Fusion systems on groups with an abelian subgroup of index p

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The search for exotic systems

Exotic fusion systems seem to offer a glimpse into what finite simple groups that don't exist should look like. More or less all exotic fusion systems are simple (or built up from exotic simple fusion systems), and many fusion systems of simple groups (at least for p small, as we shall see) are themselves simple.

The 'simplest' exotic systems were found by Ruiz and Viruel, and are on the extraspecial group 7_+^{1+2} of exponent 7. Others have been found, for example by Solomon–Benson on Sylow 2-subgroups of $\mathrm{Spin}_7(r)$ for r odd (the only known simple exotic systems at the prime 2) and by on certain 3-groups of maximal class by Díaz–Ruiz–Viruel. Another set of exotic fusion systems were constructed by Clelland and Parker, using modules for $\mathrm{GL}_2(p)$.

What the Ruiz–Viruel and the Clelland–Parker examples have in common is that the Sylow p-subgroup S in both cases possesses an abelian subgroup A of index p.

Minimal examples?

If S is abelian, then Alperin's fusion theorem, which we recall states that every map in $\mathcal{F}$ is a product of (restrictions of) automorphisms of subgroups that contain their own centralizer, proves that every map in S is a restriction of an automorphism of S.

In other words, if H is a p'-group of automorphisms of S, then we can construct the group $S \rtimes H$, and $\mathcal{F}_S(S \rtimes H)$ is a saturated fusion system on S, and all saturated fusion systems on S arise in such a way.

If however, the abelian subgroup is maximal, then we have lots of examples where this is not the case, for example $G = S_{p^2}$, where the Sylow p-subgroup is $C_p \wr C_p$, or $\mathrm{GL}_p(q)$ for $p \mid (q-1)$, or the Monster at $p = 13$, and so on.

It therefore seems like a good idea to 'classify' (in a suitable sense) all saturated fusion systems on p-groups with an abelian subgroup of index p.

What do you mean, 'classify'?

I don't want to try to classify every saturated fusion system on such groups, because in particular it would require classifying all p'-subgroups of $\mathrm{GL}_n(p)$.

The theory of **tame** and **reduced** fusion systems was started was Andersen, Oliver and Ventura and 2012. The central tenet is as follows:

A reduced fusion system is tame if an only if all saturated fusion systems reducing to it are realizable as fusion systems of finite groups.

OK, great. So what is a reduced fusion system? What is a tame fusion system for that matter?

Definition

A saturated fusion system $\mathcal{F}$ on S is **reduced** *if it has no normal subgroups, no normal subsystems on S itself, and no non-trivial morphisms $\mathcal{F} \to \mathcal{F}_T(T)$ for some $T > 1$ (i.e. $\mathrm{O}_p(\mathcal{F}) = 1$ and $\mathrm{O}^p(\mathcal{F}) = \mathrm{O}^{p'}(\mathcal{F}) = \mathcal{F}$).*

Have I thrown away too much?

No. In particular, all simple fusion systems and semisimple fusion systems are reduced. Hence if we are interested in simple fusion systems, the larger class of reduced fusion systems is still fine for us.

I seem to have a lot more room on this slide, so I can say a few words about tameness. A fusion system $\mathcal{F}$ is **tame** if there exists a finite group G with Sylow p-subgroup S, and firstly $\mathcal{F} = \mathcal{F}_S(G)$, and secondly the map

$$\kappa_G : \mathrm{Out}(G) \to \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$$

is split surjective, where $\mathcal{L}_S^c(G)$ is the centric linking system. Since I am interested in simple fusion systems, and we will not be checking whether any of these things are tame, this isn't really important for us today.

More than one abelian maximal subgroup

It seems reasonable to split the cases up into where there is more than one abelian subgroup of index p, and where there isn't.

In the case where S has more than one abelian subgroup of index p, Bob Oliver has already done this case. This will clearly contain the examples on $S = p_+^{1+2}$, and won't contain the examples on $C_p \wr C_p$.

Theorem (Oliver)

In this case, if S possesses a reduced fusion system then $S = p_+^{1+2}$.

So from now on we will assume that S contains a unique abelian subgroup of index p.

The possible essential subgroups

Let A denote the abelian subgroup of S of index p. What are the possible essential subgroups of S?

Of course, A itself might be essential, and is in some sense the most appropriate candidate. If it is not, then Oliver also dealt with this case: here there are several new exotic fusion systems, and some have a strongly closed subgroup (meaning there exists a quotient fusion system) but the fusion system is simple, which is itself interesting.

The other possible subgroups are a subgroup $P = C_p \times C_p$ and a subgroup $Q = p_+^{1+2}$, in both cases intersecting A in a specified subgroup of A, and with index p in themselves.

