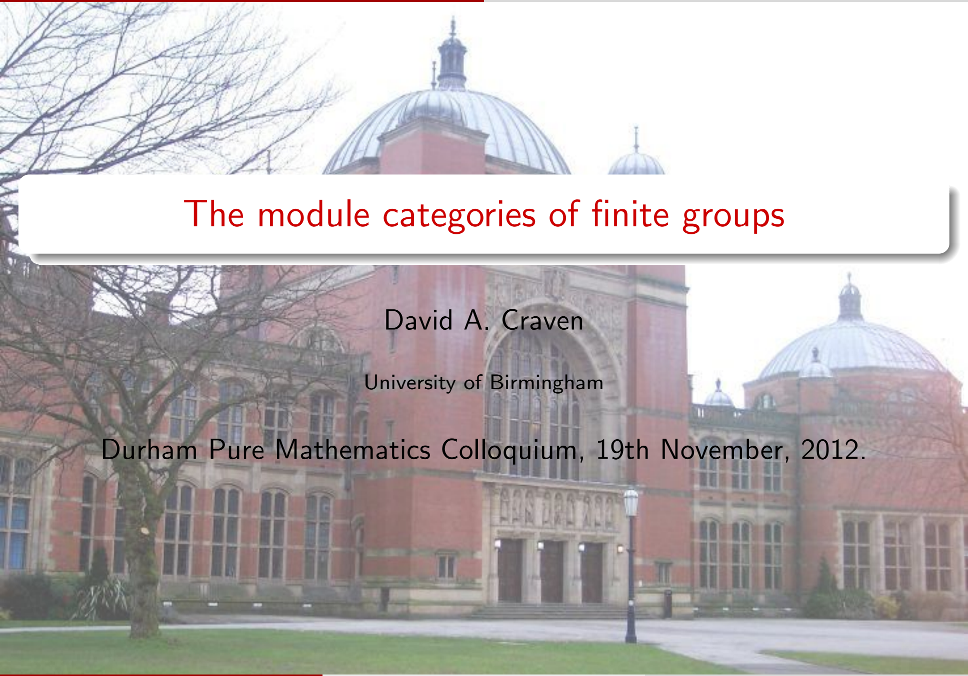

The module categories of finite groups

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Durham Pure Mathematics Colloquium, 19th November, 2012.

Notation and Conventions

Throughout this talk,

- G is a finite group,
- p is a prime,
- k of characteristic p, and
- P is a Sylow p-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

This talk is joint work with Olivier Dudas and Raphaël Rouquier.

Understanding the module category

We want to understand the finite-dimensional kG-modules for a given group G and field k. This is an abelian category (basically the morphism sets form abelian groups, we have a zero object, every injective map is the kernel of some morphism, and we have direct sums).

A very basic question, which might not have any reasonable answer in general, would be

> 'which categories can occur as module categories of finite groups?'

Some initial thoughts

- Since we have a zero object, we can tell which maps $A \to B$ are injective and surjective homomorphisms, and which are zero maps. We can also tell which modules are simple, and so can compute the invariant $l(G)$, the number of simple kG-modules. This is equal to the number of conjugacy classes of elements of G whose order is prime to p.
- The direct sum of two modules satisfies a universal property, and so if we pick two objects in the category we know which object is their direct sum, just from the category itself. Therefore we can throw away all direct sums and only study indecomposable modules.

Some initial thoughts

- If we throw away the decomposable objects and think of this category as a directed graph, the connected components of this each consist of a collection of (potentially infinitely many) indecomposable kG-modules. Each of these is called a block of the finite group.
- If the field over which the modules were taken is $\mathbb{C}$, then Maschke's theorem states that every $\mathbb{C}G$-module is a direct sum of simple modules, and so after throwing away the indecomposable modules, we are left with a directed graph consisting of a number of isolated points and no maps at all. This directed graph for an abelian group of order n has n vertices, and so we have completely described the module category in this case. This is why we said that k should have characteristic p...

The separation: finite and infinite

We now have seen examples where there are only finitely many indecomposable objects in the category.

Theorem

There are finitely many (finite-dimensional) indecomposable kG-modules if and only if the Sylow p-subgroup P of G is cyclic.

If there are only finitely many indecomposable modules, then it is conceivable that one could write down in some way all possible module categories when the Sylow p-subgroup is cyclic.

