The Combinatorial Broué Conjecture

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Groups of Lie type

Let $G = G(q)$ be a group of Lie type, e.g., $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, etc. The ordinary representation theory of G is in some sense generic in q. For example, the irreducible (complex) character degrees and their multiplicities are polynomials in q.

Obviously, if we fix $G(-)$ and vary q we get different numbers of irreducible characters. However, they split into two collections: unipotent and non-unipotent. (Very!) roughly speaking, unipotent is like the non-exceptional characters in the Brauer tree case introduced by Dudas in the last talk, and non-unipotent equals exceptional characters.

Formally, a unipotent character of $G = \mathbf{G}^F$ is a constituent of the Deligne–Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(1)$. (This probably isn't much help if you didn't know what unipotent characters were in the first place.)

The number of unipotent characters does not depend on q (their degrees do), and they have a consistent parametrization. For example, the unipotent characters of $\mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$ are labelled by partitions of n.

Modular representation theory of groups of Lie type

What can we say about the modular representation theory? To give this question some substance we first need to introduce blocks. Let k be a 'large' field of characteristic ℓ, and consider the group algebra kG. Write

$$1 = e_1 + e_2 + \cdots + e_r,$$

where the e_i are central elements such that $e_i e_j = \delta_{ij} e_i$. Any two such decompositions have a common refinement, and the e_i in the finest such decomposition are called blocks. If M is a module, then since $e_i e_j = \delta_{ij} e_i$, we see that

$$M = 1 \cdot M = e_1 \cdot M \oplus \cdots \oplus e_r \cdot M.$$

Thus if M is indecomposable, e.g., simple, $e_i \cdot M = \delta_{ij} M$ for some j. We say that M belongs to e_j. Extend this to sums of modules belonging to the same block. Then submodules and quotients of modules belonging to e_j belong to e_j. Since $e_j \cdot kG$ belongs to e_j obviously, every block has at least one simple module belonging to it.

Unipotent blocks

Let $\mathcal{O}$ be a local ring of characteristic 0, whose quotient by its maximal ideal is k, and write K for the field of fractions of $\mathcal{O}$. The decomposition

$$1 = e_1 + e_2 + \cdots + e_r,$$

lifts to a decomposition of 1 in $\mathcal{O}G$, and if N is a simple KG-module (which are the same as the $\mathbb{C}G$-modules if K is 'large') we again must have $e_i \cdot N = \delta_{ij} N$, so that a simple KG-module belongs to a block as well.

A unipotent block is a block of kG (or $\mathcal{O}G$) that has unipotent characters belonging to it. Since the unipotent characters are independent of q, it seems reasonable to ask that the unipotent blocks are independent of q. What can this mean?

Choosing our ℓ and q

Let $G = G(q)$ be a group of Lie type: the order of G is

$$|G| = q^N \prod_{d \in I} \Phi_d(q)^{a_d}.$$

If $\ell \mid |G|$ then either $\ell \mid q$, which leads to one theory, or $\ell \nmid q$, in which case $\ell \mid \Phi_d(q)$ for some d. We are mostly interested in the case where there is no other d' such that $\ell \mid \Phi_{d'}(q)$; in this case, the Sylow ℓ-subgroup P is abelian, homocyclic, of rank a_d. In particular, if $a_d = 1$ then P is cyclic.

We will always assume that ℓ divides exactly one $\Phi_d(q)$ from now on. Write $\bar{\ell} = |G(q)|_\ell$, the ℓ-part of $|G(q)|$. The unipotent blocks of kG do not depend on q **or** ℓ, as long as the d involved is the same.

Comparing primes

If q and q' are different, but ℓ stays the same (as does the power $\bar{\ell}$), we can ask whether the unipotent blocks of $G(q)$ and $G(q')$ are (for example) Morita equivalent. However, if the prime ℓ, or even just the prime power $\bar{\ell}$, differs for q and q', we will not get a Morita equivalence, and we must search for some other definition of 'independent', one that cannot be dependent on an equivalence of categories.

