# Nervous Categories

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The Martino–Priddy conjecture was recently solved by Bob Oliver (about 2002/03). It says the following.

**Theorem A** Let $G$ and $H$ be a finite groups, with Sylow $p$-subgroups $P$ and $Q$ respectively. Then

$$BG_p^\wedge \simeq BH_p^\wedge \iff \mathcal{F}_P(G) = \mathcal{F}_Q(H).$$

The aim of this talk is to explain what this line of symbols means. This talk consists of three parts.

(i) A description of $BG$, the classifying space of $G$.

(ii) A description of $\mathcal{F}_S(G)$, the fusion system of a finite group.

(iii) Some discussion of Bousfield–Kan $p$-completions.

## 1 $BG$

$BG$ is the nerve of a one-point category. So what is the nerve of a category? let $\mathscr{C}$ be a small category. Define $|\mathscr{C}|$, the (geometric realization of the) nerve of $\mathscr{C}$, to the constructed as follows.

$$|\mathscr{C}| = \left( \coprod_{n \geqslant 0} \coprod_{c_0 \to c_1 \to \cdots} \Delta^n \right) / \sim .$$

Start with $Ob(\mathscr{C})$ as the vertices.

For all non-identity morphisms $c_0 \to c_1$, add an edge between the corresponding vertices. (These are the 1-simplices.)

For all commutative triangles $c_0 \to c_1 \to c_2$, add the 2-simplex

$$\begin{array}{ccc} & c_1 & \\ {}^{\phi}\nearrow & & \searrow^{\psi} \\ c_0 & \xrightarrow[\psi\phi]{} & c_2 \end{array}$$

Continue for all commutative tetrahedra, and so on. Then $|\mathscr{C}|$ is the complex corresponding to these.

**Example 1.1** Let $\Delta(n)$ be the category with objects the integers $\{0,\dots,n\}$ and $\mathrm{Hom}(i,j)$ empty if $i > j$, and consisting of a single element if $i \leqslant j$. Then $|\Delta(n)|$ is the $n$-simplex.

Let $F$ be a functor from $\mathscr{C}$ to $\mathscr{D}$. Then $F$ induces a continuous map $|F| : |\mathscr{C}| \to |\mathscr{D}|$. If $F$ and $F'$ are functors with a natural transformation $F \to F'$, this induces a homotopy $|F| \simeq |F'|$.

If $\mathscr{C}$ has an initial or terminal object then $|\mathscr{C}|$ is contractible. The way to see this geometrically is that if $*$ is a terminal object, then all simplices with $*$ as a vertex can be collapsed, and similarly for initial objects.

Let $G$ be a discrete group (e.g., $G$ finite). Let $\mathscr{E}(G)$ be the category whose objects are $G$ and a unique morphism $g \to h$ for $g, h \in G$ (labelled by $hg^{-1}$). Since every object is initial (and terminal), $|\mathscr{E}(G)|$ is contractible. Let $\mathscr{B}(G)$ be the category with one object $o_G$, and $\mathrm{Hom}_{\mathscr{B}(G)}(o_G, o_G) = G$, with composition given by multiplication in $G$. Notice that $G$ acts on $\mathscr{E}(G)$, and we may form the quotient

$$\mathscr{E}(G)/G \cong \mathscr{B}(G).$$

Thus $|\mathscr{E}(G)|/G \cong |\mathscr{B}(G)|$. (Note also that $\pi_1(|\mathscr{B}(G)|) = G$.)

Let $G$ be a discrete group. A *classifying space* for $G$ is a path-connected space $BG$ such that $\pi_1(BG) = G$ and $\widetilde{BG} = EG$ is contractible. Any two classifying spaces are homotopy equivalent.

Notice that $|\mathscr{B}(G)|$ is a classifying space for $G$!

## 2 $\mathcal{F}_S(G)$

**Definition 2.1** Suppose that $P \leqslant H \leqslant G$. We say that $H$ *controls fusion* in $P$ with respect to $G$ if whenever $A, B \subseteq P$ are conjugate in $G$ via $g$, they are conjugate in $H$ via $h$ with $c_g = c_h$ on $A$ (where $c_g$ denotes conjugation by $g$).

**Lemma 2.2 (Burnside)** If $S \in \mathrm{Syl}_p(G)$ and $S$ is abelian, then $\mathrm{N}_G(S)$ controls fusion in $S$ with respect to $G$.

**Lemma 2.3** Suppose that $S \in \mathrm{Syl}_p(G)$ is a TI subgroup (i.e., if $g \in G \setminus \mathrm{N}_G(S)$ then $S^g \cap S = 1$). Then $\mathrm{N}_G(S)$ controls fusion in $S$ with respect to $G$.