From categories to graphs

If M and N are indecomposable (particularly simple) kG-modules, then one can see whether there is a non-trivial extension between M and N from the module category, since we can find short exact sequences

$$0 \to N \to E \to M \to 0.$$

In particular, this allows us to tell whether $\mathrm{Ext}^1(M, N)$ is non-zero. By the theorem of Brauer all Ext^1-spaces between simple modules are either 0 or 1-dimensional. We therefore get sequences

$$M_1, M_2, \dots, M_r$$

such that $\mathrm{Ext}^1(M_i, M_{i+1})$ is non-zero. A theorem of Brauer states that a given simple module appears in at most two of these sequences.
This allows us to draw a planar-embedded graph: the edges of the graph are labelled by the simple kG-modules, and we order the edges around each vertex according to the Ext-sequences above. (Since each module lies on at most two of these, this makes sense.)

The Brauer tree

It turns out that the graph above is a forest, and specifying the Brauer trees of a group is equivalent to specifying its module category.

As an example, the module category of the cyclic group of order p is a line:

$$M_p \to M_{p-1} \to \cdots \to M_1 = k \to 0.$$

(Here we omit composite arrows.) The Brauer tree is a single edge.

If $G = C_p \rtimes C_d$ then there are d simple kG-modules, and the module category has d lines of length p emanating from the zero module. The Brauer tree is a star.

p-soluble groups completed

In fact, a general theorem now gives us the Brauer trees of all soluble groups: it states that the situation on the previous slide is typical.

Theorem

If G is a p-soluble group then the Brauer trees of G are all stars with the same number of edges.

This result suggests we focus on groups that are not soluble, the most common being simple groups.

Unfolding the tree

The star with d edges can be thought of as d copies of a single edge, all identified at a single vertex. Such a process is called an unfolding of the tree sonsisting of a single edge. Hence another way of writing the previous theorem is that the Brauer trees of p-soluble groups is that they are unfoldings of an edge. Another result that makes us focus on simple groups is the following, proved by Feit in 1984.

Theorem

The Brauer trees of any finite group are unfoldings of trees for quasisimple groups.

The classification of finite simple groups now means we might be able to actually answer this question: simply determine all Brauer trees for all finite simple groups, and then we are done!

Recall that the classification of finite simple groups states that every finite simple group is an alternating group, a group of Lie type (like $\mathrm{GL}_n(q)$) or one of twenty-six sporadic groups.

The alternating groups

It is easy to see, since all characters of S_n are real, that the Brauer trees of S_n are lines. It is also easy to show that, if χ lies in a p-block of cyclic defect, then χ restricts to an irreducible ordinary character of A_n, so the Brauer trees of A_n are also lines.

Much more recently, Jürgen Müller about 10 years ago computed the Brauer trees of the double cover of the alternating groups, and found that they were unfoldings of lines. Apart from the double covers of the alternating groups, there are exceptional triple covers for A_6 and A_7, and these can easily be determined.

So alternating groups are done!

The sporadic groups

For sporadic groups, the only real way to deal with them is direct computation, and so far this has been done for all but the Baby Monster and Monster.

One way to remove this obstacle is to assume that $p > 71$, in which case there is no sporadic group with a non-trivial Sylow p-subgroup. Eventually, we aim to get all of the Brauer trees for these groups.

Groups of Lie type

If G is a group of Lie type, say $G = G(q)$, then we could have that $p \mid q$, or that $p \nmid q$. If $p \mid q$ and G has cyclic Sylow p-subgroups, then $G = \mathrm{PSL}_2(q)$ and the Brauer tree is a line.

If G is classical (i.e., $\mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$, $\mathrm{P\Omega}_{2n+1}(q)$, $\mathrm{P\Omega}^{\pm}_{2n}(q)$, $\mathrm{PSU}_n(q)$), then the Brauer trees are lines.

So we are left with the case where G is an exceptional group of Lie type, i.e., G_2, F_4, E_6, E_7, E_8, ${}^2E_6$, ${}^2B_2$, ${}^2G_2$, ${}^2F_4$, ${}^3D_4$.

The order of G is

$$|G| = q^N \prod_{d \in I} \Phi_d(q)^{a_d}.$$

If $p \mid |G|$ then $p \mid \Phi_d(q)$ for some d. In light of the previous slide, let us simplify matters and assume that $p > 71$. This means that p divides exactly one $\Phi_d(q)$.

The Φ_d-cyclotomic theory

Broadly speaking, if $p \mid \Phi_d(q)$ and $p' \mid \Phi_d(q')$ then the representation theory of $G(q)$ and $G(q')$ at the primes p and p' respectively are 'the same'. The unipotent characters are parameterized independently of q, and whose distribution into the unipotent blocks is dependent only on d.

The Brauer tree of a unipotent block should be only dependent on d, and not on p and q, when $p \mid \Phi_d(q)$.

The principal block, containing the trivial module, is a unipotent block, so you may just think about the principal block if you want.

