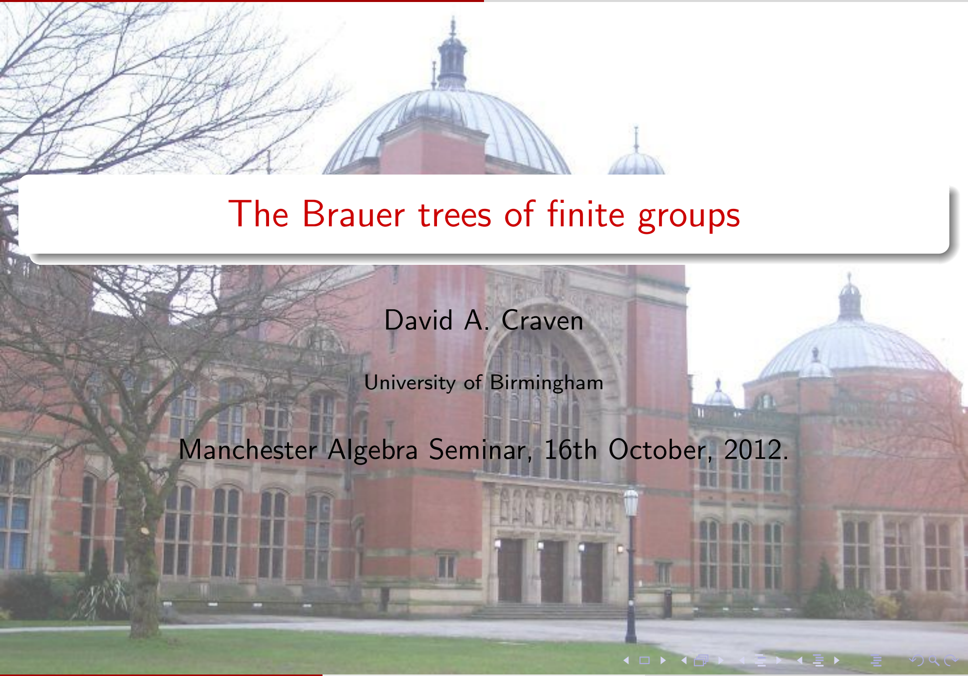

The Brauer trees of finite groups

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University of Birmingham

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Notation and Conventions

Throughout this talk,

- G is a finite group,
- p is a prime,
- K is a field of characteristic 0 and k of characteristic p (more later), and
- P is a Sylow ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

This talk is joint work with Olivier Dudas and Raphaël Rouquier.

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The first case behaves as $K = \mathbb{C}$ does. The second is much more difficult. We normally write K for a field of characteristic 0 (say $\mathbb{C}$) and k for a field of characteristic $p > 0$ (say $\bar{\mathbb{F}}_p$).

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$$kG = B_1 \oplus B_2 \oplus \cdots \oplus B_r.$$

The B_i are called blocks of kG. A large part of representation theory involves studying these blocks.

Blocks are locally controlled

Since kG is a sum of ideals, 1 can be written as a sum of elements of these ideals: $1 = e_1 + e_2 + \cdots + e_r$.

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Theorem (Brauer)

The map Br_D induces a bijection between blocks of kG with defect group D and blocks of $k\mathrm{N}_G(D)$ with defect group D.

Partitioning the kG-modules

Let M be a kG-module. On the previous slide we saw that $1 = e_1 + \cdots + e_r$ where the e_i are primitive central idempotents, so that $e_i e_j = \delta_{i,j} e_i$.

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Let $x \in G$ have p'-order. In a matrix representation of the action of x on M, the eigenvalues are p'-roots of unity, and the trace is the sum of these. Fix a monomorphism from k^* to the p'-roots of unity in K (k and K should be large enough for this to work). Map the eigenvalues over to K and add them there, giving a Brauer character, defined only on p'-elements.

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The irreducible Brauer characters $\psi_1, \dots, \psi_s$ (i.e., characters of simple kG-modules) form a basis of the class functions on the p'-elements of G. Hence every ordinary character can be written as a linear combination of the ψ_i

$$\chi = \sum a_i \psi_i.$$

The a_i are actually in $\mathbb{Z}_{\geq 0}$. If χ is irreducible then all constituents come from the same block, and χ belongs to the block as well.

