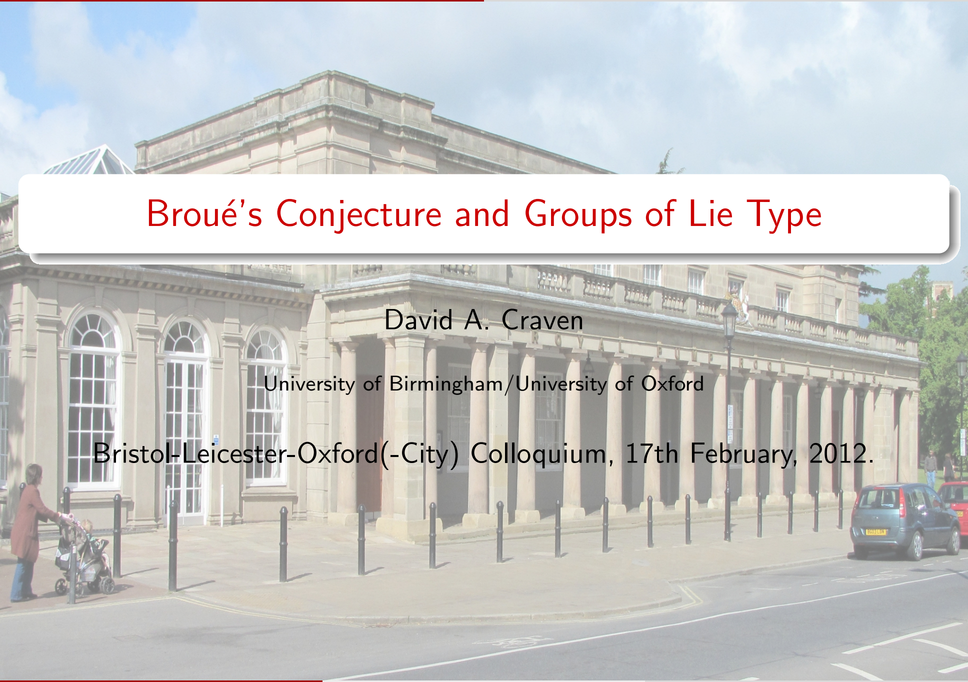

Broué's Conjecture and Groups of Lie Type

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Notation and Conventions

Throughout this talk,

- G is a finite group,
- ℓ is a prime,
- k is a field of characteristic ℓ,
- B is a block of kG, with defect group D and Brauer correspondent b;
- P is a Sylow ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

Some of this talk is joint work with Raphaël Rouquier.

Representation Theory is Local

Many features of the modular representation of a finite group are conjectural, some not even conjectural. Broadly, they fall into three categories:

- finiteness conditions;
- numerical conditions;
- structural conditions.

As an example of the first, we have Donovan's conjecture.

As examples of the second, we have the Alperin–McKay conjecture, Alperin's weight conjecture, and Brauer's height-zero conjecture.

As an example of the third, we have Broué's conjecture.

Representation Theory is Local

Some of the conjectures before (Alperin–McKay, Alperin's weight, Broué) relate the structure of a block B of kG to the structure of its Brauer correspondent b, a block of $k\mathrm{N}_G(D)$, where D is a defect group of B. Write $\ell(B)$ for the number of simple B-modules.

Alperin's weight conjecture gives a precise conjecture about $\ell(B)$ in terms of local information (normalizers of ℓ-subgroups). If D is abelian, the conjecture reduces to

$$\ell(B) = \ell(b).$$

Broué's conjecture gives a structural understanding of Alperin's weight conjecture.

Conjecture (Broué, 1988)

Let G be a finite group, and let B be a ℓ-block of G with abelian defect group D. If b is the Brauer correspondent of B in $\mathrm{N}_G(D)$, then B and b are derived equivalent.

Principal Blocks Are Good

In representation theory, one standard method of proof is to reduce a conjecture to finite simple groups and then use their classification.

In general, there is no (known) reduction of Broué's conjecture to simple groups, but for principal blocks there is.