**Theorem 2.4 (Frobenius's normal $p$-complement theorem)** Suppose that $S \in \mathrm{Syl}_p(G)$. The following are equivalent:

(i) $G$ is $p$-nilpotent (i.e., there exists $K \trianglelefteqslant G$ such that $G = K \rtimes S$.

(ii) $S$ controls fusion on $S$ with respect to $G$.

Fusion is important in many situations in group theory.

**Theorem 2.5 (Solomon)** There is no finite group $G$ with Sylow 2-subgroup $S$ that of $\mathrm{Spin}_7(3)$, such that all involutions are conjugate in $G$ and all fusion in $S$ induced by $\mathrm{Spin}_7(3)$ is induced by $G$.

The interesting thing about this theorem is its proof. It relies on extracting 3-local information, not proving the the fusion of elements in a Sylow 2-subgroup is impossible.

**Definition 2.6** Let $G$ be a finite group and $S \in \mathrm{Syl}_p(G)$. The *fusion system* of $G$ on $S$ is the category $\mathcal{F}_S(G)$, with objects all subgroups of $S$, and as morphisms the set

$$\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) = \{c_g : P \to Q \mid g \in G \text{ and } P^g \leqslant Q\}.$$

Fusion systems can, however, be defined more axiomatically and abstractly.

**Definition 2.7** Let $P$ be a finite $p$-group. A *fusion system* on $P$ is a category $\mathcal{F}$, with objects all subgroups of $P$, and morphism set $\mathrm{Hom}_{\mathcal{F}}(Q, R)$ consisting of injective group homomorphism satisfying three conditions:

(i) $\mathcal{F}_P(P) \subseteq \mathcal{F}$.

(ii) If $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$, then the induced isomorphism $\phi : Q \to \phi(Q)$ lies in $\mathrm{Hom}_{\mathcal{F}}(Q, \phi(Q))$.

(iii) If $\phi \in \mathrm{Hom}_{\mathcal{F}}(Q, R)$ is an isomorphism, then $\phi^{-1} \in \mathrm{Hom}_{\mathcal{F}}(R, Q)$.

Fusion systems themselves are loose objects, and look far from coming from finite groups. One point is that in groups all $p$-automorphisms of subgroups come from a Sylow $p$-subgroup of $G$. A *saturated* fusion system satisfies two more axioms, one concerning $p$-automorphisms in $\mathrm{Aut}_{\mathcal{F}}(Q)$, and the other allowing us to use induction by ensuring that certain isomorphisms between subgroups extend to overgroups.

**Proposition 2.8** if $G$ is a finite group, then $\mathcal{F}_S(G)$ is a fusion system on $S$.

We may now restate Solomon's theorem.

**Theorem 2.9 (Solomon, Levi, Oliver)** There exists a saturated fusion system $\mathcal{F}$ on $S \in \mathrm{Syl}_2(\mathrm{Spin}_7(3))$ such that $\mathcal{F}_S(\mathrm{Spin}_7(3)) \subseteq \mathcal{F}$ and all involutions are $\mathcal{F}$-conjugate. Furthermore, $\mathcal{F}$ is not the fusion system of any finite group.

We may also restate Frobenius's theorem.

**Theorem 2.10 (Frobenius)** If $G$ is a finite group and $S \in \mathrm{Syl}_p(G)$, then $G = K \rtimes S$ for some $p'$-group $K$ if and only if $\mathcal{F}_S(G) = \mathcal{F}_S(S)$.

The control of fusion statements of Burnside and about TI Sylow $p$-subgroups can be restated as $\mathcal{F}_S(G) = \mathcal{F}_S(\mathrm{N}_G(S))$.

Why study fusion systems? They also exist for blocks of finite groups, have connections to topology via considering $|\mathcal{F}_S(G)|$, and allow one to discuss fusion in finite groups in abstract terms.

# 3 Bousfield–Kan Completions

The idea is to understand ordinary homotopy theory by studying the mod-$p$ components. Thus we use the $p$-completion to get information on mod-$p$ components, together with a completion over the rationals, then piece the components back together to get back some information about the original space.

What is a $p$-completion? It is a functor on topological space $\lambda : X \to X_p^\wedge$ that maps mod-$p$-cohomology equivalences to homotopy equivalences. A space $X$ is $p$-complete if $\lambda : X \to X_p^\wedge$ is a homotopy equivalence, and $p$-good if $X_p^\wedge$ is $p$-complete. If $\pi_1(X)$ is finite then $X$ is $p$-good, so $BG$ is $p$-good for $G$ a finite group.