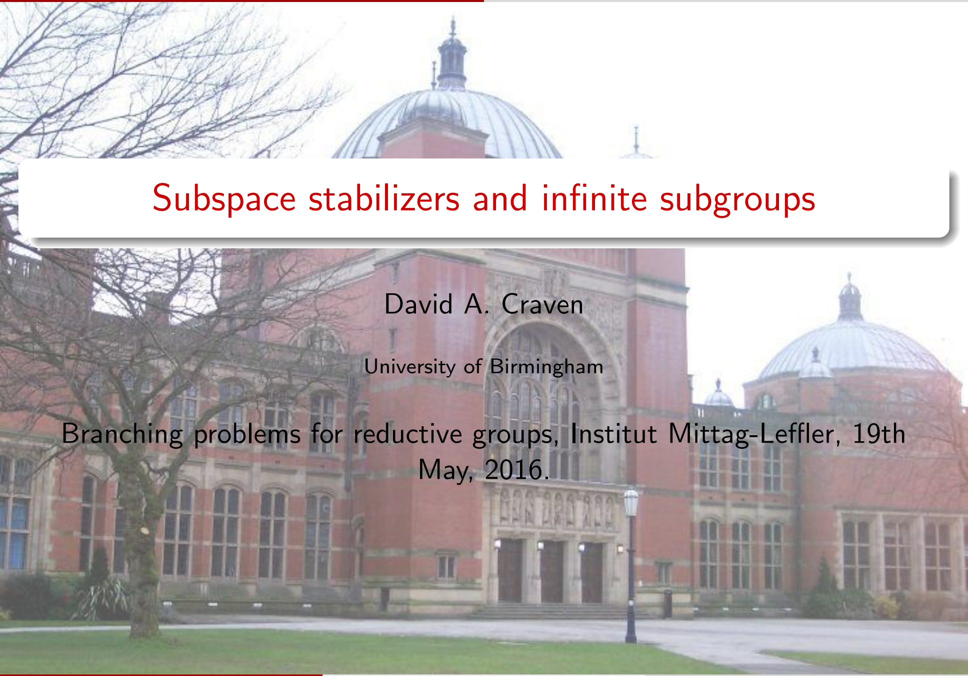

Subspace stabilizers and infinite subgroups

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Branching problems for reductive groups, Institut Mittag-Leffler, 19th May, 2016.

Maximal subgroups of finite groups

Let G be a finite group. Suppose that we want to classify maximal subgroups of G: Aschbacher and Scott, in 1985, proved that we can understand the maximal subgroups of every finite group if we can understand the maximal subgroups of $V \rtimes G$, where G is an almost simple group and V is an irreducible $\mathbb{F}_p G$-module. The maximal subgroups of this are either VM for M maximal in G, or a complement to V in G. We thus need to understand the following two situations:

1. maximal subgroups of almost simple groups;
2. $H^1(G, V)$ for G almost simple and V irreducible.

It turns out that (2) is much harder than we first thought. (Thanks, Frank.) Maybe (1) is something we can make progress on.

Maximal subgroups of finite simple groups

Let's go through the classification of the finite simple groups:

1. Sporadic groups: all known except the Monster, where there is a candidate list.
2. Alternating groups: here O'Nan–Scott deals with almost everything, modulo all maximal subgroups of all almost simple groups, but at least they have smaller order.
3. Classical groups: thanks Gerhard, Colva, Kay.
4. Exceptional groups: known except for almost simple subgroups, not necessarily acting irreducibly or even indecomposably.

$\mathrm{PSL}_2(q_0) < G(q)$

If M is an almost simple subgroup of an exceptional algebraic group G, and M is not a Lie type group in the same characteristic as M, then the fact that M has a 248-dimensional non-trivial representation means that there are decent bounds on $|M|$.

But if $\mathrm{char}(M) = \mathrm{char}(G)$ then this doesn't work. In particular, if $M = \mathrm{SL}_2(q_0)$ then q_0 can be arbitrarily large.