This we have four possibilities, assuming A is essential. If neither P nor Q is essential then A is a normal centric subgroup, and so $\mathcal{F}$ is constrained (and anyway not reduced).

The action on A

Let $G = \mathrm{Aut}_{\mathcal{F}}(A)$. We will assume that A is elementary abelian in the rest of this talk. Thus A becomes an $\mathbb{F}_p G$-module. What kind of structure does this module and the group G have?

Firstly, since A is fully normalized and has index p in S, G has a Sylow p-subgroup U of order p. We get the following conditions on A and G:

1. $|\mathrm{Aut}_G(U)| = p - 1$
2. The action of $1 \neq x \in U$ on A has a single non-trivial Jordan block.
3. A has no trivial quotients, i.e., $[G, A] = A$.
4. $C_A(G) \leq [U, A]$, which is slightly weaker than A having no trivial submodules.

Can these modules exist?

If $G = S_n$ for $n \leq p < 2n$ then the Sylow p-subgroup of G has order p, and if M denotes the non-trivial factor in the permutation module, then M satisfies the second, third and fourth properties. Clearly G satisfies the first, so we get an example.

Let G have a Sylow p-subgroup U of order p, and let A be an $\mathbb{F}_p G$-module. We say that A is inactive if U acts on A with only one non-trivial indecomposable summand, and A is completely inactive if $\mathrm{Aut}_G(U)$ has order $p-1$. We want to understand completely inactive modules.

The first thing to notice is that inactivity is inherited by restriction to subgroups, duality, submodules and quotients. Thus if G has no non-trivial simple (completely) inactive modules, and no self-extensions of the trivial module (e.g., if G is simple), then it has no (completely) inactive modules. And neither does any group containing G.

Understanding G

Suppose that A is a (faithful) simple inactive module of dimension n, yielding an embedding of G into $\mathrm{GL}_n(q)$. Suppose that $G/Z(G)$ is not an almost simple subgroup of $\mathrm{PGL}_n(q)$. This means that G falls into one of a few geometrically defined classes of maximal subgroups, e.g., parabolic subgroups, direct products of GL_ms, wreath products, etc.

As A is simple, this gets rid of things like parabolics and products of groups. If A is not absolutely irreducible then the action of U on A would have multiple non-trivial Jordan blocks, and the same if A were writeable as $X \otimes Y$ for X, Y of dimension at least 2. Thus A is not in extension type subgroups or wreath products.

We continue like this until $G \leq C_{d-1} \wr S_n$ is a collection of monomial matrices, a couple of central products inside extraspecial type maximal subgroups, or is almost simple (modulo the centre). Thus we want to understand completely inactive modules for almost simple groups.

$GL_2(p)$

Since U has order p, if G is Lie type in defining characteristic then G is of type $PSL_2(p)$. For $GL_2(p)$ there are simple modules of dimension $1, \ldots, p$, and each of these is completely inactive. These yield the Clelland–Parker examples.

However, there are more modules for $GL_2(p)$. If M is a module of dimension $i > 1$, then M has extensions with two other modules N_1 and N_2, of dimensions $p + 1 - i$ and $p - 1 - i$. This yields indecomposable modules of dimension $p + 1$ and $p - 1$, both completely inactive also. Apart from a couple of modules with 1-dimensional socle, these are all completely inactive modules for G.

The indecomposable modules of dimension $p - 1$ yield new, exotic fusion systems, whereas almost all of those of dimension $p + 1$ fail a technical condition that I haven't told you about, which is satisfied whenever $\dim A \leq p$. (This comes back later.)

Alternating and sporadic groups

For alternating and sporadic groups, there is a useful result that we can apply that will make our lives much easier.

Proposition

If a simple group G is either of alternating or sporadic type, and $p > 3$ divides $|G|$, then G is generated by two elements of order p.

This is important: if M is an inactive module then the socle of the action of an element x of order p has codimension at most $p-1$ (since the non-trivial block has dimension at most p). If $G = \langle x, y \rangle$ then the intersection of $C_M(x)$ and $C_M(y)$ has codimension at most $2p-2$.

Thus if M is simple then $\dim M \leq 2p-2$. If $\dim M \geq 2p$ then M has at least two trivial submodules, and a fact about groups with cyclic Sylow p-subgroup is that if A is simple and B is indecomposable, then $\mathrm{Hom}(A, B)$ is at most 1-dimensional.

For groups of Lie type in non-defining characteristic, it looks as if, for $p > 5$ dividing $|G|$, they are also generated by two elements of order p. However, we are some way from proving this statement, so we cannot use it.

We need another way to bound the dimension of an inactive module.

Proposition

If $U \in \mathrm{Syl}_p(G)$ and $\mathrm{C}_G(U)$ is abelian, then the dimension of any inactive module is at most $2p - 1$.

