The Major Problems in Group Representation Theory

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In group representation theory, there are many unsolved conjectures, most of which try to understand the involved relationship between a finite group G and the normalizers of p-subgroups $\mathrm{N}_G(Q)$, where Q is often the Sylow p-subgroup, but will frequently be smaller than the Sylow.

Alperin's fusion theorem proves that the conjugacy of elements in a given Sylow p-subgroup is governed by the normalizers of non-trivial p-subgroups. In the abelian case, an old theorem of Burnside proves that $\mathrm{N}_G(P)$ controls fusion in P, where $P \in \mathrm{Syl}_p(G)$, so we should expect the structure there to be fairly transparent compared to other groups.

In this lecture we will see a variety of conjectures linking the representation theories of finite groups and normalizers of p-subgroups. The first of these were numerical, linking the (complex) character degrees of G with $\mathrm{N}_G(P)$, and then became more structural. The most structural of them all – Broué's conjecture – details the precise nature of the control of the representation theory of G be $\mathrm{N}_G(P)$ in the case where P is abelian. A common generalization of Alperin's weight conjecture and Broué's conjecture would be a very interesting development.

By K we denote a field, and G is a finite group. All KG-modules are finite-dimensional.

1 Preliminaries on Representation Theory

1.1 Blocks

As is well known, in an algebraically closed field K of characteristic 0, and p where $p \nmid |G|$, the representation theory of G over K satisfies the following two facts:

(i) The number of simple KG-modules is equal to the number of conjugacy classes of G; and

(ii) Every KG-module is the direct sum of simple modules.

The second one can be thought of as saying that the group algebra KG, thought of as a free module, can be decomposed into simple modules, or that the group algebra KG, thought of as an algebra, can be written as the direct sum of (simple) matrix algebras.

Now let p be a prime dividing $|G|$. Things are different:

(i) The number of simple KG-modules is equal to the number of conjugacy classes of G, whose elements have order prime to p; and

(ii) Every KG-module is **NOT** the direct sum of simple modules.

This means that the group algebra is not decomposable as the sum of matrix algebras. We can write KG as a direct sum of 2-sided ideals, each of which cannot be written as the sum of two 2-sided ideals. These indecomposable ideals are called *blocks*. Since we can write

$$KG = B_1 \oplus B_2 \oplus \cdots \oplus B_r,$$

we get a decomposition

$$1 = e_1 + e_2 + \cdots + e_r,$$

with each $e_i \in B_i$, e_i a central idempotent that is primitive (i.e., $e_i = e + e'$ with e, e' central idempotents mean e or e' is 0) and $e_i e_j = 0$ if $i \neq j$.

If M is a KG-module, then $M = M \cdot e_1 \oplus M \cdot e_2 \oplus \cdots \oplus M \cdot e_r$ is a direct sum decomposition of M into summands. Thus if M is any indecomposable module then M can be thought of as belonging to a block. Notice that if M is an indecomposable module belonging to the block B (with idempotent e) then $Me = M$, so that if N is a submodule or quotient of M then $Ne = N$ as well. Thus if any modules belong to a block then at least one simple module does. Since B itself, viewed as a module, belongs to B, we see that every block contains some simple modules, and (usually) some non-simple indecomposable modules.

The *principal block* is the block containing the trivial module.

1.2 Relating Characteristics 0 and p

Defining the character of an irreducible representation of a finite group in characteristic 0 is simple as the trace of a matrix. For irreducible representations in characteristic p, however, this doesn't work, since then the character of a p-dimensional representation would take 0 on the trivial element of the group, which would be silly.

Let K be a field of characteristic p and fix a primitive root of unity ζ. Identify ζ with a corresponding root of unity in $\mathbb{C}$, which will be how we make a character in $\mathbb{C}$ from a representation over K. If A is a matrix of order n then its eigenvalues are nth roots of unity, and so we can write them as powers of ζ, and then send them to $\mathbb{C}$. Summing them up, we get the trace of the matrix if $K = \mathbb{C}$, and generally we get a *modular character* or *Brauer character*.

However, since ζ has order prime to p (in fact, of order $p^n - 1$ for some n if K is finite) then this can only work if the matrix A has order prime to p. Thus modular characters are only defined on p'-elements, which are called *p-regular*. As we remarked before, there are the same number of simple KG-modules as there are conjugacy classes of p-regular elements, and in fact the modular characters form a basis for all such class functions.