Liebeck–Seitz, $t(G)$, and a solution

In 1998, Liebeck and Seitz defined a quantity $t(G)$ and proved a theorem about it. The exact definition of $t(G)$ is not important. Here is the important bit:

G	G_2	F_4	E_6	E_7	E_8
$t(G)$	12	68	124	388	1312

Theorem

Let x be a semisimple element in G of order greater than $t(G)$. There exists an infinite subgroup X such that x and X stabilize the same subspaces of the adjoint module $L(G)$.

Consequently, any maximal subgroup $M = \mathrm{PSL}_2(q_0)$ of $G = G(q)$ is either known or $q_0 \leq \gcd(2, p-1) \cdot t(G)$.

Stabilizing subspaces

Let G be an algebraic group and let V be a module for G.

Let H be a subgroup of an algebraic group G. We define H to be fabulous for V if there exists an infinite subgroup X containing H such that X and H stabilize the same subspaces of V. (Notice that if $H \not\leq X$, replace X by $\langle X, H\rangle$.)

Say that x is fabulous if $\langle x\rangle$ is.

Thus Liebeck–Seitz says that any element of order greater than $t(G)$ is fabulous for $L(G)$.

Their proof actually shows that there are elements of order $t(G)$ that are not fabulous, so this bound is tight.

A fabulous foundation

We still let G be an algebraic group and V be a module for G. Let T be a maximal torus of G, and arrange things so that T acts diagonally on $V = \mathrm{span}\{e_1, \ldots, e_n\}$. Let x be an element of T.

The eigenspaces E_j of x are spanned by subsets of the e_i. There are only finitely many possibilities for the E_j, and so finitely many subgroups $U_1, \ldots, U_m$ that stabilize exactly the same subspaces as the various elements x of T. In particular, there are only finitely many U_j that are finite. Let $\mathcal{X}_V$ denote the set of element orders from these U_j.

If y stabilizes the same subspaces as x, then y acts diagonally so lies in T.

Proposition

Every element of order not in $\mathcal{X}_V$ is fabulous. If $n \in \mathcal{X}_V$ then there is an element of T of order n that is not fabulous.

Liebeck–Seitz proved that $\mathcal{X}_{L(G)} \subseteq \{1, \ldots, t(G)\}$ and $\max \mathcal{X}_{L(G)} = t(G)$. (In fact, they determine $\mathcal{X}_{L(G)}$ exactly in terms of the root lattice of G.)

G_2

Suppose that we want to compute $\mathcal{X}_{\mathrm{V}_{\min}}$ for $G = G_2$. The first thing we need is a representation of the torus. Easiest: take $A_2 \leq G_2$ and think of its maximal torus. If $x \in A_2$ has order n, and $\zeta^n = 1$, then x has eigenvalues

$$\zeta^a, \zeta^b, \zeta^{-a-b}, \zeta^{-a}, \zeta^{-b}, \zeta^{a+b}, 1,$$

on $\mathrm{V}_{\min}$, since A_2 acts as $L(01) \oplus L(10) \oplus L(00)$. We want to set various of these eigenvalues to be equal (i.e., the eigenspaces of x to not be 1-dimensional) and see what effect that has on ζ. For example, if $\zeta^a = \zeta^{-a}$ then $\zeta^a = \pm 1$. If the same is true for ζ^b then $o(x) = 1, 2$.

In other words, we need to take the abelian group

$$\langle a, b \mid r_1, r_2, \dots \rangle$$

where the r_i are of the form $a = -a$, $a = -a - b$, $a = 0$, and so on. If this abelian group is infinite then any element x that stabilizes exactly those eigenspaces is fabulous. If this is finite we compute its exponent.

Proof for G_2

By using the S_3 acting on the eigenvalues, we can cut down the possibilities. Suppose that 1 is equal to another eigenvalue, say ζ^a. The eigenvalue exponents are $0, b, -b, 0, -b, b, 0$, and we have an infinite abelian group still, so we need to set two more eigenvalues to be equal. Thus $b = 0$ ($o(x) = 1$) or $b = -b$ ($o(x) = 2$).

