Tensor Products of Modular Representations in Groups of Lie Type

David A. Craven

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Although the block structure of finite groups – particularly those of the groups of Lie type – has received considerable attention in recent years, the tensor structure of the category of finite-dimensional KG-modules (where G is a finite group and K is a field of characteristic p) has experienced much less growth.

The results that I will talk about today are the beginnings of a theory that has not yet been written. At this point in time, it is hard to see the outline of this edifice, never mind the details: even the conjectures are conjectural. Yet enough can be seen to indicate that there is something going on in the tensor structure of the module category of a finite group.

1 Algebraic Modules

Alperin defined the concept of an algebraic module. It is defined to be a module M that satisfies a polynomial with integer coefficients in the Green ring. This is equivalent to there being a finite list of indecomposable modules $M_1, \dots, M_n$, such that any indecomposable summand of $M^{\otimes i}$ is one of the M_j for some j. Algebraic modules are nice, in the sense that most of the things that you are likely to want to do to modules behave well with respect to algebraicity. For example, sums, tensor products, and summands, of algebraic modules are algebraic. Algebraicity is preserved under inflation, restriction, and induction, and sources and Green correspondents of algebraic modules are algebraic.

The only thing you are likely to want to do that hasn't been mentioned is extend modules to overgroups.

Example 1.1 Let G be the group $\mathrm{SL}_2(8) \rtimes C_3 = {}^2G_2(3)$, and let $H = G'$. Then H has cyclic Sylow 3-subgroups, and so all modules are algebraic. (There are only finitely many inddecomposable modules.) However, as we shall see later, the 7-dimensional natural (simple) KG-module M is non-algebraic. Clearly the restriction of M to H remains simple, and so M is a non-algebraic extension of an algebraic module.

No examples are known in characteristic 2, although this is unlikely to mean that there aren't any.

There are some interesting theorems known about algebraic modules. Berger in 1976 proved that simple modules for soluble groups are algebraic, and this was extended in 1979 by Feit to p-soluble groups. More recently, I have examined the interplay between algebraicity and the concepts in Auslander–Reiten theory. We will very briefly cover two of them, because it turns out that they are useful later on.

Theorem 1.2 Let G be a finite group and let K be a field of characteristic p. Let M be an indecomposable, non-periodic KG-module. Then at most one of the modules $\Omega^i(M)$ is algebraic.

This can be extended to Auslander–Reiten quiver. We need a very special case of this collection of results.

Theorem 1.3 Let G be a finite group, and let B be a block of KG with defect groups $C_p \times C_p$ for p odd. (Then the components of the Auslander–Reiten quiver are of type A_∞.) Let M be a non-periodic, algebraic, indecomposable module. Then M cannot lie on the second row of the Auslander–Reiten quiver.

2 Defining-Characteristic Theory

In 1979, Alperin proved the following result.

Theorem 2.1 (Alperin) Let $G = \mathrm{SL}_2(2^n)$, and let K be a field of characteristic 2. Then all simple KG-modules are algebraic.

Not long after this, Kovacs extended this result to odd primes.

Theorem 2.2 (Kovacs) Let $G = \mathrm{SL}_2(p^n)$, and let K be a field of characteristic p. Then all simple KG-modules are algebraic.

For $\mathrm{SL}_3(p)$, it was remarked as 'well-known' by Berger that the natural module is non-algebraic. Using this, it is not difficult to prove the following result.

Theorem 2.3 Let G be one of the groups

(i) $\mathrm{SL}_d(p^n)$ ($d \geqslant 3$),

(ii) $\Omega^{\pm}_{2d}(p^n)$ ($d \geqslant 4$),

(iii) $\mathrm{Sp}_{2d}(p^n)$ ($d \geqslant 3$),

(iv) $\Omega_{2d+1}(p^n)$ ($d \geqslant 3$),

(v) $\mathrm{SU}_d(p^n)$ ($d \geqslant 6$), and

(vi) ${}^3D_4(p^n)$.

Then the natural module for G is non-algebraic.

Of course, there are a few groups missing here, but before we talk about them, let's consider the exceptional groups of Lie type.

