha^x_4\alpha^y_1\alpha^z_5 ba^{-1} - \alpha_2\alpha^x_4\alpha^y_2\alpha^z_4 b - \alpha_3\alpha^x_1\alpha^y_1\alpha^z_2 + \alpha_3\alpha^x_1\alpha^y_5\alpha^z_4 \\
+\alpha_3\alpha^x_2\alpha^y_3\alpha^z_2 - \alpha_3\alpha^x_2\alpha^y_5\alpha^z_6 a^{-1} + \alpha_3\alpha^x_3\alpha^y_2\alpha^z_2 + \alpha_3\alpha^x_3\alpha^y_3\alpha^z_3 - \alpha_3\alpha^x_3\alpha^y_5\alpha^z_5 a^{-1} - \alpha_3\alpha^x_3\alpha^y_6\alpha^z_6 a^{-1} + \alpha_3\alpha^x_4\alpha^y_1\alpha^z_6 ba^{-1} \\
-\alpha_3\alpha^x_4\alpha^y_3\alpha^z_4 b - \alpha_4\alpha^x_2\alpha^y_4\alpha^z_2 b^{-1} + \alpha_4\alpha^x_2\alpha^y_5\alpha^z_1 a^{-1}b^{-1} - \alpha_4\alpha^x_3\alpha^y_4\alpha^z_3 b^{-1} + \alpha_4\alpha^x_3\alpha^y_6\alpha^z_1 a^{-1}b^{-1} - \alpha_4\alpha^x_4\alpha^y_1\alpha^z_1 a^{-1} \\
+\alpha_4\alpha^x_4\alpha^y_4\alpha^z_4 + \alpha_4\alpha^x_5\alpha^y_1\alpha^z_3 - \alpha_4\alpha^x_5\alpha^y_6\alpha^z_4 + \alpha_4\alpha^x_6\alpha^y_1\alpha^z_2 - \alpha_4\alpha^x_6\alpha^y_5\alpha^z_4 + \alpha_5\alpha^x_1\alpha^y_4\alpha^z_2 ab^{-1} - \alpha_5\alpha^x_1\alpha^y_5\alpha^z_1 b^{-1} \\
+\alpha_5\alpha^x_4\alpha^y_3\alpha^z_1 - \alpha_5\alpha^x_4\alpha^y_4\alpha^z_6 - \alpha_5\alpha^x_5\alpha^y_2\alpha^z_2 a - \alpha_5\alpha^x_5\alpha^y_3\alpha^z_3 a + \alpha_5\alpha^x_5\alpha^y_5\alpha^z_5 + \alpha_5\alpha^x_5\alpha^y_6\alpha^z_6 - \alpha_5\alpha^x_6\alpha^y_3\alpha^z_2 a + \alpha_5\alpha^x_6\alpha^y_5\alpha^z_6 \\
+\alpha_6\alpha^x_1\alpha^y_4\alpha^z_3 ab^{-1} - \alpha_6\alpha^x_1\alpha^y_6\alpha^z_1 b^{-1} + \alpha_6\alpha^x_4\alpha^y_2\alpha^z_1 - \alpha_6\alpha^x_4\alpha^y_4\alpha^z_5 - \alpha_6\alpha^x_5\alpha^y_2\alpha^z_3 a + \alpha_6\alpha^x_5\alpha^y_6\alpha^z_5 - \alpha_6\alpha^x_6\alpha^y_2\alpha^z_2 a \\
-\alpha_6\alpha^x_6\alpha^y_3\alpha^z_3 a + \alpha_6\alpha^x_6\alpha^y_5\alpha^z_5 + \alpha_6\alpha^x_6\alpha^y_6\alpha^z_6 + c(-\alpha_1\alpha^x_2\alpha^y_2\alpha^z_1 + \alpha_1\alpha^x_2\alpha^y_4\alpha^z_5 - \alpha_1\alpha^x_6\alpha^y_1\alpha^z_5 b + \alpha_1\alpha^x_6\alpha^y_2\alpha^z_4 ab \\
+\alpha_2\alpha^x_2\alpha^y_2\alpha^z_3 - \alpha_2\alpha^x_2\alpha^y_6\alpha^z_5 a^{-1} - \alpha_3\alpha^x_1\alpha^y_1\alpha^z_3 + \alpha_3\alpha^x_1\alpha^y_6\alpha^z_4 + \alpha_3\alpha^x_2\alpha^y_2\alpha^z_2 + \alpha_3\alpha^x_2\alpha^y_3\alpha^z_3 - \alpha_3\alpha^x_2\alpha^y_5\alpha^z_5 a^{-1} \\
