Simple groups, simple representations, simple fusion systems

David A. Craven

University of Birmingham

THE FINAL CONFERENCE
'Finite simple groups, their fusion systems and representations'
Semester: 'Local representation theory and simple groups'
Centre Interfacultaire Bernoulli, Lausanne, Switzerland
12th–16th December, 2016

What's past is prologue

Let $\mathcal{F}$ be a saturated fusion system on a finite p-group S. We have heard about simple fusion systems, and exotic fusion systems. For $p = 2$, the classification of simple fusion systems states that a simple fusion system $\mathcal{F}$ is either the fusion system of a finite simple group, the Solomon fusion system, or maybe some others.

For odd primes, there is a large menagerie of exotic simple fusion systems that are known, and more seem to be found under every stone we overturn.

The idea is to come up with some broad characterization of these. I will try to do something like this at the end of the talk. Hopefully.

Generators maketh a group

Let G be a finite simple group. If x and y are elements of G, do x and y generate G? We've already seen this:

Theorem (Dixon, Kantor–Lubotzky, Liebeck–Shalev)

Probably.

After this, one may ask the same question, placing restrictions on x and y. For example, if we fix $x \neq 1$, is there a y that generates G together with x, and how likely is that to happen? Or we can specify the order of x, or of x and y. If $o(x) = o(y) = 2$ then the answer is 'no', of course, so what about $o(x) = 2$, $o(y) = 3$?

Theorem (Liebeck–Shalev, Lübeck–Malle)

You cannot for ${}^2B_2(2^{2a+1})$ *(obviously), or some* $\mathrm{PSp}_4(q)$*s, and finitely many other groups. All the rest, you can.*

3 is not the magic number

If $o(x) = r$ and $o(y) = s$, if $G = \langle x, y\rangle$ then G is (r, s)-generated. All non-Suzuki exceptional groups are $(2, 3)$-generated.

If one replaces '3' by 'p', the answer is probably that all simple groups are $(2, p)$-generated if $p \geq 5$ divides $|G|$, but as far as I know, this has not been completely solved. There is work of Liebeck–Shalev on (r, s)-generation, showing that large enough rank groups of Lie type are probabilistically (r, s)-generated (i.e., the probability of random generation by elements of orders r and s tends to 1 as q tends to ∞), but in general the question remains open.

Baby steps

I wanted actual $(2, p)$-generation, not probabilistic generation, so the Lübeck–Malle result on $(2, 3)$-generation of exceptional groups is fine for me, but the probabilistic methods in Liebeck–Shalev would need to be made completely explicit to guarantee $(2, 3)$-generation.

For low-rank exceptional groups I did this, getting the following small result.

Theorem

Let G be a finite exceptional group of twisted rank less than 4. If $p \mid |G|$ is odd then G is $(2, p)$-generated (and the probability tends to 1 as $q \to \infty$).

I stopped at 4 because the maximal subgroups of F_4 and above are not completely known. (I would guess now that there is enough information to do F_4, E_6, ${}^2E_6$ and E_7 in this way, but not E_8 yet.)

Generalizing generation by generic generators: generalities

Suppose that G is generated by u and an element v of order n. Then

$$\langle u, u^v, u^{v^2}, \dots, u^{v^{n-1}} \rangle$$

generates a normal subgroup H whose quotient is generated by Hv, hence G/H is cyclic of order at most n. If $\mathrm{Hom}(G, Z_n) = 0$, we have that G is generated by at most n conjugates of u.

For $u \in G$, write $\alpha(u)$ for the minimal number of conjugates of u needed to generate the normal closure of u in G. Thus $\alpha(u) \le n$ in the example above. $(2, p)$-generation implies that G is also generated by two conjugates of an element of order p.

Time to act

Let G be a finite group, and let M be a finite-dimensional module for G in characteristic p. Let u be a p-element of G.

By computing the Jordan normal form of u on M, we can understand the action of u, and in particular how many Jordan blocks there are in this action. If there is a unique Jordan block of size greater than 1, we say that u acts minimally actively on M, and that M is minimally active.

(This has also been called 'almost cyclic' in recent literature, particularly work by Zalesskii and coauthors.)

Lemma

If M has no trivial submodule, then $\dim(M) \leq \alpha(u) \cdot (o(u) - 1)$.

