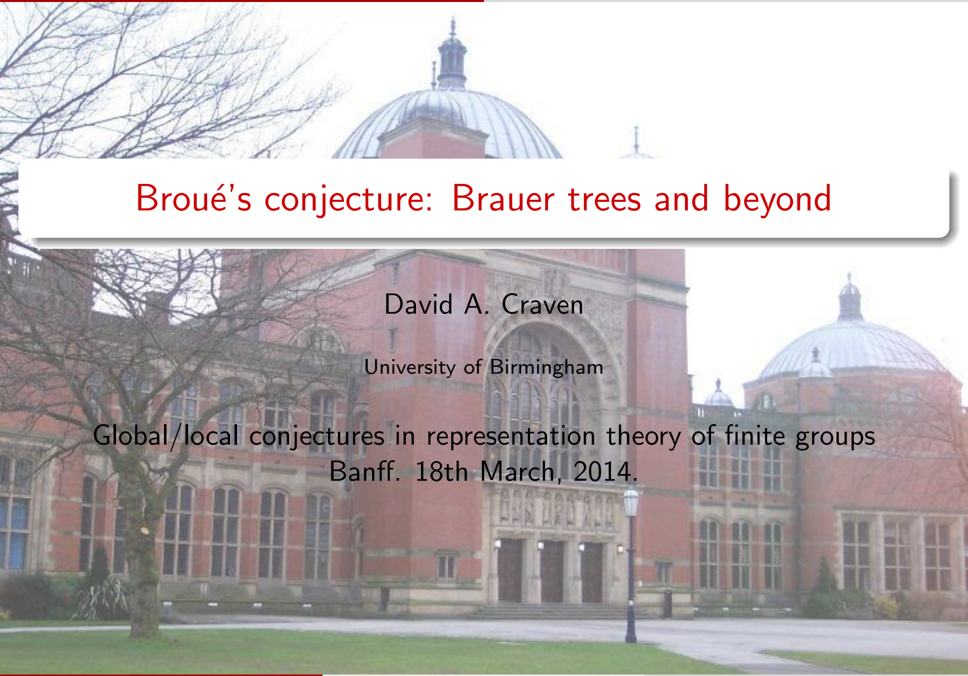

Broué's conjecture: Brauer trees and beyond

David A. Craven

University of Birmingham

Global/local conjectures in representation theory of finite groups
Banff. 18th March, 2014.

Notation and Conventions

Throughout this talk,

- G is a finite group,
- ℓ is a prime,
- k is a field of characteristic ℓ,
- B is a block of kG, with defect group D and Brauer correspondent b;
- P is a Sylow ℓ-subgroup of G,
- Q is a general ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored. Green is chosen so you can't read it, and if I mess something up I can just say it's in some green text that you didn't see.

This talk concerns joint work with Olivier Dudas (for Brauer trees) and Raphaël Rouquier (for the trees and generic behaviour).

A reminder

Broué's conjecture states that if the block B has an abelian defect group, then it is derived equivalent to the Brauer correspondent b.

In general we have no real plan to solve this except by going through the classification of finite simple groups. Even in this case there is not a full reduction of the conjecture to simple groups, but there is for principal blocks. (The block contributing the trivial module is called the principal block, and denoted by $B_0(kG)$. Its defect group is always the Sylow ℓ-subgroup P, and its Brauer correspondent is the principal block of $kN_G(P)$.)

Thus if we are considering principal blocks, we need to relate the principal block of kG with the principal block of $kN_G(P)$. Although we won't focus exclusively on principal blocks, even this case is hard enough, and all our results do apply to that case.

Principal Blocks Are Good

In representation theory, one standard method of proof is to reduce a conjecture to finite simple groups and then use the classification of the finite simple groups. In general, there is no (known) reduction of Broué's conjecture to simple groups, but for principal blocks there is.

Theorem

Let G be a finite group, and suppose that P is abelian. Then there are normal subgroups $H \leq L$ such that

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- L/H *is a direct product of simple groups and an abelian ℓ-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L compatible with automorphisms passes up to G. Thus if Broué's conjecture for principal blocks holds for all almost simple groups, it holds for all groups.

When Is Broué's Conjecture Known?