Theorem

Let G be a finite group. If P is abelian, then there are normal subgroups $H \leq L$ of G such that

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- L/H *is a direct product of simple groups and an abelian* ℓ*-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L (compatible with automorphisms of the simple components) passes up to G. Thus if Broué's conjecture for principal blocks holds for all simple groups (with automorphisms), it holds for all groups.

Classification of Finite Simple Groups

We need to know Broué's conjecture for the finite simple groups. If the Sylow ℓ-subgroups of a simple group G are abelian, then one of the following holds:

1. $G = A_n$ (Broué's conjecture known: Chuang, Kessar, Marcus, Rickard, Rouquier)
2. G is a sporadic group (Broué's conjecture known if $\ell > 11$)
3. $G = \mathrm{SL}_2(q)$ and $\ell \mid q$ (Broué's conjecture known: Okuyama)
4. $G = G(q)$ is a Lie-type group and $\ell \nmid q$.

Hence in order to prove Broué's conjecture for principal blocks, we need to understand groups of Lie type in non-defining characteristic.

Groups of Lie Type

Let $G = G(q)$ be a group of Lie type (e.g., $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{Sp}_{2n}(q)$): the order of G is

$$|G| = q^N \prod_{i \in I} \Phi_i(q).$$

Suppose that $\ell \nmid q$ divides exactly one of the cyclotomic polynomials $\Phi_d(q)$ in the product. Then the Sylow ℓ-subgroup is abelian, and contained in a Φ_d-torus.

The unipotent characters of G are certain irreducible characters of G, not depending on q. A unipotent block of G is one containing a unipotent character, such as the principal block, which contains the trivial character.

Geometric Broué

Broué's conjecture has a special version for unipotent blocks of groups of Lie type, called the geometric form.

The following is a rough statement of the conjecture.

Conjecture

Let $G = G(q)$ be a finite group of Lie type, and let D be a defect group of a unipotent block B of G. We may embed D inside a Φ_d-torus T, and there is a Deligne–Lusztig variety Y, carrying an action of G on the one side and T on the other such that

1. *the action of T can be extended to an action of $\mathrm{N}_G(T) = \mathrm{N}_G(D)$, and*
2. *the complex of cohomology of this variety induces a derived equivalence between B and its Brauer correspondent.*

Geometric Broué

In fact, if ζ is a primitive dth root of unity, then there should be a Deligne–Lusztig variety Y_ζ associated naturally to ζ, and whose complex of cohomology produces the desired equivalence.

While this is (a lot) more specific than the abstract version of Broué's conjecture, it still needs to be more specific, as the variety Y_ζ can be hideously complicated, and the extension of the action given in the conjecture is not specified at all.

This equivalence should be a perverse equivalence, which requires some combinatorial data. If these data can be extracted without analyzing the variety Y_ζ, then the derived equivalence should be able to be constructed without the variety at all, purely combinatorially.

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras, $\mathcal{A} = mod\text{-}A$, $\mathcal{B} = mod\text{-}B$.

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{A}')$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, T_1, $T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, for all i, the cohomology of $F(S_i)$ only involves T_j for $j < i$, except for one copy of T_i in degree $-\pi(i)$, and T_j can only appear in degrees less than $-\pi(j)$.

Properties of a Perverse Equivalence

1. If B is a unipotent block, then there should be a perverse equivalence from B to its Brauer correspondent b.
2. There is an algorithm that gives us a perverse equivalence from any block B for a group G to **some** algebra, and we need to check that the target is the Brauer correspondent b. (This is simply checking that the Green correspondents are the last terms in the complexes.) This algorithm is very useful!
3. The alternating sum of the cohomology of the complex X_i corresponding to S_i constructed by this algorithm gives a row of the decomposition matrix, with only S_j appearing for $\pi(S_j) < \pi(S_i)$, except for a single copy of S_i. When placed in ascending order of $\pi(-)$, this yields a lower unitriangular decomposition matrix.