Going by defect group

If the defect group of B is the trivial subgroup, then B is just a matrix algebra $M_n(k)$, mirroring the situation where the blocks of KG are all matrix algebras (Artin–Wedderburn theory). Notice that B has a single simple module, and a single associated ordinary character.

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If there is only one simple module then we take one of the ordinary characters out of the exceptional so the Brauer tree has two vertices.

Theorem

This graph is a tree with at most one exceptional node. The number of edges is equal to $s = |\mathrm{N}_G(D)/\mathrm{C}_G(D)|$, and the exceptionality is $(|D| - 1)/s$.

For a given p, there are only finitely many Brauer trees since $s \mid (p-1)$ (if we ignore the exceptionality). This raises the possibility of classifying them all, if this is even possible.

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Thus the Brauer tree is a line, with the exceptional in the middle. In fact, if G is p-soluble then the Brauer tree of any block of G is a star with exceptional in the middle. Thus our goal is achieved for p-soluble groups.

Reducing to the finite simple groups

In 1984, Walter Feit produced a reduction to the finite quasisimple groups. If T is a Brauer tree of a block then there exists a quasisimple group G and a block b of G, such that T is an unfolding of the Brauer tree of b. An unfolding of a tree is several copies of the same tree, with all exceptional vertices identified.

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Thus if we work up to unfolding then it suffices to classify the Brauer trees of the quasisimple groups.

Helpfully, there is a classification of the finite simple groups, so we can 'simply' work through all the groups on the list, classifying them as we go. In the next few slides we will summarize the work that has been done towards this.

The alternating groups

It is easy to see, since all characters of S_n are real, that the Brauer trees of S_n are lines. It is also easy to show that, if χ lies in a p-block of cyclic defect, then χ restricts to an irreducible ordinary character of A_n, so the Brauer trees of A_n are also lines.

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So alternating groups are done!

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One way to remove this obstacle is to assume that $p > 71$, in which case there is no sporadic group with a non-trivial Sylow p-subgroup. Eventually, we aim to get all of the Brauer trees for these groups.

Groups of Lie type

If G is a group of Lie type, say $G = G(q)$, then we could have that $p \mid q$, or that $p \nmid q$. If $p \mid q$ and G has a block with cyclic defect group, then $G = \mathrm{PSL}_2(q)$ and the Brauer tree is a line.

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If G is classical (i.e., $\mathrm{PSL}_n(q)$, $\mathrm{PSp}_n(q)$, $\mathrm{P}\Omega_{2n+1}(q)$, $\mathrm{P}\Omega_{2n}^{\pm}(q)$, $\mathrm{PSU}_n(q)$), then the Brauer trees are lines.

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If $p \mid |G|$ then $p \mid \Phi_d(q)$ for some d. In light of the previous slide, let us simplify matters and assume that $p > 71$. This means that p divides exactly one $\Phi_d(q)$.

The Φ_d-cyclotomic theory

Broadly speaking, if $p \mid \Phi_d(q)$ and $p' \mid \Phi_d(q')$ then the representation theory of $G(q)$ and $G(q')$ at the primes p and p' respectively are 'the same'. The unipotent characters, that are parameterized independently of q, and whose distribution into the unipotent blocks is dependent only on d.

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The representation theory of all blocks is in some sense related to unipotent blocks, although the precise mechanisms for this, and even what is precisely meant by this, remain obscure. Recently there has been much work in this direction, and we should soon understand this mechanism in much more detail.

The small exceptional groups

If G is one of $G_2(q)$, ${}^2G_2(q)$, ${}^2F_4(q)$, ${}^3D_4(q)$, or ${}^2B_2(q)$, then all Brauer trees, not just unipotent blocks, are known, by various papers which appeared mostly during the 1990s.

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- if $G = E_6(q)$, then as long as the d such that $p \mid \Phi_d(q)$ is at least 4, all blocks are known. For all primes at least 5, the Brauer trees of unipotent blocks are known. (Hiss–Lübeck–Malle)

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- If $G = F_4(q)$ or $G = {}^2E_6(q)$ then the Brauer trees of unipotent blocks are known. (Hiss–Lübeck)

This leaves the unipotent blocks of the groups $E_7(q)$ and $E_8(q)$, along with the non-unipotent blocks of several types of groups.