It's the simple things

Let G be a finite group, and let M be a faithful kG-module, for k a field of characteristic p. This yields an embedding $G \to \mathrm{GL}_n(k)$ for $n = \dim(M)$. We assume for this talk that the image of G in $\mathrm{PGL}_n(k)$ is almost simple. Thus $G_0 = G'$ is a quasisimple group, and G induces non-inner automorphisms on G_0. Since we are interested in whether a p-element u acts minimally actively, we may assume that G is the normal closure of $\langle u \rangle$, i.e., that G/G_0 is cyclic and generated by $G_0 u$. Note also that $Z(G_0)$ is a p'-group.

If G_0 is a cover of an alternating group Alt_n and u is fixed point free, then we can normally show that $\alpha(u) = 2$ and so if u acts minimally actively then

$$\dim(M) \leq \alpha(u) \cdot (o(u) - 1) < 2n.$$

From this, if u acts minimally actively on M, we get that M is the permutation module or $n \leq 9$.

Lies

Let G_0 be a quasisimple group of Lie type in characteristic $r \neq p$. 42 years ago, Landazuri–Seitz gave lower bounds for the dimension of a simple module for G_0; these were improved 23 years ago by Seitz and Zalesskii, and several papers have been written since then giving the exact minimal degrees. (Brundan, Guralnick, Himstedt, Hiss, Hoffman, Kleshchev, Lübeck, Magaard, Malle, Pentilla, Praeger, Saxl, Tiep, Zalesskii.)

Normally these are very much larger than the order of u. For bounds on $\alpha(u)$, Liebeck–Saxl 25 years ago, and later Guralnick–Saxl 13 years ago, gave bounds that more or less become $\alpha(u) \leq \ell + 3$, where ℓ is the Lie rank of G_0. Thus we get the formula

$$o(u) \cdot (\ell + 3) > \dim(M).$$

Then $o(u)$ is bounded above and $\dim(M)$ is bounded below, and these yield few options for u acting minimally actively. Better estimates for $o(u)$, $\alpha(u)$ and $\dim(M)$ eliminate more cases, and we are left with those that are minimally active, mostly Weil modules for SL_n, SU_n and Sp_{2n}.

Damned Lies

Let G_0 be a quasisimple group of Lie type in characteristic p. For large p, there are now very many highest weight modules of dimension less than p, so lots of candidates for minimally active modules.

Also lots of potential for induction. For example, suppose that u acts with a single Jordan block. This occurs if and only if there is no trivial subquotient of dimension at least 2 in the action of u on M. If u lies in the unipotent radical of any parabolic then u acts trivially on all composition factors of the restriction, so they are all 1-dimensional, not possible. For type A, this shows that only the regular could act in this way.

More or less, the only examples of minimally active modules are regular elements acting on natural modules, together with the spin module for B_3, and symmetric and exterior squares of the natural for A_2 and A_3 respectively.

Statistics

For $p \leq 17$, or for $G \neq G_0$, there are a variety of extra cases that can occur. If we require $p > 17$, we get a nice, clean statement.

Theorem

Let $p > 17$. If a quasisimple group G possesses a minimally active simple module then G is one of:

1. *the alternating group* Alt_n*;*
2. *a classical group or G_2 in defining characteristic p;*
3. $\mathrm{PSL}_2(r^a)$ *where* $p^b = (r^a \pm 1)/\gcd(2, p)$ *is a prime power (including Fermat and Mersenne primes);*
4. $\mathrm{PSL}_n(2)$, $2^n - 1$ *is a Mersenne prime;*
5. $\mathrm{PSp}_{2n}(3)$, $p^b = (3^n - 1)/2$ *is a prime power;*
6. $\mathrm{PSL}_n(q)$ *and* $\mathrm{PSU}_n(q)$, $p^b = (q^n \pm 1)/(q \pm 1)$ *is a prime power.*

Who said that?

For everything other than Lie type in defining characteristic, I have completely determined from first principles all elements that act minimally actively on a given simple module.

So have other people. There is lots of work in this area, and I am almost certain to miss some of it.

In 1995, Suprunenko determined all highest weight modules on which a unipotent element acts with a single Jordan block. She did the same thing for minimal action (allowing trivial blocks) for G classical in 2013, although complete proofs are not yet in the literature for types C and D for $p = 2$.

In 2014, Di Martino, Pellegrini and Zalesskii did everything for sporadic groups, generalizing to non-p-elements. In work in press, Di Martino and Zalesskii did Weil modules for SL_n, SU_n and Sp_{2n}.

But wait: there's more

A Ph.D. student of Bob Guralnick has nearly finished dealing with the case of all simple groups with u acting with one Jordan block.