Thus we may assume that the 1-eigenspace is 1-dimensional, and ignore it from now on. Suppose that $\zeta^a = \zeta^b$, so that the eigenvalue exponents are $a, a, -2a, -a, -a, 2a$. Setting these equal to one another yields $\alpha a = 0$ for $\alpha = 1, 2, 3, 4$, so $o(x) = 1, 2, 3, 4$.

Hence we may assume that no two of $a, b, -a-b$ are equal. We still have to make (up to automorphism) a equal to something, so $a = -a$ or $a = -b$. The second cannot happen as then $a + b = 0$. In the first case we have $-1, \zeta^b, -\zeta^{-b}, -1, \zeta^b, -\zeta^b$, and so ζ^b is equal to one of these things. This gives $o(x) = 2, 4$, and so we get

$$\mathcal{X}_{V_{\min}} = \{1, 2, 3, 4\}, \qquad \mathcal{X}_{L(G)} = \{1, \dots, 9, 10, 12\}.$$

F_4

If we want to move past G_2, this hand calculation becomes far too tedious, and for F_4 there are thousands of possible outcomes, so we need to use a computer. However, the principles are identical.

Here there are two obvious options, using the type A theme above. (Of course, understanding the torus for type A is very easy.) We can choose the A_1^4 or the $A_2\tilde{A}_2$ subgroups.

Theorem

For $G = F_4$ and $V = \mathrm{V}_{\mathrm{min}}$, we have

$$\mathcal{X}_V = \{1, \ldots, 18\}.$$

E_6

If $G = E_6$ and x is real, then x lies inside F_4, and x is fabulous for the minimal module of E_6 if and only if it is for the minimal module of F_4, so whenever $o(x) \geq 19$.

If x is non-real then we have to worry about the centre, and our result is not as nice as for F_4.

Theorem

Let $G = E_6$, and let x be a semisimple element of order $n \geq 28$ such that $6 \nmid n$ and $\langle x \rangle \cap Z(G) = 1$. Then x is fabulous.

Why $6 \nmid n$? This is a combination of using the A_5A_1 and $A_2A_2A_2$ maximal-rank subgroups. The former group is really $(\mathrm{SL}_6 \circ \mathrm{SL}_2).2$, and so we cannot easily see the top 2 from the torus. The same happens with the second group and a top 3, so together they deal with elements not divisible by 6.

Stupid corollary

There exists a maximal subgroup $J_3 \leq E_6(4)$, and this is unique up to conjugacy in $\mathrm{Aut}(E_6(4))$.
(In fact,

Lemma

Inside $E_6(k)$ there are six conjugacy classes of maximal subgroup J_3, permuted transitively by the diagonal and graph automorphisms.

which is a strengthening of Kleidman–Wilson's original result, which is only over $\mathbb{F}_4$. In particular, J_3 is only maximal in $E_6(4)$.)

Corollary

The elements of order 19 in J_3 are non-real. Moreover, $J_3 : 2$ does not embed in $E_6(k)$.

E_7

This is actually quite hard. It took about a year of CPU time to get this result, but since it's parallelizable it was fine.

Take the A_7 maximal-rank subgroup of E_7, which looks like 4.PSL.2, so we can really only get information about odd-order elements.

Theorem

Let $G = E_7$, and let x be a semisimple element of odd order. If x has order greater than 75, then x is fabulous.

This is sharp, in the sense that there are non-fabulous elements of orders $1, \ldots, 75$.

Compare this bound with 388 obtained from $t(G)$.

Decent corollaries

These new bounds have reduced significantly the possibilities for a previously unknown $\mathrm{PSL}_2(q_0)$ maximal subgroup. Indeed, the previous bounds are that $q_0 \leq \gcd(2, q_0 - 1) \cdot t(G)$, and we now replace $t(G)$ by the bounds above.

Also,

Corollary

Suppose that $G = G(q)$ is an exceptional group F_4, E_6, ${}^2E_6$ or E_7, and has a Ree or Suzuki group $H(q_0)$ as a maximal subgroup. Then

1. $q_0 = 8$, *or possibly* $q_0 = 32$ *for* $G = E_7$, *or*
2. $q_0 = 3$, *or possibly* $q_0 = 27$ *for* $G = E_7$.

With more work, in fact one can eliminate these entirely from being maximal subgroups.