Chopping and Multiplying Modules

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For the purposes of this talk, G is a finite group, K is an algebraically closed field of characteristic p, where $p \mid |G|$, and all modules are finite-dimensional.

1 Basics of Module Theory

A KG-module is simply a finite-dimensional K-vector space, with a G-action on it that satisfies some obvious conditions:

(i) $v1 = v$;

(ii) $(v + w)g = vg + wg$; and

(iii) $(\lambda v)g = \lambda(vg)$.

If V is given a basis, we can write the action of g as a matrix. For example, if $G = \langle x \rangle$, the cyclic group of order p, and $V = K^2$, then we can turn V into a KG-module by

$$x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

This example is illuminating because it demonstrates that Maschke's Theorem does not hold for modular representations.

Theorem 1.1 (Brauer) Let $l(G)$ denote the number of conjugacy classes of G consisting of elements whose order is coprime to p. Then there are exactly $l(G)$ non-isomorphic simple KG-modules.

Note that K is not necessarily required to be algebraically closed here, but if it is not algebraically closed, it has to be 'big enough'.

The *direct sum* $M \oplus N$ of two modules M and N is simply all pairs (x, y), where $x \in M$ and $y \in N$, with G acting pointwise. A *summand* of a module M is a submodule N_1, such that there exists another submodule N_2 with $N_1 \cap N_2 = 0$ and $N_1 + N_2 = M$, so that $M \cong N_1 \oplus N_2$. A module is *indecomposable* if there are no non-trivial summands. The Krull–Schmidt Theorem says that every module has an essentially unique decomposition into indecomposable summands.

If not all indecomposable modules are simple, then how many indecomposable representations are there? Let P denote a Sylow p-subgroup of G. If $P = 1$, then all indecomposable modules are simple. If P is cyclic, then not all indecomposable module are simple, but there are still only finitely many of them. If P is dihedral, quaternion or quasidihedral, then although there are infinitely many of them, they can still be classified in some meaningful way. If P is anything else, the indecomposable modules are far too complicated to be classified, and are called *wild*.

Let M and N be modules with bases X and Y. The *tensor product* $M \otimes N$ of M and N is the module whose basis is all symbols $x \otimes y$, where $x \in X$ and $y \in Y$, with group action given by

$$(x \otimes y)g = xg \otimes yg.$$

It is itself a module.

2 Algebraic Modules

The *Green ring* of KG-modules is defined to be the free abelian group on the basis set of all indecomposable KG-modules, with $M + N$ defined to be equal to $M \oplus N$, and the product of two modules defined as $M \otimes N$. Notice that not all elements of the Green ring can be thought of as modules, since they could have negative multiplicities attached; they are *virtual modules*.

The structure of the Green ring, while a commutative ring with a 1, is far from that of traditional commutative rings. For example, it is not an integral domain: in general, it has nilpotent elements. It is also in general infinite-dimensional. We can still, however, carry over some notions from algebraic number theory. One of those is algebraic modules.

A module is said to be *algebraic* if it satisfies some polynomial equation in the Green ring, with co-efficients in $\mathbb{Z}$.

Proposition 2.1 Let M be a KG-module. Then the following are equivalent:

(i) M is algebraic;

(ii) M satisfies a *monic* polynomial equation in the Green ring with co-efficients in $\mathbb{Z}$; and

(iii) there are only finitely many different indecomposable summands of the (infinite-dimensional) module
$$T(M) = M \oplus M^{\otimes 2} \oplus M^{\otimes 3} \oplus \cdots .$$

The third equivalent condition is often the easiest to use in actually deciding if modules are algebraic or not. In particular, it is very easy to use this condition to prove the following.

Lemma 2.2 Let M and N be KG-modules.

(i) M and N are algebraic if and only if $M \oplus N$ is algebraic.

(ii) If M and N are algebraic, then so is $M \otimes N$.

Hence the algebraic modules form a subring $\mathrm{Alg}(G)$ of the Green ring.

On a related note, a module is said to be *simply generated* if it is a summand of some tensor product of simple modules.

Lemma 2.3 The sum and tensor product of two simply generated modules are simply generated. Also, summands of simply generated modules are simply generated.

Again, the simply generated modules form a subring $\mathrm{SG}(G)$ of the Green ring. One natural object to study in the context of these two subrings is their intersection, $\mathrm{Alg}(G) \cap \mathrm{SG}(G)$. In the case of p-soluble groups, the answer is easy to state.

Theorem 2.4 (Berger, Feit) Let G be a p-soluble group. Then every simply generated module is algebraic. In particular, all simple modules are algebraic. Hence $\mathrm{SG}(G) \leqslant \mathrm{Alg}(G)$.

Clearly, all simply generated modules are algebraic if and only if all simple modules are algebraic. Thus we consider the question of when all simple modules are algebraic. In general, what is the kind of result that we can get here? The answer is still not known, and for example it is not even known to what extent the composition factors of a finite G affect whether the simple KG-modules are algebraic. Say that a group G has the SMA property (for a particular prime p, although this will often be elided), if all simple modules are algebraic.

Conjecture 2.5 Let G be a finite group. The G has the SMA property if and only if all composition factors of G have the SMA property.

Obviously, if a group G has the SMA property then so do all proper quotients, but this does not deal with composition factors 'at the bottom' of G. Similarly, an inductive hypothesis will deal with all non-faithful modules for a finite group G, but cannot deal with the general simple module. Very little can be said about this issue in general, at least as of yet.