Properties of a Perverse Equivalence

4. The lower triangularity of the matrix gives a bijection between the simple B-modules and some of the ordinary B-characters. If B has cyclic defect group then this association sends a simple module to the incident vertex farther from the exceptional node.
5. The ordinary characters in the upper square part are the unipotent characters when B is a unipotent block, and hence this gives a natural bijection between the unipotent characters and the simple modules of B.
6. The algorithm's output should be 'generic' in ℓ. (This is ongoing research of Rouquier and me.) This would imply that, assuming this version of the geometric version of Broué's conjecture, the decomposition numbers of unipotent blocks are independent of ℓ, for all sufficiently large ℓ. It would also suggest an answer to 'sufficiently large'.

The Perversity Function

The cohomology of the variety Y_ζ over $\bar{\mathbb{Q}}_\ell$ should have the property that each unipotent character χ should appear in exactly one degree $\pi_\zeta(\chi)$. (This degree will depend on ζ.) These degrees should be the perversity function for the perverse equivalence from B to b. (Actually, this is the cohomology with non-compact support. For compact support, take $-\pi_\zeta(\chi)$ and shift by twice the length of the variety.)

If $\ell \mid \Phi_d$, where d is the largest integer such that $\Phi_d \mid |G|$, i.e., d is the Coxeter number, then Lusztig calculated the cohomology for $\zeta = \exp(2\pi i/d)$. Let $f(q)$ be a polynomial. If $a(f)$ denotes the multiplicity of q in a factorization of f, and $A(f) = \deg(f)$, then the degree of the cohomology that χ is in is $(a(\chi(1)) + A(\chi(1)))/d$ plus half the power of $(q-1)$ in $\chi(1)$.

If $d = 1$ or $d = 2$ then Digne–Michel–Rouquier conjectured, for the principal block, that the degree of cohomology in which χ lies is $2A(\chi(1))/d$.

The Perversity Function: Connecting the Extremes

Define $\zeta = e^{2k\pi i/d}$, and for f a polynomial in q write $\phi_\zeta(f)$ for the number of non-zero zeroes of f (with multiplicity) of argument at most that of ζ (with argument in $[0, 2\pi)$), with the exception that positive reals count for $1/2$ (as their argument is 'both' 0 and 2π). Write $a(f)$ for the multiplicity of the zero at 0. Write $\pi_\zeta(f) = k(\deg f + a(f))/d + \phi_\zeta(f)$.

Conjecture

If f denotes the relative degree of χ, then $\pi_\zeta(f)$ is the unique degree in the cohomology with non-compact support of Y_ζ in which χ appears, and the $\pi_\zeta(\chi)$ form the perversity function for a perverse equivalence from B to b.

This conjecture has been shown to hold in a variety of situations, both for the perverse equivalences and for the Deligne–Lusztig variety Y_ζ. For example Olivier Dudas has shown for $\mathrm{GL}_n(q)$ that this formula holds for all d if and only if it holds for $d = 1$.

Pulling π Downstairs

The quantity $\pi_\zeta(\chi)$ is defined for unipotent characters in B, but for the algorithm computing perverse equivalences it needs to be defined on the simple b-modules, which can be thought of as the ordinary characters of the automizer $E = N_G(P)/PC_G(P)$ if B is the principal block and in general $E = \mathrm{N}_G(D, b)/\mathrm{C}_G(D)$.

This means we need a bijection between the unipotent characters of B and the ordinary characters of E. Recall that E is a complex reflection group, and its action on the torus T is as complex reflections.

The object we need for this is the cyclotomic Hecke algebra, which in one specialization gives the characters of B and in another gives the characters of b.

The Cyclic Case, I

The case where the defect group is cyclic is one where we can say the most. Here the π-function and bijection are both fully understood.

Theorem

Suppose that G is of Lie type, B is a unipotent block, and D is cyclic. If G does not have type E_7 or E_8 (and even then in many cases) the 'combinatorial form' of Broué's conjecture is true, with $\pi(-) = \pi_\zeta(-)$ and bijection given by mapping χ to $\omega_\chi \zeta^{(a(\chi)+A(\chi))/\ell(b)}$ (ω_χ is a root of unity, normally ± 1), with the Brauer tree of b (a star) being represented on the complex plane.