If $d \geq 1$ is an integer, then we are comparing blocks of $kG(q)$ and $k'G(q')$, where $\ell \mid \Phi_d(q)$ and $\ell' \mid \Phi_d(q')$; we say that e and e' are from the same Φ_d-block if the unipotent characters in e and e' have the same labels, so that a Φ_d-block is a set of unipotent blocks. The weight of a Φ_d-block is the rank of any defect group of a block from the Φ_d-block. Blocks with cyclic defect group have weight 1.

A guiding example: Brauer trees

The example we can use to guide our thinking is the Brauer tree. There, if there are e simple modules, the exceptionality ε satisfies $\varepsilon = (\bar{\ell} - 1)/e$, so if we fix the tree with exceptionality then we fix $\bar{\ell}$. However, it might make sense to fix the tree **without** exceptionality, and this allows us to compare primes.

Fix $e \geq 1$, let Λ be the set of all powers $\bar{\ell}$ of primes ℓ such that $e \mid (\ell - 1)$, and fix a tree with planar embedding T, with e edges and a fixed exceptional node. A generic block $\hat{B}$ is the set of all Brauer tree algebras with the tree T, and with exceptionality $(\bar{\ell} - 1)/e$ for $\bar{\ell} \in \Lambda$.

Two blocks with cyclic defect group are generically equivalent if they belong to the same generic block.

Generic equivalence for cyclic blocks

The following theorem summarizes the results over several decades and a dozen mathematicians.

Theorem

With the possible exception of thre Φ_d-blocks of $E_8(q)$ ($d = 15, 18, 24$), a Φ_d-block of weight 1 is a generic block. In each case, the Brauer tree, together with labellings by the unipotent characters, is known.

So far, so good. However, there are plenty of blocks that do not have cyclic defect groups, and so we need to be able to deal with those as well.

In 1989, Rickard proved that any two Brauer tree algebras with the same number of edges and same exceptionalities are derived equivalent. In particular, he produced an algorithm to produce a derived equivalence from a given Brauer tree to the star with exceptional node in the middle, and this algorithm made no mention of the exceptionality.

General genericity

Rickard's derived equivalence depends only on the shape of the Brauer tree, so B and B' are generically equivalent if the derived equivalence defined by Rickard works for both algebras. the Brauer tree of the group $k(Z_{\bar{\ell}} \rtimes Z_e)$ is the star with exceptional node in the centre.

Rickard's equivalence is defined by a set of combinatorial data. What would allow us to define genericity is a derived equivalence defined purely combinatorially, and we would state that two blocks are generically equivalent if the same combinatorial data can be used to define a derived equivalence with a group algebra $k(P \rtimes E)$, where this is somehow defined generically. So all we need is a general definition of a type of derived equivalence, defined combinatorially, that extends Rickard's definition to all unipotent blocks of groups of Lie type possessing abelian defect groups.

This need is satisfied by perverse equivalences.

What is a perverse equivalence?

Let A and A' be finite-dimensional algebras, $\mathcal{A} = \textit{mod-}A$, $\mathcal{A}' = \textit{mod-}A'$.

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, $T_1, T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \ldots, S_i$, and $D^b_i(\mathcal{A})$ denotes the subcategory of $D^b(\mathcal{A})$ with support modules in $\mathcal{A}_i$, then

- F induces equivalences $D^b_i(\mathcal{A}) \to D^b_i(\mathcal{A}')$, and
- $F[\pi(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{A}'_i/\mathcal{A}'_{i-1}$.

Note that mod-A' is determined, up to equivalence, by A, π, and the ordering of the S_i.

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An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, T_1, $T_2, \ldots$, T_r, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, for all i, the cohomology of $F(S_i)$ has only one copy of T_i in degree $-\pi(i)$, and any other T_j can only appear in degrees between $-\pi(i)$ and $-\pi(j) - 1$.

The geometric Broué conjecture

Broué's conjecture has a special version for unipotent blocks of groups of Lie type, called the geometric form.

Conjecture

Let $G = G(q)$ be a finite group of Lie type, and let D be an abelian defect group of a unipotent block B of G. We may embed D inside a Φ_d-torus T, and there is a Deligne–Lusztig variety Y, carrying an action of G on the one side and T on the other, whose complex of cohomology Γ has the following properties:

1. *the action of T can be extended to an action of $\mathrm{N}_G(T) = \mathrm{N}_G(D)$;*
2. *the complex induces a derived equivalence between B and its Brauer correspondent.*

The geometric Broué conjecture

In fact, if $\kappa \geq 1$ is prime to d, then there should be a Deligne–Lusztig variety $Y_{\kappa/d}$ associated naturally to κ, and whose complex of cohomology produces the desired equivalence.

