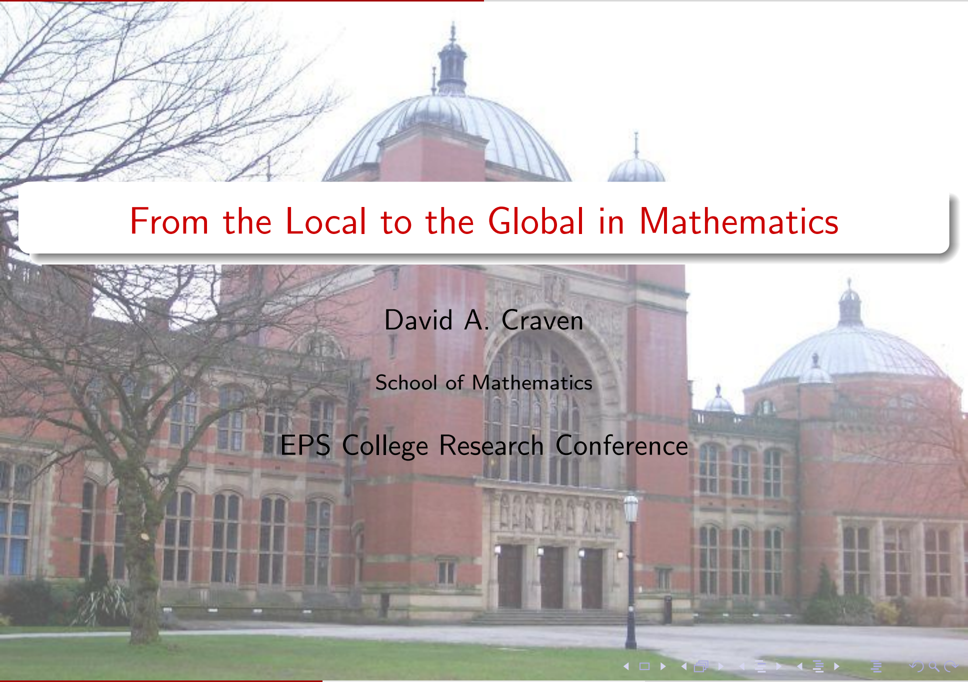

From the Local to the Global in Mathematics

David A. Craven

School of Mathematics

EPS College Research Conference

The Local-to-Global Principle

The Local-to-Global Principle

Idea

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The Local-to-Global Principle

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Global method:

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The exact relationship between these two groups is the subject of much research over the last few decades.

The Local-to-Global Principle in Mathematics

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Group Theory Briefly

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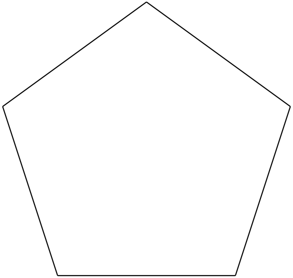
Groups come in various guises: they can be defined abstractly, as permutations of some set, like $\{1, \ldots, 15\}$, or as matrices, over the complex numbers say. Writing a group as matrices is a representation of the group.

An Example: The Dihedral Group D_5

Consider a regular pentagon:



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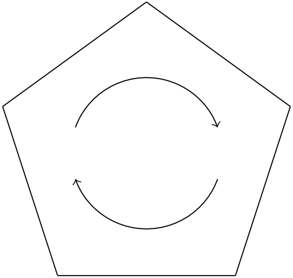

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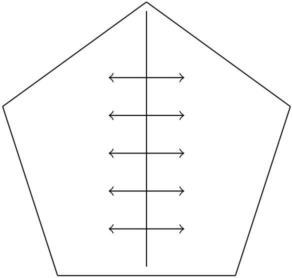

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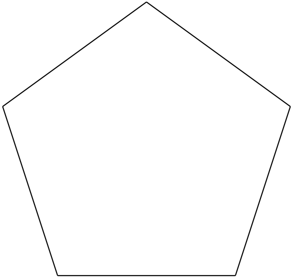

We can rotate the pentagon by multiples of $2\pi/5$, or we can reflect the pentagon in one of five lines. There are five rotations and five reflections, yielding a group of order 10. We can represent this group as rotations and reflections of the plane, so that the reflection through the vertical becomes

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Subgroups

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As an example of a subgroup, the set of rotations of the pentagon is a subgroup of the dihedral group. The set of reflections is not, because the product of two different reflections is a rotation.

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Now we need to find a group with D_5 as a 5-local subgroup.

Character Tables

D_5	1	ref	rot	rot^2
χ_1	1	1	1	1
χ_2	1	-1	1	1
χ_3	2	0	$2\cos(2\pi/5)$	$2\cos(4\pi/5)$
χ_4	2	0	$2\cos(4\pi/5)$	$2\cos(2\pi/5)$

$A_5 = I$	1	$(1,2)(3,4)$	$(1,2,3)$	$(1,2,3,4,5)$	$(5,4,3,2,1)$
χ_1	1	1	1	1	1
χ_2	3	-1	0	$-2\cos(2\pi/5)$	$-2\cos(4\pi/5)$
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Almost Theorem

For any finite group, and any prime p, the number of characters of degree not divisible by p is the same as that of its p-local subgroup. Furthermore, the number of characters whose degree has remainder $\pm i$ on division by p also is the same between the two groups.

Broué's Conjecture

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Broué's conjecture is the main direction of research in this area. Recently a lot of progress has been made in this area, and we are hopeful that we can prove this, and extend it to get a complete structural understanding of the local-global principle for the representation theory of finite groups.

So What Use Is All This Stuff?

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