Broué's conjecture is known for quite a few groups:

- A_n, S_n (Chuang–Rouquier, Marcus);
- $\mathrm{GL}_n(q)$ (Chuang–Rouquier);
- D cyclic, $C_2 \times C_2$ (Rickard; Erdmann, Rouquier);
- G finite, $\ell = 2$, B principal;
- G finite, $\ell = 3$, $|P| = 9$, B principal (Koshitani, Kunugi, Miyachi, Okuyama, Waki);
- $\mathrm{SL}_2(q)$, $\ell \mid q$ (Chuang, Kessar, Okuyama)
- various low-rank Lie type groups $L(q)$ with $\ell \nmid q$.

How Do You Find Derived Equivalences?

There are four main methods to prove that B and b are derived equivalent.

1. Okuyama deformations: using many steps, deform the Green correspondents of the simple modules for B into those for b. This works well for small groups.
2. Rickard's Theorem: randomly find complexes in the derived category of b related to the Green correspondents of the simple modules for B, and if they 'look' like simple modules (i.e., Homs and Exts behave nicely) then there is a derived equivalence $B \to b$.
3. More structure: if B and b are more closely related (say Morita or Puig equivalent) then they are derived equivalent. More generally, find another block B' for some other group, an equivalence $B \to B'$, and a (previously known) equivalence $B' \to b$.
4. Perverse equivalence: build a derived equivalence up step by step in an algorithmic way.

Groups of Lie type

Let $G = G(q)$ be a group of Lie type, e.g., $\mathrm{GL}_n(q)$, $\mathrm{Sp}_{2n}(q)$, etc. The ordinary representation theory of G is in some sense generic in q. For example, the irreducible (complex) character degrees and their multiplicities are polynomials in q.

The order of G is

$$|G| = q^N \prod_{d \in I} \Phi_d(q)^{a_d}.$$

If $\ell \mid |G|$ then either $\ell \mid q$, which leads to one theory, or $\ell \nmid q$, in which case $\ell \mid \Phi_d(q)$ for some d. We are mostly interested in the case where there is no other d' such that $\ell \mid \Phi_{d'}(q)$; in this case, the Sylow ℓ-subgroup P is abelian, homocyclic, of rank a_d. In particular, if $a_d = 1$ then P is cyclic.

Unipotent characters

Obviously, if we fix $G(-)$ and vary q we get different numbers of irreducible characters. However, they split into two collections: unipotent and non-unipotent. (Very) roughly speaking, if a decomposition matrix of a block is lower triangular, then the unipotent characters are the ones in the top square of the matrix and the non-unipotent characters are the ones in the rest of the matrix, often repeating rows, like exceptional characters in blocks with cyclic defect groups.

Formally, a unipotent character of $G = \mathbf{G}^F$ is a constituent of the Deligne–Lusztig character $R_{\mathbf{T}}^{\mathbf{G}}(1)$. (This probably isn't much help if you didn't know what unipotent characters were in the first place.)

The number of unipotent characters does not depend on q (their degrees do), and they have a consistent parametrization. For example, the unipotent characters of $\mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$ are labelled by partitions of n.

Unipotent blocks

A unipotent block is a block of kG (or $\mathcal{O}G$) that has unipotent characters belonging to it. Since the unipotent characters are independent of q, it seems reasonable to ask that the unipotent blocks are independent of q. What can this mean?

We will always assume that ℓ divides exactly one $\Phi_d(q)$ from now on. Write $\overline{\ell} = |G(q)|_\ell$, the ℓ-part of $|G(q)|$. The distribution of unipotent characters into the unipotent blocks of kG do not depend on q **or** ℓ, as long as the d involved is the same.

Comparing primes

If q and q' are different, but ℓ stays the same (as does the power $\bar{\ell}$), we can ask whether the unipotent blocks of $G(q)$ and $G(q')$ are (for example) Morita equivalent. However, if the prime ℓ, or even just the prime power $\bar{\ell}$, differs for q and q', we will not get a Morita equivalence, and we must search for some other definition of 'independent', one that cannot be dependent on an equivalence of categories.