While this is (a lot) more specific than the abstract version of Broué's conjecture, it still needs to be more specific, as the variety $Y_{\kappa/d}$ can be hideously complicated (and gets worse as κ grows).

This equivalence should be perverse. If the associated data can be extracted without analyzing the variety $Y_{\kappa/d}$, then the derived equivalence should be able to be constructed without the variety at all, purely combinatorially.

Cyclotomic Hecke algebras

Let's stick with the cyclic case for now. The (specialized) cyclotomic Hecke algebra $\mathcal{H}(Z_e, \mathbf{u})$ is an algebra over $R = \mathbb{C}(q^{1/2})$, and is given by

$$R[T]/(T - u_1)(T - u_2)\dots(T - u_e),$$

where $u_i = \omega_i q^{v_i}$ for roots of unity ω_i and semi-integers v_i. (For GL_n we have $\omega_i = 1$, for classical groups $\omega_i = \pm 1$, for exceptional untwisted it goes up to sixth roots, and can go up twelfth roots for Ree and Suzuki groups.)

This is invariant under global multiplication by a root of unity or power of q. Arrange them so that $u_1 = q^{v_1}$, $v_1 \geq v_i$ for all i.

The generic degree associated to the parameter u_i is

$$\frac{u_1}{u_i} \prod_{\substack{j=1 \\ j \neq i}}^{e} \frac{(u_1 - u_j)}{(u_i - u_j)}.$$

Cyclotomic Hecke algebras

To every unipotent block there is associated a cyclotomic Hecke algebra (not just the cyclic case), and the generic degrees are actually the degrees (as polynomials in q) of the unipotent characters (up to a fixed polynomial, which is 1 for the principal block).

Let $\kappa \geq 1$ be prime to d, B be a unipotent block of weight w, with automizer (normalizer modulo centralizer) E, and let $\mathcal{H} = \mathcal{H}(E, \mathbf{u})$ be its associated cyclotomic Hecke algebra. The specialization $q \mapsto \mathrm{e}^{2\pi\mathrm{i}\kappa/d}$ turns $\mathcal{H}$ into the group algebra $\mathbb{C}E$. This gives us a natural bijection between the irreducible representations of the cyclotomic Hecke algebra $\mathcal{H}$ and the irreducible characters of E.

Since k has characteristic ℓ, and $P = Z_{\ell^a}^w$ is an ℓ-group, the simple kH-modules for $H = (Z_{\ell^a}^w) \rtimes E$ are in natural bijection with the irreducible characters of E.

generic degrees $\leftrightarrow$ characters of $\mathcal{H}$ $\leftrightarrow$ characters of E $\leftrightarrow$ kH-modules

Combinatorial Broué

In order to get a perverse equivalence, we need a bijection between the simple modules for a block B and its Brauer correspondent b.

- The simple b-modules are in **non-canonical** bijection with the simple modules for kH;
- the simple kH-modules are in bijection with the characters of the cyclotomic Hecke algebra;
- the characters of the cyclotomic Hecke algebra are in natural bijection with the unipotent characters in B via the generic degrees, which are in natural bijection with the simple B-modules.

The perversity function $\pi(-)$ can be calculated from the generic degrees (C., 2011).

Theorem (C. (2012))

This bijection and perversity function yield a perverse equivalence for unipotent blocks with cyclic defect groups whenever the Brauer tree is known.

Combinatorial Broué: Example

$G = {}^2F_4(q^2)$, $\ell \mid \Phi'_{24}(q)$. (By Φ'_{24} we mean the polynomial factor of Φ_{24} with ζ_{24} as a root.)