There is an in-preparation manuscript of Testerman–Zalesskii that also deals with $G = G_0$ of Lie type in defining characteristic, including the exceptional groups.

If the Sylow p-subgroup has order p and $\mathrm{Aut}_G(\langle u \rangle)$ has order $p-1$ then this case was dealt with, for all finite groups, in joint work with Bob Oliver and Jason Semeraro.

There's also work of, for example, Tiep–Zalesskii and Kleshchev–Zalesskii in which multiple blocks may be non-trivial, but none can have size p.

Fusing these themes

Let S be a finite p-group with an elementary abelian subgroup A of index p, and let $\mathcal{F}$ be a saturated fusion system on S. Suppose that A is essential: the action of $\mathrm{Aut}_{\mathcal{F}}(A)$ on A turns A into a $\mathbb{F}_p G$-module, where $G = \mathrm{Aut}_{\mathcal{F}}(A)$.

Notice that G has a Sylow p-subgroup U of order p, generated by u. It turns out that u acts minimally actively on A. If (G, A) satisfy some other conditions (one is that $|\mathrm{Aut}_G(U)| = p - 1$) then one always gets a saturated fusion system $\mathcal{F}$ on S with $\mathrm{Aut}_F(A) = G$. Normally these fusion systems are simple, and normally they are exotic.

Opportunity knocks

All of these fusion systems are obtained from a group fusion system by adding another essential subgroup or more, all the time either $p^2 \rtimes \mathrm{SL}_2(p)$ or $p_+^{1+2} \rtimes \mathrm{SL}_2(p)$. In almost all exotic examples, this is the case, with the rest involving either a $q^2 \rtimes \mathrm{SL}_2(q)$ (i.e., a maximal parabolic of $\mathrm{SL}_3(q)$) or a $q_+^{1+2} \rtimes \mathrm{SL}_2(q)$ instead. In other words, we have that for a simple fusion system $\mathcal{F}$, one of the following is true:

1. $\mathcal{F} = O^{p'}(\mathcal{F}_S(G))$ for some finite simple group G;
2. $\mathcal{F}$ is a Solomon fusion system;
3. $\mathcal{F}$ is obtained from a group fusion system by adding essential subgroups of the form q^2 or q_+^{1+2};
4. maybe some others.

As a first approximation, we call a simple fusion system opportunistic if it is obtained from another saturated fusion system by the addition of a single essential subgroup of the form q^2 or q_+^{1+2}.

Con-fusion

Why opportunistic? The construction of opportunistic fusion systems should be controlled by a 'tree of groups'. Such generic constructions have been considered before, by Broto–Levi–Oliver, Robinson, Semeraro and Parker–Semeraro, for example. If certain compatibility conditions are met, one can always perform a construction. The idea is that $\mathcal{F}$ is constantly looking for an opportunity to get bigger, and seizes it as soon as it is presented to it.

The definition of opportunistic should be extended to include these tree structures, so that a fusion system is opportunistic if there exists a non-trivial tree, and such that the edge groups are very small.

Question

Suppose that $\mathcal{F}$ was given no opportunities in odd characteristic. Is $\mathcal{F} = O^{p'}(\mathcal{F}_S(G))$ for some G?

Could you be any more vague?

OK, here is a type of saturated fusion system: let $E_1, \ldots, E_r$ be well-chosen representatives of the classes of essential subgroups of S. We want to know if, for each $1 \leq i, j \leq r$ there exists $E \leq E_i \cap E_j$ such that

1. E is normal in E_i, E_j, and stabilized by $\mathrm{Aut}_{\mathcal{F}}(E_i)$, $\mathrm{Aut}_{\mathcal{F}}(E_j)$;
2. the restriction maps $\phi : \mathrm{Aut}_{\mathcal{F}}(E_i) \to \mathrm{Aut}_{\mathcal{F}}(E)$ and $\phi : \mathrm{Aut}_{\mathcal{F}}(E_j) \to \mathrm{Aut}_{\mathcal{F}}(E)$ have kernel a p-subgroup (necessarily of inner automorphisms).

For example, if $\mathcal{F}$ is the fusion system of $\mathrm{GL}_n(q)$ in defining characteristic then this is satisfied.

It should be true that such saturated fusion systems always come from finite groups. We then build all simple fusion systems from these opportunistically. It is likely that there are severe restrictions on which group fusion systems can be extended opportunistically.