In order for this theorem to make sense, for non-principal blocks anyway, we have to fix a rotation of the Brauer tree of b, to decide which non-exceptional b-character is placed at the position 1 (in $\mathbb{C}$). This can be done by taking Green correspondents of a simple B-module with smallest π_ζ-function.

The Cyclic Case, II

The theorem suggests that we should think of a Brauer tree as being embedded in $\mathbb{C}$, not in $\mathbb{R}^2$: one of the important directions this research might take is a generalization of the Brauer tree to (some other) abelian defect groups.

The method of proof of the theorem is fairly simple: using the π-function and bijection, we construct the Brauer tree of the block, and compare it to the known one (when it is known, i.e., not for some blocks of E_7 and E_8). Combinatorial Broué's conjecture holds if and only if the Brauer tree is correct.

Notice that this allows us to make conjectures as to the shape of the Brauer tree in the remaining cases, and this has led to some outstanding cases being resolved.

An Example: $\mathrm{PSL}_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
3	$q(q^2+q+1)$	1	1			
4	$q^2(q^2+1)$		1	1		
5	$q^3(q^2+q+1)$	1	1	1	1	
6	q^6	1			1	1

$$\begin{array}{ll}
X_2: & 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0. \\
X_5: & 0 \to \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(234) \oplus M_{4,1} \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0. \\
X_3: & 0 \to \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(5) \oplus M_{1,1} \to M_{1,1} \oplus M_{1,2} \to C_3 \to 0. \\
X_4: & 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(4) \to M_{4,2} \to C_4 \to 0.
\end{array}$$

ℓ-Extended Finite Groups

Let H be a finite group, and let ρ be a faithful complex representation of H. It is well known that there exists an algebraic number field K, with ring of integers $\mathcal{O} = \mathcal{O}_K$, such that $H \leq \mathrm{GL}_n(\mathcal{O})$ and this embedding induces ρ.

Let ℓ be an integer with $\gcd(\ell, |H|) = 1$, such that the map $\mathcal{O} \to \mathbb{Z}/\ell\mathbb{Z}$ induces a faithful representation of H over $\mathbb{Z}/\ell\mathbb{Z}$ via ρ. Write M for the $\mathbb{Z}/\ell\mathbb{Z}H$-module, and $G_\ell = M \rtimes H$.

- $k(G_\ell)$ is a polynomial in ℓ, and $k(G_\ell) \cdot |H|$ is a monic polynomial in ℓ with integer coefficients.
- If H is a reflection group and ρ is its natural representation over $\mathbb{Z}$, then the second coefficient of $k(G_\ell) \cdot |H|$ is $3N$, where N is the number of reflections in H. (A similar formula exists for complex reflection groups.)

Being Generic: An Example

Let $G = \mathrm{PSU}_3(q)$, $\ell \mid (q+1)$. There are three simple modules in the principal block, as $\mathrm{N}_G(P)/\mathrm{C}_G(P) \cong S_3$. G has a permutation representation on q^3+1 points, let Q be a Sylow ℓ-subgroup of the point stabilizer, so that $|Q| = \ell$.

π	Ord. Char.	1_1	2_1	1_2
0	1	1		
2	$q(q-1)$		1	
3	q^3	1	2	1

Let M_1 be the trivial source module $23/13/23/\cdots/23$ with vertex Q, $\dim M_1 = 3\ell$. Let M_2 be the relatively projective summand of $(1/1/1)\uparrow_Q^N$ with head 23, $\dim M_2 = 9\ell$. For $\ell = 5, 7, 17$ we have the following:

$$\begin{array}{ll} X_3: & 0 \to \mathcal{P}(3) \to \Omega(M_1) \to C_3 \to 0. \\ X_2: & 0 \to \mathcal{P}(2) \to \mathcal{P}(22) \to \mathcal{P}(2) \oplus M_2 \to C_2 \to 0. \end{array}$$