If $d \geq 1$ is an integer, then we are comparing blocks of $kG(q)$ and $k'G(q')$, where $\ell \mid \Phi_d(q)$ and $\ell' \mid \Phi_d(q')$; we say that e and e' are from the same Φ_d-block if the unipotent characters in e and e' have the same labels, so that a Φ_d-block is a set of unipotent blocks. The weight of a Φ_d-block is the rank of any defect group of a block from the Φ_d-block. Blocks with cyclic defect group have weight 1.

A guiding example: Brauer trees

The example we can use to guide our thinking is the Brauer tree. There, if there are e simple modules, the exceptionality ε satisfies $\varepsilon = (\bar{\ell} - 1)/e$, so if we fix the tree with exceptionality then we fix $\bar{\ell}$. However, it might make sense to fix the tree **without** exceptionality, and this allows us to compare primes.

Fix $e \geq 1$, let Λ be the set of all powers $\bar{\ell}$ of primes ℓ such that $e \mid (\ell - 1)$, and fix a tree with planar embedding T, with e edges and a fixed exceptional node. A generic block $\hat{B}$ is the set of all Brauer tree algebras with the tree T, and with exceptionality $(\bar{\ell} - 1)/e$ for $\bar{\ell} \in \Lambda$.

Two blocks with cyclic defect group are generically equivalent if they belong to the same generic block.

Generic equivalence for cyclic blocks

The following theorem summarizes the results over several decades and a dozen mathematicians.

Theorem

A Φ_d-block of weight 1 is a generic block. In each case, the planar-embedded Brauer tree is known.

So far, so good. However, there are plenty of blocks that do not have cyclic defect groups, and so we need to be able to deal with those as well.

In 1989, Rickard proved that any two Brauer tree algebras with the same number of edges and same exceptionalities are derived equivalent. In particular, he produced an algorithm to produce a derived equivalence from a given Brauer tree to the star with exceptional node in the middle, and this algorithm did not depend on the exceptionality.

Techniques to determine the trees

There are various arguments that are already known:

- Parity argument: the sum of adjacent ordinary characters is projective, and hence has degree divisible by ℓ. The irreducible characters have height 0 however, by standard theory. Hence they have degree $\pm a$ modulo ℓ for some a.
- Degree argument: the sum of the dimensions of the simple modules incident to a character is the degree of the character. This eliminates many potential trees.
- Real stem: the real characters in the block form a line, and complex conjugation is a reflection in this line.
- Tensor product: if we are in the principal block it is easy to compute $\Omega^n k$ from the tree, and since $\Omega^n k \otimes M \cong \Omega^n M$ plus projectives, we can exclude some structures this way.
- Induction: sends projectives to projectives.

The main new technique

These arguments aren't enough. The main new argument brought to the table here is the Deligne–Lusztig variety, whose cohomology is sometimes computable, particularly for the Coxeter torus.

What really works here is considering the variety associated to the Coxter torus, but where the prime ℓ **does not** divide the Coxeter polynomial. We can often prove that this variety has cohomology that is torsion free, and so we get some information about some higher extensions.

Since the previous techniques pin down the real stem, the location of the non-cuspidals, and give us only a few possibilities for the location of the cuspidals, an element of $\mathrm{Ext}^n(M, k)$ should give us enough information to find M.

And it does. Except for which of $E_8[\theta]$ and $E_8[\theta]^2$ are two nodes in two blocks, for Φ_{15} and Φ_{18}.

What is a perverse equivalence?

Let A and A' be finite-dimensional algebras.

An equivalence $F : D^b(mod\text{-}A) \to D^b(mod\text{-}A')$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, $T_1, T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}_{\geq 0}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \ldots, S_i$, and $D^b_i(\mathcal{A})$ denotes the subcategory of $D^b(\mathcal{A})$ with support consisting of modules in $\mathcal{A}_i$, then

- F induces equivalences $D^b(\mathcal{A}_i) \to D^b(\mathcal{A}'_i)$, and
- $F[\pi(i)]$ induces a Morita equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{A}'_i/\mathcal{A}'_{i-1}$.

Note that mod-A' is determined, up to Morita equivalence, by A, π, and the ordering of the S_i.

The geometric Broué conjecture

Broué's conjecture has a special version for unipotent blocks of groups of Lie type, called the geometric form.