This follows from the theory of canonical characters, which implies that there are at most $(p-1)\chi(1)$ trivial summands in the restriction of an inactive module to U, where $\chi \in \mathrm{Irr}(C_G(U))$.

Now, if we could only find a way to make the centralizer $C_G(U)$ abelian.

Induction to the rescue

Obviously the centralizer isn't abelian in all cases. But we can set up an induction using the following result.

Proposition

Suppose that $G = G(q^{\delta_G})$ is a group of Lie type. If $U \in \mathrm{Syl}_p(G)$ has order p then either $C_G(U)$ is abelian (p is regular semisimple) or there exists $H = H(q^{\delta_H})$ a subgroup of G such that $U \leq H$, $C_H(U)$ is abelian and $\mathrm{Aut}_G(U) = \mathrm{Aut}_H(U)$.

As an example, if $G = \mathrm{GL}_n(q)$ and $p \mid \Phi_d(q)$, then $H = \mathrm{GL}_d(q)$ or $H = \mathrm{GL}_{d+1}(q)$ will work.

Thus we now simply have to construct all modules for groups of Lie type of dimension at most $2p-1$, where $p \mid \Phi_d(q) \mid q^d - 1$, and where $|\mathrm{Aut}_G(U)|$ is of order at most $4dt$ where t is the maximal size of a graph automorphism (this follows from knowledge of normalizers of Φ_d-tori in Lie type groups).

All the modules

So $p \mid \Phi_d(q^t) \mid q^{td} - 1$ and $|\mathrm{Aut}_G(U)| \leq 4dt$.

The twin statements $p - 1 \geq 4dt$ and $p \leq q^{td} - 1$ already put strong conditions on p, q and d. Throw in Landazuri–Seitz lower bounds on dimensions of modules for groups of Lie type, e.g., $\dim M \leq q^{(n-1)t} - 1$ for $\mathrm{GL}_n(q^t)$ and we get a finite, and small, list of possibilities.

Assume G is not alternating or $\mathrm{PSL}_2(p)$. We have one of:

1. $G = \mathrm{SL}_2(8) : 3 = {}^2G_2(3)$ or $G = 6 \cdot \mathrm{PSL}_3(4)$ and $p = 7$;
2. $G = \mathrm{PSU}_3(3).2 = G_2(2)$ or $G = 6_1 \cdot \mathrm{PSU}_3(4).2_2 = G_{34}$ and $p = 7$;
3. $G = \mathrm{PSU}_3(4) : 4$ and $p = 13$;
4. $G = \mathrm{PSU}_4(2) = \mathrm{PSp}_4(3)$ and $p = 5$;
5. $G = \mathrm{PSU}_5(2).2$ and $p = 11$;
6. $G = \mathrm{Sp}_4(4).4$ and $p = 17$;
7. $G = \mathrm{Sp}_6(2)$ and $p = 5, 7$ or $G = 2 \cdot \mathrm{Sp}_6(2)$ and $p = 7$;
8. $G = 2 \cdot \Omega_8^+(2)$ and $p = 7$;
9. $G = G_2(3).2$ or $G = {}^2B_2(8) : 3$ and $p = 13$.

Do all of these give exotic fusion systems?

No.

The ones of dimension at most p do, and there are a lot of those, but if the dimension is more than p then there is another technical condition on the action of $\mathrm{Aut}_G(U)$ on the socle of A and on U that needs to be satisfied. This fails for (for example) $6 \cdot \mathrm{Suz}$ and $p = 11$, where there is a 12-dimensional module, but is satisfied by the group $\mathrm{GL}_2(p)$, where A is a $(p+1)$-dimensional indecomposable module whose top is the natural module.

The complete list of groups and modules is now known, except for groups G lying inside $C_{p-1} \wr S_n$.

What is this technical condition?

If we have a non-trivial G-action, we can consider $Z_0 = C_A(U) \cap [U, A]$. This 1-dimensional subspace is an $N_G(U)$-module. We also have the conjugation action of $N_G(U)$ on U. This yields an action of $N_G(U)$ on

$$Z_0 \times U \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

If $\dim(A) > p$, then instead of $N_G(U)$ we need to take the centralizer in this of $C_A(U)/Z_0$, which is of course much smaller. Write μ for the image in $Z_0 \times U$.

Write Δ_i for the twisted diagonal subgroup $\{(x^i, x) \mid x \in \mathbb{Z}_p\}$. If μ contains Δ_0 or Δ_{-1} then we get an exotic fusion system. If it contains both (in the case where $\dim(A) \leq p$) we can potentially build others. Depending on the image of μ, either $P = C_p \times C_p$ or Qp_+^{1+2} or both are essential.