Character	$\omega_i q^{aA/e}$	$k=5$	$k=11$	$k=13$	$k=19$
$\phi_{1,0}$	1	0	0	0	0
${}^2B_2[\psi^3];1$	$\psi^7 q$	4	10	12	18
${}^2F_4^{\mathrm{II}}[-\mathrm{i}]$	$-\mathrm{i}q^2$	8	18	22	32
${}^2F_4[-\theta^2]$	$-\theta q^2$	8	18	22	32
${}^2B_2[\psi^5];1$	ψq	4	10	12	18
$\phi_{2,1}$	q^2	7	17	21	31
${}^2B_2[\psi^3];\varepsilon$	$\psi^7 q^3$	9	21	25	37
${}^2F_4[-\theta]$	$-\theta^2 q^2$	8	18	22	32
${}^2F_4^{\mathrm{II}}[\mathrm{i}]$	$\mathrm{i}q^2$	8	18	22	32
${}^2B_2[\psi^5];\varepsilon$	ψq^3	9	21	25	37
$\phi_{1,8}$	q^4	10	22	26	38
${}^2F_4^{\mathrm{II}}[-1]$	$-q^2$	10	20	24	32

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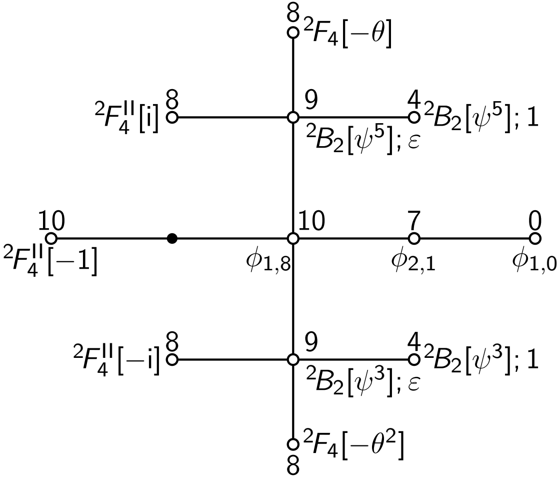

Beyond the cyclic case

Now suppose that we have a Φ_d-block of weight 2. We have the perversity function and bijection, so we can define a perverse equivalence. However, our desired derived equivalence must lift a stable equivalence, which unlike the cyclic case is not 'trivial' (i.e., induction and restriction). This requires us to come up with a complex of relative projective modules to insert.

The automizer E of the block B is a complex reflection group, and its action on $P = Z_{\ell^a}^2$ is as complex reflections. A complex reflection group comes with a set of reflection hyperplanes, i.e., subgroups Q of $Z_{\ell^a}^2$, and the centralizer in E of Q is another complex reflection group. In fact, $\mathrm{C}_G(Q)$ is the direct product of Q and another group of Lie type G_1. The group $\mathrm{N}_G(Q)$ is (in general a sub)direct product of $Q \rtimes Z_e$ and G_1, possibly with some automorphisms.

The group G_1 has a perverse equivalence, since it's a group of Lie type. This will be used.

Digging around to get a complex

1. Let S be a simple B-module, with Green correspondent C, an indecomposable b-module. Write $H = \mathrm{N}_G(D)$, the normalizer of the defect group of B.
2. For Q a reflection hyperplane of the action of $E = \mathrm{N}_G(D)/\mathrm{C}_G(D)$ on D, write $\mathrm{C}_G(Q) = Q \times G_1$, and $\mathrm{C}_H(D) = Q \times (R \rtimes A)$, for $|R| = Z_{\ell^a}$ and A an ℓ'-group. Write $\mathrm{N}_H(D) = (Q \times (R \rtimes A))B$ for B some ℓ'-group acting on Q, and also possibly on $R \rtimes A$.
3. Consider the projective-free part of the restriction of C to the subgroup QAB of H. This should be a sum of tensor products of QB-modules X_i with AB-modules Y_i (which is an AB-character since AB is an ℓ'-group). These AB-modules Y_i should be characters of Green correspondents of simple kG_1-modules $\bar{S}_i$ in $\mathrm{N}_{G_1}(R)$.
4. The $\bar{S}_i$ have images U_i under the perverse equivalence for G_1, which are complexes of projective $\mathrm{N}_{G_1}(R)$-modules. The contribution of Q to the stable equivalence is the sum of the complexes $X_i \otimes U_i$.