If χ is a complex character, write $\chi = \sum_\phi a_\phi \phi$, where the sum is over all modular characters. It can be shown that the subset of the modular characters such that $a_\phi \neq 0$ comes from exactly one block, and so we say that that complex character also belongs to the block.

What we have done therefore is partitioned all indecomposable KG-modules, all irreducible modular characters and all irreducible complex characters into blocks.

We give an example: let G be the group A_5, and let $p = 5$. Over $\mathbb{F}_4$ there are four simple modules, of dimensions 1, 2, 2, and 4. (Over $\mathbb{F}_2$, we cannot get the 2-dimensionals, and instead get another 4-dimensional.) If x denotes an element of order 5 in G, then the characteristic polynomial of the 4-dim representation is $\Phi_5(x) = x^4 + x^3 + x^2 + x + 1$, and the sum of the roots of this is -1. Going to the ATLAS of Brauer Characters, we indeed see that the value of this 4-dimensional representation on a 5-element is -1. (Note that to find all roots to this polynomial, we needed to expand the field to $\mathbb{F}_{16}$, even though the representation itself is definable over $\mathbb{F}_2$.

1.3 Defect Groups

The defect group of a block B is a p-subgroup that controls the structure of B in a very strict sense. If M is a module for G, then we can always see M inside $M \downarrow_H \uparrow^G$ for any subgroup H. What is more is that we might want to find M as a *summand* of $M \downarrow_H \uparrow^G$. This cannot be true for any H, since for example if this is true for $H = 1$ then M is a summand of a free module. (Such modules are called *projective*, and there is one of these for each simple module.)

Lemma 1.1 If M in an indecomposable module and P is a Sylow p-subgroup of G, then $M \mid M \downarrow_P \uparrow^G$.

Define a module to be *relatively H-projective* if M is a summand of $M \downarrow_H \uparrow^G$. The lemma above says that every module is relatively P-projective. A *vertex* of an indecomposable module M is a p-subgroup D such that M is relatively D-projective but not relatively Q-projective for any $Q < D$. These exist by the previous lemma; what is not obvious – it is a theorem of Green – is that vertices are unique up to G-conjugacy.

Now consider all of the indecomposable modules lying in the block B; each of these has a vertex, and this collection of p-subgroups has a unique largest member under inclusion, which we call a *defect group* of B (defined up to conjugacy in G). Since the trivial module has vertex a Sylow p-subgroup of G, the principal block has defect groups the Sylow p-subgroups of G.

Finally, the *Brauer correspondent* of a block B is a unique specified block b of $\mathrm{N}_G(D)$, where D is a defect group of B (and b). It can be determined by choosing an indecomposable module of B with vertex D, and restricting it to $\mathrm{N}_G(D)$: this restriction will decompose into a collection of summands, and exactly one of those will have vertex D, the rest having vertex of strictly smaller order. The block to which this module belongs is the Brauer correspondent. (This correspondence is well defined, by the Burry–Carlson–Puig theorem.) If B is the principal block of KG then its Brauer correspondent is the principal block of $K\,\mathrm{N}_G(P)$, where P is a Sylow p-subgroup of G (Brauer's third main theorem).

2 The Conjectures

There are many conjectures in representation theory, but we will focus on three. The first, the McKay conjecture (and its variants) relate the degrees of irreducible complex characters of a block with the irreducible characters of its Brauer correspondent.

Having considered the complex characters, we now want to shift to the modular characters in the block. Alperin's weight conjecture related the number of simple B-modules with the structure of the Brauer correspondent again.

The third conjecture attempts to offer a structural understanding of Alperin's weight conjecture, at least in the case where the defect group is abelian. Broué's conjecture is the claim that the derived categories of a block and its Brauer correspondent are equivalence, if the defect group is abelian.

3 The McKay Conjecture

In 1971, McKay observed the following fact about several simple groups.

Conjecture 3.1 (McKay, 1971) Let G be a finite simple group, and let P be a Sylow 2-subgroup of G. The number of characters of odd degree for G is equal to that for $\mathrm{N}_G(P)$.

In 1973, a very similar-looking result was proved by Isaacs.

