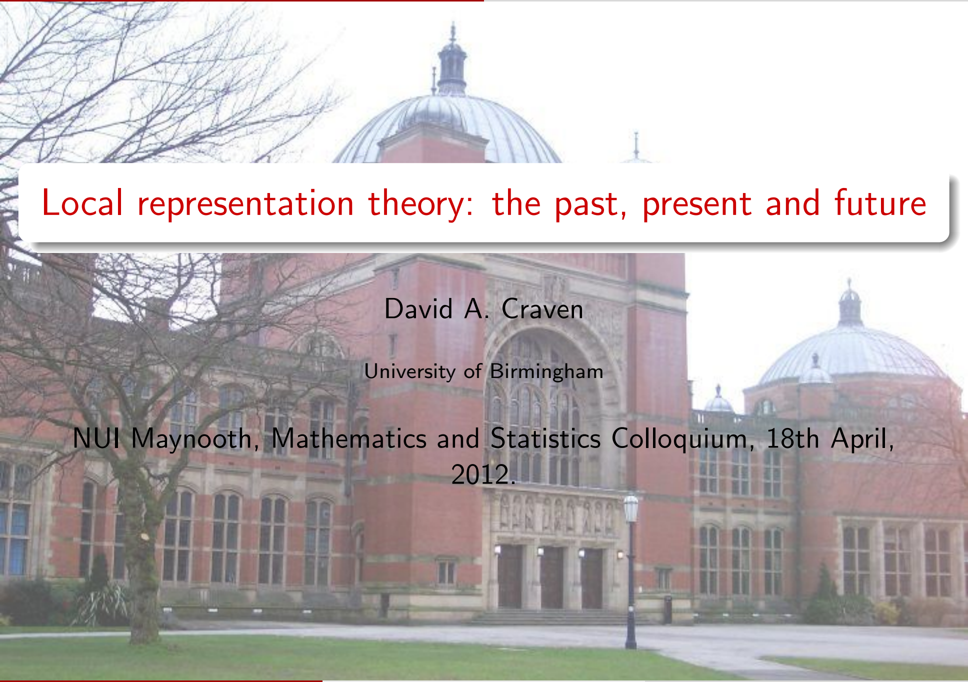

Local representation theory: the past, present and future

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Decomposing the Group Algebra

If K is a field of characteristic 0, then Maschke's theorem states that the group algebra KG is a semisimple ring. In fact, it is only necessary that $|G|$ is invertible in K, so we get two cases:

1. $\mathrm{char}(K) = p$ does not divide $|G|$
2. $\mathrm{char}(K) = p$ divides $|G|$

The first case behaves as $K = \mathbb{C}$ does. The second is much more difficult.

The ring is no longer semisimple, but write it as a sum of ideals, as fine a decomposition as possible.

$$KG = B_1 \oplus B_2 \oplus \cdots \oplus B_r.$$

The B_i are called blocks of KG. A large part of representation theory involves studying these blocks.

Blocks are locally controlled

Since KG is a sum of ideals, 1 can be written as a sum of elements of these ideals: $1 = e_1 + e_2 + \cdots + e_r$. The e_i are central idempotents of KG.

Let H be a subgroup of G. The projection map $\mathrm{Br}_H(-) : Z(KG) \to Z(K\mathrm{C}_G(H))$ has the following property.

Theorem

If H is a p-subgroup of G and $\mathrm{char}(K) = p$, then Br_H is a surjective ring homomorphism.

The image of any e_i under Br_P is either a central idempotent or zero. A defect group for B_i is a maximal p-subgroup D with $\mathrm{Br}_D(e_i) \neq 0$.

Theorem (Brauer)

The map Br_D induces a bijection between blocks of KG with defect group D and blocks of $K\mathrm{N}_G(D)$ with defect group D.

The Brauer Correspondent

To every block B of KG there exists a p-subgroup D, the defect group, and a block b of $K\mathrm{N}_G(D)$, the Brauer correspondent. How are these two blocks related?