Theorem 2.4 Let (G, M) be one of the pairs

(i) $G = G_2(p^n)$, and M is the 7-dimensional natural KG-module (p odd) or the 6-dimensional natural KG-module ($p = 2$),

(ii) $G = {}^2G_2(3^n)$, and M is the 7-dimensional natural KG-module,

(iii) $G = F_4(p^n)$, and M is the 26-dimensional natural KG-module ($p \neq 3$) or the 25-dimensional natural KG-module ($p = 3$),

(iv) $G = {}^2F_4(2^n)$, and M is the 26-dimensional natural KG-module,

(v) $G = E_6(p^n)$, and M is the 27-dimensional natural KG-module,

(vi) $G = {}^2E_6(p^n)$, and M is the 27-dimensional natural KG-module,

(vii) $G = E_7(p^n)$, and M is the 56-dimensional natural KG-module, and

(viii) $G = E_8(p^n)$, and M is the 248-dimensional natural KG-module.

Then M is non-algebraic.

The remaining groups are $\mathrm{SU}_d(p^n)$ for $d = 3, 4, 5$, $\mathrm{Sp}_4(p^d)$, and $Sz(2^n)$. In these cases nothing is known in general, although for a few of small examples things are known; for example, $\mathrm{Sp}_4(2) = S_6$ has all simple modules characteristic 2 (and 3). The same is true for $\mathrm{SU}_3(3)$. For $\mathrm{SU}_5(2)$, the 10-dimensional simple module is non-algebraic, for example.

One should note that the cases where we definitely know that the simple modules are algebraic tend to be those with abelian Sylow p-subgroups. We will see something similar in the next section.

3 Non-Defining-Characteristic Theory

Here is an interesting proposition.

Proposition 3.1 Let G be one of the groups ${}^2G_2(3^n)$, and let K be a field of characteristic 2. Then all simple KG-modules are algebraic.

The groups $\mathrm{SL}_2(2^n)$, ${}^2G_2(3^n)$, $\mathrm{PSL}_2(q)$ (for $q \equiv 3, 5 \bmod 8$), and J_1 are the simple groups with abelian Sylow 2-subgroups. We have dealt with the first two collections, and know that all simple modules for these groups are algebraic. We can do one better.

Theorem 3.2 Let G be a group with an abelian Sylow 2-subgroup, and let K be a field of characteristic 2. Then all simple KG-modules are algebraic.

The groups of Lie type in the theorem above are not the hardest part of this theorem: J_1 is, and needs a computer to produce.

There are two directions one can go if one wants to expand Theorem 3.2: either to 2-blocks, and not finite groups, and to groups with abelian Sylow p-subgroups. Both directions appear fruitful; one more fruitful than the other.

Consider extending it to 2-blocks of finite groups with abelian defect groups. While it is clear that blocks with cyclic defect groups have only finitely many indecomposable modules in them, and so all (not necessarily simple) modules with cyclic vertex are algebraic. The next defect group is $C_2 \times C_2$.

Theorem 3.3 (C, Eaton, Kessar, Linckelmann) Let B be a 2-block of a finite group, and suppose that B has Klein four defect groups. Then all simple B-modules are algebraic.

Proof: (Sketch) Firstly, reduce to quasisimple groups, extended by (possibly) some automorphisms. Then sporadic and alternating groups die easily, so one reduces to trying to prove this result for the groups of Lie type in odd characteristic. For symplectic and orthogonal groups the result follows from reality of conjugacy classes, as it does for ${}^2G_2(q)$, $G_2(q)$, and ${}^3D_4(q)$. The groups ${}^2F_4(q)$ and ${}^2B_2(q)$ are obvious, so we reduce to linear and unitary groups on the one hand, and the higher-rank exceptional groups in the other.

For the groups $F_4(q)$, the blocks of interest are real. For the groups ${}^\varepsilon E_6(q)$, $E_7(q)$, and $E_8(q)$, they do not have blocks of defect 1, and using centralizers of involutions we can pin down the structure of the blocks of defect 2. This leaves the 2-blocks of defect 0 that extend to Klein four defect blocks under automorphisms. Here we note that $4 \nmid |\operatorname{Out}(E_8(q))|$, so we look at $E_7(q)$ and ${}^\varepsilon E_6(q)$. For $E_7(q)$, we have that the automorphisms are diagonal and field automorphisms, and the presence of a field automorphism can be exploited. For $E_6(q)$, it is much more complicated, and the graph automorphism can be used to great effect.