Theorem 3.2 (Isaacs, 1973) Let G be a group of odd order, and let P be a Sylow p-subgroup of G. The number of characters of G of order not divisible by p is equal to that of $\mathrm{N}_G(P)$.

The McKay conjecture doesn't have a reduction to simple groups, at least for now. *However*, there is a more complicated statement which, if it is true for simple groups, would imply the McKay conjecture for all finite groups. We **approximately** say that a group G, that is maximal quasisimple with $p \nmid |\mathrm{Z}(G)|$, is *good* for the prime p if we have the following situation:

(i) If $P \in \mathrm{Syl}_p(G)$, then there is a subgroup $\mathrm{N}_G(P) \leqslant N < G$ such that $\mathrm{Aut}(G) = \langle \mathrm{N}_{\mathrm{Aut}(G)}(N), \mathrm{Inn}(G) \rangle$;

(ii) There is an $\mathrm{N}_{\mathrm{Aut}(G)}(P)$-equivariant bijection f between the p'-characters of N and those of G; and

(iii) Let χ be a p'-character of G. Then there is a group H between G and $\mathrm{Aut}(G)$ stabilizing χ and with $\mathrm{C}_H(G)$ is abelian. If characters χ and $f(\chi)$ have extensions $\bar{\chi}$ and $\overline{f(\chi)}$ to include $\mathrm{C}_H(G)$, which cover the same characters of $\mathrm{C}_H(G)$. Both $\bar{\chi}$ and $\overline{f(\chi)}$ are invariant in H and $\mathrm{N}_H(N)$ respectively, and the associated elements in the Schur multipliers of $H/\langle \mathrm{C}_H(G), G\rangle$ and $\mathrm{N}_H(N)/\langle \mathrm{C}_H(G), N\rangle$ are 'equal'.

This final criterion is really technical and quite difficult to prove, and this is why the following theorem is not as exhaustive as it could be.

Theorem 3.3 The following groups are good:

(i) all alternating groups and all sporadic groups for all primes (Malle);

(ii) ${}^2B_2(q)$ and ${}^2G_2(q)$ for all primes and all q (Isaacs–Malle–Navarro);

(iii) $\mathrm{PSL}_2(q)$ for all primes;

(iv) the Tits group for all primes;

(v) (almost) the exceptional groups for all non-defining primes (Malle–Späth). (First two conditions done.)

Years to Solution? Less than ten.

4 Alperin's Weight Conjecture

The McKay conjecture, and its refinements, talk about the number of p'-characters in a block of a finite group as being related to the number of characters of p'-degree in the normalizer of the defect group.

Having described the number of ordinary (i.e., complex) characters in a p-block, we move to the characters over $K = \bar{\mathbb{F}}_p$. Write $k(G)$ (or $k(B)$) for the number of complex characters for G (or for the block B), and $\ell(G)$ (or $\ell(B)$) for the number of simple modules over a field of prime characteristic (or the number of simple B-modules). Finally, denote (non-standardly) the number of *projective* simple modules by $\ell_P(G)$; remember that this is the number of blocks of defect 0.

Conjecture 4.1 (Alperin) Let G be a finite group and let K be a field of characteristic p. We have

$$\ell(G) = \sum_Q \ell_P(\mathrm{N}_G(Q)/Q),$$

where the sum is taken over representatives of all conjugacy classes of p-subgroups Q (including 1).

This conjecture has been checked for numerous simple and close-to-simple groups, but this isn't very helpful in this case, as there is no reduction of this conjecture to groups associated with simple groups.

Well, this might not be true. In the 1990s, Dade announced a series of ever-increasingly complicated conjectures, refinements of both the McKay conjecture and Alperin's weight conjecture, called things like Dade's Projective Conjecture, and Dade's Invariant Conjecture. The last of these, the Inductive Conjecture, is supposed to be inductive, in the sense that it reduces the problem to simple groups (with stuff top and bottom). However, no proof of this has ever been circulated, and it is at least believed not to be the case.

Another approach, using the Grothendieck groups of fusion systems, is being pioneered by Puig, but unfortunately nobody can understand his stuff yet. If the community at large can get to grips with this, maybe this will lead to a solution, but I wouldn't hold my breath.

Years to Solution? At least twenty, unless the Puig approach succeeds.