This question led to some difficult and delicate character-theoretic statements that were important in the classification of the finite simple groups.

Since 1970, a series of conjectures has emerged that attempt to pin down the precise relationship between these two objects. These often involve the characters of a group. One can associate each irreducible character of G to a block of KG; write $k(B)$ for the number of these. If χ belongs to B with defect group D, then $|P : D|$ divides $\chi(1)$. The height of χ is the quantity

$$\log_p |\chi(1)|_p / |P : D|_p.$$

Local-Global Conjectures

The height of χ is the quantity $\log_p |\chi(1)|_p / |P : D|_p$.

Conjecture (McKay conjecture)

If P is a Sylow p-subgroup of G, then the number of irreducible characters of p'-degree of G and $\mathrm{N}_G(P)$ are the same.

Conjecture (Alperin–McKay conjecture)

The number of height-zero characters of a block and its Brauer correspondent are equal.

We can also control the defect group somewhat using heights.

Conjecture (Brauer's height-zero conjecture)

All characters of a block B have height zero if and only if its defect group is abelian.

Combining these two conjectures, we get that $k(B) = k(b)$ whenever the defect group is abelian. What is going on here?

Broué's Conjecture

Broué's abelian defect group conjecture explains this numerical coincidence.

Conjecture

If B is a block with abelian defect group then B and its Brauer correspondent b are derived equivalent.

At the moment there is no corresponding statement, even conjecturally, in the non-abelian case, although this is a subject of some speculation at the moment; a 'global' local-global conjecture would unify all of the current conjectures in this area.

Reducing to the Finite Simple Groups

In general, Broué's conjecture has not (yet!) been reduced to some statement about simple groups. If B is the principal block – the block to which the trivial character belongs – then there is a reduction, however.

Theorem

Let G be a finite group. If P is abelian, then there are normal subgroups $H \leq L$ of G such that

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- L/H *is a direct product of simple groups and an abelian* ℓ*-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L (compatible with automorphisms of the simple components) passes up to G. Thus if Broué's conjecture for principal blocks holds for all simple groups (with automorphisms), it holds for all groups.

What are the Finite Simple Groups?

1. Alternating groups (Broué's conjecture known by Chuang–Kessar–Marcus–Rouquier)
2. Sporadic groups (Broué's conjecture known whenever $p > 11$, and for a few other cases)
3. Groups of Lie type, e.g., $\mathrm{GL}_n(q)$, where $p \mid q$ (Broué's conjecture known by Okuyama)
4. Groups of Lie type where $p \nmid q$

This last collection of simple groups is therefore the case that needs solving. Luckily, here we have a specific form of the conjecture, coming from the geometry of Deligne–Lusztig varieties.

Using and Losing the Geometry

The Deligne–Lusztig varieties Y_ζ have actions of the group of Lie type G on the one side, and the Sylow p-subgroup P on the other. The complex of cohomology of this variety should induce a derived equivalence between the two objects.

The main problem is that this variety is very difficult to understand, and so this approach, a priori, appears to make things no simpler.

Recently though, a new approach has been suggested by Raphaël Rouquier and me: construct the cohomology of the variety (conjecturally) without looking at the variety. On then proves that this cohomology induces a derived equivalence combinatorially.

The benefit of this is that one gets a candidate for a derived equivalence, rather than merely asserting that one exists.

Does This Work?

An obvious question is: have any new derived equivalences been found, and any new cases of Broué's conjecture been proved?

The short answer is 'yes'. Groups such as $D_4(q)$, ${}^3D_4(q)$, $\mathrm{Sp}_8(q)$, which were too big for previous methods have fallen to this new idea.

A longer answer is: The groups $\mathrm{GL}_3(q)$ (for example) have order $q^3(q-1)^2(q+1)$. If $p \nmid q$ and $p > 2$ then p can divide only one of $q+1$ and $q-1$; the representation theory of $\mathrm{GL}_3(q)$ in characteristic p should only depend (in some broad sense) on which of the cyclotomic polynomials that p divides. The perverse equivalences suggested by Deligne–Lusztig varieties have this property that they do not depend on p, and only on '$p \mid (q+1)$', whereas previous methods were generally for a fixed prime (normally 3 or 5).