The representation theory of all blocks is in some sense related to unipotent blocks, although the precise mechanisms for this, and even what is precisely meant by this, remain obscure. Recently there has been much work in this direction, and we should soon understand this mechanism in much more detail.

The small exceptional groups

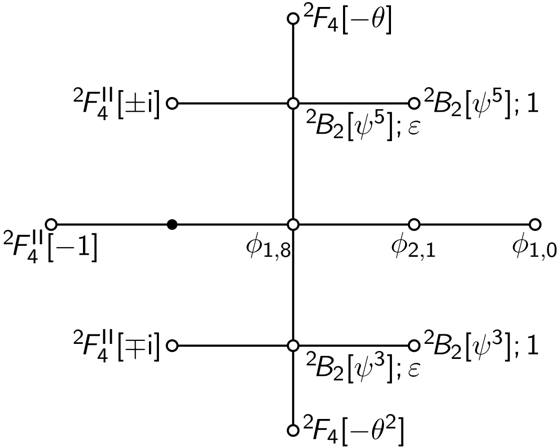
If G is one of $G_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, or ${}^2B_2(q)$, then all Brauer trees are known, by various papers which appeared mostly during the 1990s. Some other cases were explored in other papers:

- if $G = E_6(q)$, then as long as the d such that $p \mid \Phi_d(q)$ is at least 4, all blocks are known. For all primes at least 5, the Brauer trees of unipotent blocks are known. (Hiss–Lübeck–Malle)
- If $G = F_4(q)$ or $G = {}^2E_6(q)$ then the Brauer trees of unipotent blocks are known. (Hiss–Lübeck)

This leaves the unipotent blocks of the groups $E_7(q)$ and $E_8(q)$, along with the non-unipotent blocks of several types of groups.

An example

$G = {}^2F_4(q^2)$, $p \mid \Phi'_{24}(q)$. (By Φ'_{24} we mean the polynomial factor of Φ_{24} with ζ_{24} as a root.)



Deligne–Lusztig varieties enter

Recently, Deligne–Lusztig varieties have been found to actually be of practical, rather than just theoretical, help with solving problems like finding Brauer trees. The Deligne–Lusztig variety associated to the Coxeter torus (i.e., the largest d such that $\Phi_d(q)$ divides $|G(q)|$) has a particularly nice structure, and this is closely related to the relatively simple structure of the Brauer tree for these d.

Hiss, Lübeck and Malle gave a conjecture on the shape of the Brauer tree, based on the cohomology of this variety: the tree consists of lines emanating from the exceptional node, and each ray consists of characters with the same eigenvalue of Frobenius with the planar embedding in terms of increasing argument as a complex number. This is the HLM conjecture.

The HLM conjecture follows from the known cohomology of the Deligne–Lusztig variety, **if** it could be proved that, over a p-adic ring $\mathbb{Z}_p$, the cohomology is torsion-free. This is definitely not true for other d, but seemed to be true for d the Coxeter number.

The HLM conjecture

The previously unknown Brauer trees of unipotent blocks were for

- ${}^2G_2$, $d = 12''$
- F_4, $d = 12$
- ${}^2F_4$, $d = 24''$
- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 3$
- E_7, all d including $d = 18$
- E_8, all d including $d = 30$

(Here, red denotes a Coxeter case.)

Theorem (Dudas (2011))

The HLM conjecture is true for ${}^2G_2$, $d = 12''$ *and* F_4, $d = 12$.

Theorem (Dudas–Rouquier (2012))

The HLM conjecture is true.

Removing the lines

The previously unknown Brauer trees were for

- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 12$
- E_7, all $d \neq 18$
- E_8, all $d \neq 30$

Proposition (C. (2012))

Many of the trees for E_7 and E_8 are lines, or Morita equivalent to cases solved by Dudas and Dudas–Rouquier.

This leaves

- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 12$
- E_7, $d = 9, 10, 14$
- E_8, $d = 9, 12, 14, 15, 18, 20, 24$

The Coxeter variety for non-Coxeter primes

We can take the Deligne–Lusztig variety associated to the Coxeter torus T, and study it even when the prime p **does not** divide $|T|$. This gives us enough information that, with a few extra arguments, we get the following theorem.

Theorem (C.–Dudas–Rouquier (2012))

The Brauer trees of all unipotent blocks with cyclic defect group, for any group of Lie type, are known.

In three cases, ${}^2F_4(q)$, $d = 12'$, $E_8(q)$ $d = 15$ and $d = 18$, we do not have the complete labelling of the vertices in the planar-embedded Brauer tree. In each case, there is a pair of cuspidal characters that cannot (yet) be distinguished. In the case of ${}^2F_4(q)$, the character labelling isn't actually well defined.

Another example

$G = E_8(q)$, $p \mid \Phi_{15}(q)$.