An example

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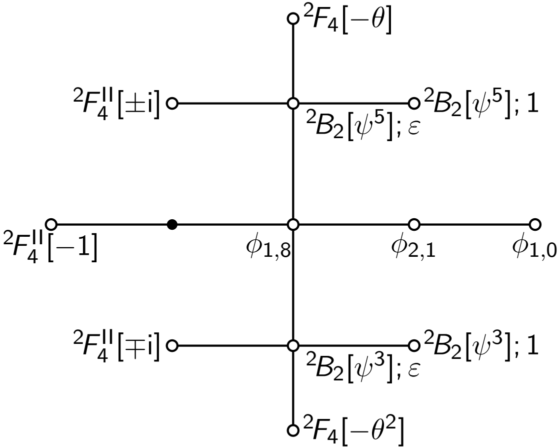

Deligne–Lusztig varieties enter

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The HLM conjecture follows from the known cohomology of the Deligne–Lusztig variety, **if** it could be proved that, over a p-adic ring $\mathbb{Z}_p$, the cohomology is torsion-free. This is definitely not true for other d, but seemed to be true for d the Coxeter number.

The HLM conjecture

The previously unknown Brauer trees of unipotent blocks were for

- ${}^2G_2$, $d = 12''$
- F_4, $d = 12$
- ${}^2F_4$,$d = 24''$
- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 3$
- E_7, all d including $d = 18$
- E_8, all d including $d = 30$

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This leaves

- ${}^2E_6$, $d = 12$, $q \not\equiv 1 \bmod 12$
- E_7, $d = 9, 10, 14$
- E_8, $d = 9, 12, 14, 15, 18, 20, 24$

The Coxeter variety for non-Coxeter primes

We can take the Deligne–Lusztig variety associated to the Coxeter torus T, and study it even when the prime p **does not** divide $|T|$. This gives us enough information that, with a few extra arguments, we get the following theorem.

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Theorem (C.–Dudas–Rouquier (2012))

The Brauer trees of all unipotent blocks with cyclic defect group, for any group of Lie type, are known.

In three cases, ${}^2F_4(q)$, $d = 12'$, $E_8(q)$ $d = 15$ and $d = 18$, we do not have the complete labelling of the vertices in the planar-embedded Brauer tree. In each case, there is a pair of cuspidal characters that cannot (yet) be distinguished. In the case of ${}^2F_4(q)$, the character labelling isn't actually well defined.

Another example

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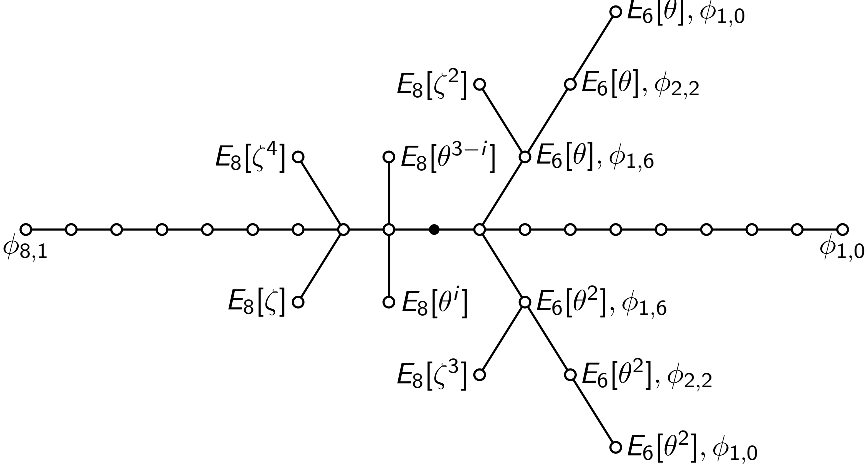

Moving to all blocks

Suppose that a principal block of a finite group G has cyclic defect group. Then G itself has cyclic Sylow p-subgroups, and the restricted structure of such groups allows us to prove the following corollary.

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What about the non-unipotent blocks for groups of Lie type? A theorem of Bonnafé and Rouquier reduces the problem to the quasi-isolated blocks, but even for $F_4(q)$ and $p \mid \Phi_3(q)$ this is difficult. At the moment this is too far, but it should eventually be soluble in the future.