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras, $\mathcal{A} = mod\text{-}A$, $\mathcal{B} = mod\text{-}B$.

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, $T_1, T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, for all i, the cohomology of $F(S_i)$ only involves one copy of T_i in degree $-\pi(i)$, and T_j can only appear in degrees less than $-\pi(j)$ (and between $-\pi(i)$ and 0).

How Can a Perverse Equivalence Help?

It turns out that there is a **unique** perverse equivalence with a given bijection between the simple modules and perversity function.

In fact, given any algebra A' and a function $\pi(-)$ on the simple A'-modules, there is an algorithm to construct the algebra A such that there is a perverse equivalence $A \to A'$ with these properties. We then must check that the algebra A is the block B of the group G, where A' is the algebra b of the Brauer correspondent. This is much easier than constructing a derived equivalence.

Completing the Description

In fact, in recent work, I have identified (conjecturally, but since the whole thing is a conjecture this isn't a problem) the perversity function needed for groups of Lie type. I have also identified (again, conjecturally) the bijection needed, and so everything is now understood. All we have to do is prove that the resulting map is a derived equivalence.

I have completed this project whenever the defect group D is cyclic. Although Broué's conjecture was previously known, it was the 'wrong' derived equivalence, in the sense that it is not compatiable with larger-rank groups.

Although a general method to do this is still absent, this seems to be a reasonable goal, and as a consequence we get almost every conjecture available for these blocks.

Beyond Abelian

Broué's conjecture only deals with the case where D is abelian. If $\mathrm{N}_G(P)$ controls fusion in P with respect to G then $\ell(B)=\ell(b)$ again, but we don't always have a derived equivalence (e.g., the principal block of $\mathrm{Sz}(8)$). Something more complicated must happen, even in this case.

We can use Alperin's weight conjecture to try to guide us. Another way of writing Alperin's weight conjecture is

$$|\mathrm{Irr}_{\text{non-proj}}(B)| = \sum_{\sigma\in\mathcal{R}/G} (-1)^{|\sigma|+1} w(B_\sigma),$$

where

- $\mathcal{R}$ is the poset of radical chains, not starting at 1.
- B_σ is the corresponding block of $\mathrm{N}_G(\sigma)$
- $w(B_\sigma)$ is the number of non-projective irreducible B_σ-characters that are V_σ-projective
- V_σ is the minimal element of σ.

Algebraic Topology, Enter Stage Right

Now we compare this to results from homotopy theory, factorizing the classifying space BG, at least up to p-completion, over the normalizers of various collections of p-subgroups. The following theorem is only approximately true.

Theorem (Normalizer decomposition)

Let $\mathcal{A}_p(G)$ denote the poset of all p-subgroups of G or any ample collection, and let $\mathcal{N}$ denote the set of all chains in $\mathcal{A}_p(G)$, up to G-conjugation. We have that the map

$$\underrightarrow{\operatorname{Hocolim}}_{\sigma \in \mathcal{N}} B[N_G(\sigma)] \to BG$$

is a mod-p cohomology equivalence.

The Future

Ordinary cohomology passes through a homotopy colimit (via Bousfield–Kan spectral sequences) to create an alternating sum, as with Alperin's weight conjecture. However, Hochschild cohomology $HH^1(-)$, which can count the number of simple modules, is not (in general) functorial with respect to homotopy colimits, so a naïve attack of Alperin's weight conjecture won't work.

Question

What is the analogue of the derived category that will extend Broué's conjecture to the non-abelian case?

Our guide to this problem should be the theory of homotopy colimits over fusion systems, associating a space at each point of the orbit category of the centric radical subgroups. This is, in a very real sense, the major problem in group representation theory.