-\alpha_3\alpha^x_2\alpha^y_6\alpha^z_6 a^{-1} + \alpha_3\alpha^x_3\alpha^y_2\alpha^z_3 - \alpha_3\alpha^x_3\alpha^y_6\alpha^z_5 a^{-1} + \alpha_3\alpha^x_4\alpha^y_1\alpha^z_5 ba^{-1} - \alpha_3\alpha^x_4\alpha^y_2\alpha^z_4 b - \alpha_4\alpha^x_2\alpha^y_4\alpha^z_3 b^{-1} \\
+\alpha_4\alpha^x_2\alpha^y_6\alpha^z_1 a^{-1}b^{-1} + \alpha_4\alpha^x_6\alpha^y_1\alpha^z_3 - \alpha_4\alpha^x_6\alpha^y_6\alpha^z_4 + \alpha_5\alpha^x_1\alpha^y_4\alpha^z_3 ab^{-1} - \alpha_5\alpha^x_1\alpha^y_6\alpha^z_1 b^{-1} + \alpha_5\alpha^x_4\alpha^y_2\alpha^z_1 - \alpha_5\alpha^x_4\alpha^y_4\alpha^z_5 \\
-\alpha_5\alpha^x_5\alpha^y_2\alpha^z_3 a + \alpha_5\alpha^x_5\alpha^y_6\alpha^z_5 - \alpha_5\alpha^x_6\alpha^y_2\alpha^z_2 a - \alpha_5\alpha^x_6\alpha^y_3\alpha^z_3 a + \alpha_5\alpha^x_6\alpha^y_5\alpha^z_5 + \alpha_5\alpha^x_6\alpha^y_6\alpha^z_6 - \alpha_6\alpha^x_6\alpha^y_2\alpha^z_3 a \\
+\alpha_6\alpha^x_6\alpha^y_6\alpha^z_5) + c^{-1}(-\alpha_1\alpha^x_3\alpha^y_3\alpha^z_1 + \alpha_1\alpha^x_3\alpha^y_4\alpha^z_6 - \alpha_1\alpha^x_5\alpha^y_1\alpha^z_6 b + \alpha_1\alpha^x_5\alpha^y_3\alpha^z_4 ab - \alpha_2\alpha^x_1\alpha^y_1\alpha^z_2 + \alpha_2\alpha^x_1\alpha^y_5\alpha^z_4 \\
+\alpha_2\alpha^x_2\alpha^y_3\alpha^z_2 - \alpha_2\alpha^x_2\alpha^y_5\alpha^z_6 a^{-1} + \alpha_2\alpha^x_3\alpha^y_2\alpha^z_2 + \alpha_2\alpha^x_3\alpha^y_3\alpha^z_3 - \alpha_2\alpha^x_3\alpha^y_5\alpha^z_5 a^{-1} - \alpha_2\alpha^x_3\alpha^y_6\alpha^z_6 a^{-1} + \alpha_2\alpha^x_4\alpha^y_1\alpha^z_6 ba^{-1} \\
-\alpha_2\alpha^x_4\alpha^y_3\alpha^z_4 b + \alpha_3\alpha^x_3\alpha^y_3\alpha^z_2 - \alpha_3\alpha^x_3\alpha^y_5\alpha^z_6 a^{-1} - \alpha_4\alpha^x_3\alpha^y_4\alpha^z_2 b^{-1} + \alpha_4\alpha^x_3\alpha^y_5\alpha^z_1 a^{-1}b^{-1} + \alpha_4\alpha^x_5\alpha^y_1\alpha^z_2 - \alpha_4\alpha^x_5\alpha^y_5\alpha^z_4 \\
-\alpha_5\alpha^x_5\alpha^y_3\alpha^z_2 a + \alpha_5\alpha^x_5\alpha^y_5\alpha^z_6 + \alpha_6\alpha^x_1\alpha^y_4\alpha^z_2 ab^{-1} - \alpha_6\alpha^x_1\alpha^y_5\alpha^z_1 b^{-1} + \alpha_6\alpha^x_4\alpha^y_3\alpha^z_1 - \alpha_6\alpha^x_4\alpha^y_4\alpha^z_6 - \alpha_6\alpha^x_5\alpha^y_2\alpha^z_2 a \\
-\alpha_6\alpha^x_5\alpha^y_3\alpha^z_3 a + \alpha_6\alpha^x_5\alpha^y_5\alpha^z_5 + \alpha_6\alpha^x_5\alpha^y_6\alpha^z_6 - \alpha_6\alpha^x_6\alpha^y_3\alpha^z_2 a + \alpha_6\alpha^x_6\alpha^y_5\alpha^z_6)
\end{gathered}
$$