Conjecture

Let $G = G(q)$ be a finite group of Lie type, and let D be an abelian defect group of a unipotent block B of G. We may embed D inside a Φ_d-torus T, and there is a Deligne–Lusztig variety Y, carrying an action of G on the one side and T on the other, whose complex of cohomology Γ has the following properties:

1. *the action of T can be extended to an action of $\mathrm{N}_G(T) = \mathrm{N}_G(D)$;*
2. *the complex induces a derived equivalence between B and its Brauer correspondent.*

The geometric Broué conjecture

In fact, if $\kappa \geq 1$ is prime to d, then there should be a Deligne–Lusztig variety $Y_{\kappa/d}$ associated naturally to κ, and whose complex of cohomology produces the desired equivalence.

While this is (a lot) more specific than the abstract version of Broué's conjecture, it still needs to be more specific, as the variety $Y_{\kappa/d}$ can be hideously complicated (and gets worse as κ grows).

This equivalence should be perverse. If the associated data can be extracted without analyzing the variety $Y_{\kappa/d}$, then the derived equivalence should be able to be constructed without the variety at all, purely combinatorially.

From geometry to combinatorics

1. The perversity function $\pi_{\kappa/d}$ is known, and is a dependent only on the cyclotomic polynomials dividing the generic degrees of the unipotent characters in the block, so independent of q and ℓ, dependent only on d.
2. The ordering of the simple modules on the block and its Brauer correspondent is given by the specialized cyclotomic Hecke algebra, so dependent only on d, not on q and ℓ.
3. The stable equivalence that should be lifted can also be determined just using local data.

Now all we need to know is how the perverse equivalence depends on ℓ. (It cannot depend on q since the algorithm for computing perverse equivalences is carried out in the normalizer of the defect group.)

Genericity

Let E be a finite subgroup of $\mathrm{GL}_n(R)$ for some ring of integers R in an algebraic number field. (E.g., E a complex reflection group in its reflection representation.) If $\mathbb{Z}_{\bar{\ell}}$ has the right roots of unity, then there is a map from R to $\mathbb{Z}_{\bar{\ell}}$ that allows us to construct the group $H_{\bar{\ell}} = (\mathbb{Z}_{\bar{\ell}})^n \rtimes E$. (E.g., the normalizer of a defect group in a group of Lie type modulo the $\mathrm{O}_{\ell'}$.) The group algebras $kH_{\bar{\ell}}$ form a generic block in the sense above.

Theorem (C.–Rouquier, last week)

The image under a perverse equivalence with fixed perversity function of the generic block above is a generic block.

What?

In other words, for $\bar{\ell}$ and $\bar{\ell}'$ large enough, under the same perverse equivalence, the images of these two algebras have the following properties:

- Their decomposition matrices are equal (up to i-exceptionality for $i \geq 1$)
- The complexes that are the images of the simple modules under the perverse equivalences have the same projective modules and the same cohomologies.

Corollary

If the combinatorial Broué conjecture is true then the decomposition matrices for unipotent blocks do not depend on q or ℓ, but only on d, for all sufficiently large $\bar{\ell}$ (i.e., the power of the prime).

Donovan's conjecture

Having shown that Broué's conjecture implies the stabilization of decomposition numbers, and the triangularity of the decomposition matrix (this is true for any perverse equivalence), what about Donovan's conjecture?

Example

Let E be a finite group lying in $\mathrm{GL}_n(R)$ with $n > 1$. Assume that the representation has no trivial summands. For sufficiently large $\bar{\ell}$, as one ranges over all perversity functions the decomposition numbers of the resulting algebras are unbounded.

The restrictions on this theorem should be reducible to $n > 1$ and $E \neq 1$, with a bit more work, and might also be able to remove the restriction on ℓ with the exception of $\ell = 2$ and $E = \mathbb{Z}_3$ (so $H_2 = A_4$).

Donovan's conjecture

What this says in some sense is that Donovan's conjecture is a statement about group theory, not about representation theory. There are infinitely many Morita classes of algebra inside the derived equivalence class of the local block, so either CFSG or prime switching are likely to be needed to solve this.

It also shows that the cyclic case is really special, because as soon as you leave the cyclic case in almost any direction the decomposition numbers can explode, and the reason they don't (if they don't) is because finite groups are very restrictive.