- There is a unique V_4 subgroup of $\mathrm{Out}(H/\mathrm{Z}(H))$ up to conjugation, and so we may assume that P consists of the graph automorphism, field automorphism, and their product.
- Since the graph automorphism inverts the centre of $3.E_6(q)$, and P is meant to centralize $\mathrm{Z}(H)$, we see that we may assume that $\mathrm{Z}(H) = 1$.
- The adjoint group is actually $E_6(q).3$, and we can take this group instead. (This makes the Deligne–Lusztig theory easier.) The dual group is the simply connected group $L = 3.E_6(q)$.
- By a theorem of Feit and Zuckerman, all semisimple elements of L are real. (Remember that the semisimple elements of L label the Lusztig series.)
- Let B be a block of defect zero in KH, and let M denote the unique simple module. Then M comes from a Lusztig series with (real) semisimple label s.
- Since M has trivial vertex, the centralizer of s is a torus. Hence the Lusztig series contains a single element, namely the module M.
- Finally, as s is real, the dual of M also lies in the series defined on s, and so M is self-dual, as required.

For the linear and unitary groups one has to be even more sneaky, and we cannot go into serious details here. $\square$

A common generalization of Theorems 3.2 and 3.3 is conjectural.

Conjecture 3.4 Let B be a 2-block of a finite group, and suppose that B has abelian defect groups. Then all simple B-modules are algebraic.

Moving in the other direction, let's consider other primes, and the most natural case is $p = 3$. Here, we have the following theorem.

Theorem 3.5 Let G be a simple group of Lie type such that $27 \nmid |G|$. Let B be the principal 3-block of KG, where K is a field of characteristic 3. Then all simple B-modules are algebraic.

One would like the following theorem: let G be a finite group, and suppose that $9 \mid |G|$. Then all $B_0(G)$-modules are algebraic. However, this is not true.

Example 3.6 Let G be one of the groups M_{11} and M_{23}, and let K be a field of characteristic 3. Then there are non-algebraic simple KG-modules in the principal block.

We can recover the theorem above if one excludes M_{11} and M_{23} from being composition factors. So the groups of Lie type behave much better than the sporadic groups. However, while the sporadic group problems seem to be located here, in the sense that other sporadic groups (so far) are holding up much better for other primes, when $p = 5$ there is a problem.

Example 3.7 (Kawata, Michler, Uno) Let $G = F_4(2)$ and K be a field of characteristic 5. Then there is a simple module of dimension 875823 on the second row of its Auslander–Reiten quiver. Hence it cannot be algebraic.

It's not clear whether this is a $p = 5$ thing, a $p \geqslant 5$ thing, or whether it's a $p^n \neq 9$ thing. At the moment there is not enough theory for any reasonable conjectures to be formulated.

What if the defect groups are not abelian? In this case, very little is known, and only then for $p = 2$.

Theorem 3.8 Let G be the group $\mathrm{PSL}_2(q)$, and let K be a field of characteristic 2. Then all simple KG-modules are algebraic if and only if $q \not\equiv 7 \bmod 8$, and in the case where $q \equiv 7 \bmod 8$ the two $(q-1)/2$-dimensional simple modules in the principal block are non-algebraic.

We can move up to slightly higher ranks, or rather to semidihedral Sylow 2-subgroups.

Theorem 3.9 Let K be a field of characteristic 2.

(i) Let G be the group $\mathrm{PSL}_3(q)$ for $q \equiv 3 \bmod 4$. Then all simple KG-modules are algebraic if and only if $q \equiv 3 \bmod 8$.

(ii) Let G be the group $\mathrm{PSU}_3(q)$ for $q \equiv 1 \bmod 4$. Then all simple KG-modules are algebraic.

[The only other simple group with semidihedral Sylow 2-subgroups – the group M_{11} – has algebraic simple modules in characteristic 2.] The simple groups $\mathrm{PSL}_3(q)$ and $\mathrm{PSU}_3(q)$ with wreathed Sylow 2-subgroups would be the next logical step to go to, if one were simply nibbling at the edges. That's all that's going to happen right now though, until something important and radical happens.