You don't want to see the length-4 determinant.

But we can do it anyway

Using the splitting theorem and a page of calculations, we get the following theorem.

Theorem

There are no non-trivial units of length at most 2 in $K\Gamma$ over any field K.

Using the two theorems on $K\Gamma$, and a computer, we can also get the next result.

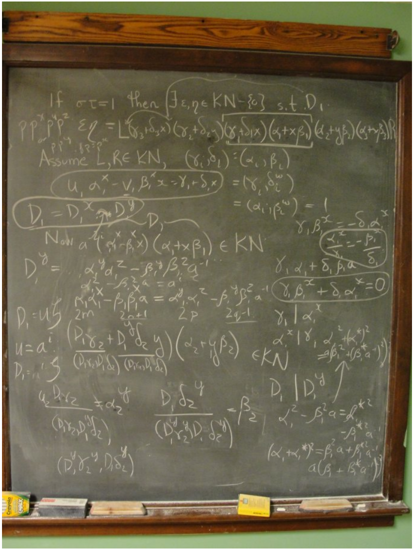
Theorem

The Promislow set is not the support of a unit in $K\Gamma$, for any field K.

This is not the 'correct' way to prove that it is not a unit. This would be by proving that there are no length-3, non-trivial units.

The Promislow set is not of length 3. However, we can apply an outer automorphism and make it length 3.

And then this happened...



Another false dawn

The blackboard on the previous slide fails to produce a proof, however.

The prime divisors that we first thought collapsed, allowing us to move parts of the unit around, failed to collapse upon writing the proof down a second time.

However, each failed attempt gives us more information on the structure of the units in the group ring, and hopefully brings us closer to a correct solution.

Conclusion

- We have a splitting theorem.
- We know the length of the inverse and have the determinant condition.
- We proved that the Promislow set is not the support of a unit.
- We have partial results on the structure of general units, both in $K\Gamma$ and for other supersoluble torsion-free groups.
- It also appears that all aspects of the unit conjecture are hard.

So is the Unit Conjecture true for all groups?

For the zero divisor conjecture (that if $xy = 0$ then either $x = 0$ or $y = 0$ in the group ring), there are methods that can be used to extend results from supersoluble groups to polycyclic, and soluble groups. The same might be true for the Unit Conjecture; the strategy is therefore to prove the result for supersoluble groups, then extend to soluble groups using similar techniques.

On the other hand, the structure of torsion-free groups in general is very wild, and without nice conditions like virtual solubility, there might not be much stopping a unit from appearing.

> *"Doublethink means the power of holding two contradictory beliefs in one's mind simultaneously, and accepting both of them."*
>
> *George Orwell*

The search continues...