5 Broué's Abelian Defect Group Conjecture

If the normalizer of the Sylow p-subgroup controls fusion then Alperin's weight conjecture for the principal block states that the number of simple B_0-modules is equal to that of the normalizer of the Sylow. More generally, if D is an abelian p-subgroup, and B is a block with defect group D, then the number of simple B-modules is equal to that of its Brauer correspondent.

Broué's conjecture attempts to give a structural explanation for this fact. We will not define exactly what a derived category is here, but it is roughly like taking all complexes of modules in B and formally inverting morphisms that induce isomorphisms in (co)homology. (It needs to be checked that this process makes sense, the same as the process of taking a field of quotients for an integral domain.)

Broué's conjecture asserts the following.

Conjecture 5.1 Let G be a finite group, and let B be a p-block with defect group D. If b denotes the Brauer correspondent of B in $\mathrm{N}_G(D)$, then the derived categories of B and b are equivalent.

Some motivation for this conjecture comes from an alternative definition of a defect group, which is sometimes used in this approach to representation theory.

Much more than this is suspected, however. There are many different ideas about how to build up the complexes, and with what extra structure these derived equivalences should be endowed. In the case of principal blocks, we again have a reduction to finite simple groups, and here a surprising amount is known.

Theorem 5.2 Let G be a finite simple group with an abelian Sylow p-subgroup.

(i) If $p = 2$ then Broué's conjecture is true for B_0.

(ii) If $p = 3$ and G is not the O'Nan group, then Broué's conjecture is true for B_0.

(iii) If $p = 5$ and G is an alternating group, $\mathrm{PSL}_n(q)$ for some n and q, $G_2(q)$, and a few other groups, then Broué's conjecture is true for B_0.

(iv) If G is an alternating group or $\mathrm{PSL}_n(q)$ for some n and q, the Broué's conjecture is true for all blocks of G.

We will now focus on the case where the block is the principal block.

One of the aspects of Broué's conjecture that is not done so well in the literature is that each of the derived equivalences here should be the composition of several *perverse* equivalences. We will define a perverse equivalence now: recall that if $\mathcal{C}$ is a category, then a *full* subcategory is is subcategory $\mathcal{D}$ such that if A and B are objects in $\mathcal{D}$ then $\mathrm{Hom}_{\mathcal{D}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$. A *Serre* subcategory is a full subcategory that is closed under extensions, subobjects and quotients; i.e., if $0 \to A \to B \to C \to 0$ is a short exact sequence, then B belongs to the category if and only if both A and C do.

Definition 5.3 Let A and A' be K-algebras, with simple modules $\mathcal{S}$ and $\mathcal{S}'$. Let $F : D^b(A) \to D^b(A')$ be an equivalence of categories. We say that F is *perverse* if there exist labellings $\mathcal{S} \to \{1, \dots, n\}$ and $\mathcal{S}' \to \{1, \dots, n\}$, and a function $p : \{1, \dots, n\} \to \mathbb{Z}$, such that

(i) F induces an equivalence of categories $D_i^b(A) \to D_i^b(A')$, where $D_i^b(A)$ is the subcategory of $D^b(A)$ of complexes of modules whose support is in the Serre subcategory of A generated by $S_1, \dots, S_i$; and

(ii) $F[p(i)]$ induces an equivalence $A_i/A_{i-1} \to A'_i/A'_{i-1}$.

The idea here is that we assign each simple module in mod A to a simple module in mod A', and $p(i)$ tells you the number of terms in the complex in $D^b(A')$ associated with S_i.

In practice, there is an algorithm that can be guaranteed to construct a perverse equivalence between $D^b(b_0)$ – b_0 is the principal block of $\mathrm{N}_G(P)$ – and *some* algebra, and by choosing the function p carefully (or just by trying lots) we hope that we can get the Green correspondents of the S_i at the end of the complex.

We give an example with $G = M_{11}$. This has seven simple modules in B_0, of dimensions 1, 5, 5, 10, 10, 10, and 24. The function p takes values $0, 2, 3, 4, 5, 6, 7$ on the seven modules, and the complexes can be explicitly described. However, in order to describe the perverse equivalence it suffices to tell you the function p and the algorithm that we use to calculate the complexes, which is so explicit that I have implemented it on a computer.

Years to Solution? Between ten and twenty, for principal blocks.