3 Examples of Algebraic Modules

Here we give two classes of examples of algebraic modules.

3.1 Permutation Modules

Let M be a module. Suppose that M has a basis $X = \{x_1, \ldots, x_n\}$, such that, for all i and for all $g \in G$, we have $x_i g = x_j$ for some j. Then M is called a *permutation module*.

Lemma 3.1 Let M_1 and M_2 be two permutation modules with permutation bases X and Y respectively.

(i) The direct sum of M_1 and M_2 is a permutation module, with permutation basis $X \cup Y$.

(ii) The tensor product of M_1 and M_2 is a permutation module, with permutation basis $X \times Y$.

Notice that there are only finitely many indecomposable permutation modules, up to isomorphism. Hence we get the following.

Theorem 3.2 Any permutation module is algebraic.

3.2 Projective Modules

Recall that a free A-module is a direct sum of copies of A, thought of as a module for itself. Then a *projective module* is a summand of a free module. In particular, a projective indecomposable module is an indecomposable summand of A.

Now specialize to $A = KG$; the projective indecomposable modules are in 1-1 correspondence with simple modules, with the projective corresponding to M having socle and top isomorphic to M. They are the summands of KG, viewed as a KG-module.

Lemma 3.3 Let KG be a group algebra.

(i) If P_1 and P_2 are projective modules, then so is $P_1 \oplus P_2$.

(ii) If P is projective and M is any KG-module, then $P \otimes M$ is projective.

(iii) Summands of projective modules are projective.

(iv) If P is a submodule of a KG-module M, then P is a summand of M.

Theorem 3.4 Let P be a projective module. Then P is algebraic, and P is simply generated if and only if $\mathrm{O}_p(G) = 1$.

4 Induction and Restriction

Let H be a subgroup of G, and let M be a KG-module. We can simply think of M as a KH-module, and get the *restriction* of M to H, denoted by $M\downarrow_H$. Note tht $\dim M = \dim M\downarrow_H$. Dual to this is *induction*, which takes a KH-module N, and produces a KG-module, namely the module

$$N \otimes_{KH} KG.$$

This module has dimension $|G : H| \dim N$.

Lemma 4.1 Suppose that M is an algebraic KG-module, and let H be a subgroup of G. Then $M\downarrow_H$ is an algebraic KH-module. Conversely, suppose that N is an algebraic KH-module. Then $N\uparrow^G$ is an algebraic KG-module.

Let M denote an indecomposable KG-module. It turns out that there is a certain p-subgroup P (determined only up to conjugacy) and a certain indecomposable KP-module S such that M is a summand of $S\uparrow^G$, and that this property does not hold for any smaller subgroups Q of P. The subgroup P is called a *vertex*, and the module S is called a *source*.

Theorem 4.2 Let M be an indecomposable KG-module, and let P and S be its vertex and source. Then M is algebraic if and only if S is.

Notice that if P is cyclic, we have mentioned previously that there are only finitely many indecomposable KP-modules, and so all KP-modules are algebraic.

Example 4.3 Recall that a module is projective if and only if it is a summand of KG. It turns out that, if T denotes the trivial module for the trivial subgroup of G, then

$$KG \cong T\uparrow^G .$$

Since the trivial subgroup is certainly the smallest p-subgroup you can get, every summand of KG has vertex 1 and source T. Now, the summands of KG are simply the projective modules, and so every projective module has source T. Clearly T is algebraic (since $T \otimes T = T$), and so all projective modules are algebraic, as we asserted earlier.

It also turns out that a module is a permutation module if and only if its source is trivial (but not necessarily its vertex). This is another way to see that the permutation modules are also algebraic.

5 The groups $\mathrm{SL}_2(q)$

Now let's look at the special linear groups of dimension 2. Now unless p is the defining characteristic (i.e., $p^a = q$ for some a) or $p = 2$, the Sylow p-subgroup is cyclic, and so the group is uninteresting (and also possesses the SMA property). Let us examine the two possibilities for p in turn.

5.1 $q = p^a$

This case is often called the *defining characteristic* case, since this is the characteristic of the field over which the group is defined. A recurrent theme of defining-characteristic representations of groups of Lie type is the Steinberg Tensor Product Theorem. Let F be a field of order p^n. Then there is an automorphism σ (the *Frobenius twist*) of F, sending x to x^p. The automorphism σ obviously has order n.

Now think of, for example, $G = \mathrm{SL}_2(9)$, where $p = 3$. There is an obvious 2-dimensional 'natural' module over GF(9), namely to think of each element g of G as being represented by the matrix g itself. We can get another module by taking the matrix g and applying the Frobenius twist σ to each of its entries. This again forms a GF(9)G-module.

Theorem 5.1 (Steinberg) Let $G = G(p^a)$ be a group of Lie type (for example, $G = \mathrm{SL}_n(p^a)$). Then there is a set $\{M_1, \ldots, M_r\}$ of *fundamental modules* such that every simple module M can be written as

$$M = N_1 \otimes N_2^{\sigma} \otimes N_3^{\sigma^2} \otimes \cdots \otimes N_a^{\sigma^{a-1}},$$

where the N_i are fundamental modules.

In the case where $G = \mathrm{SL}_2(p^a)$, there are $p - 1$ different fundamental modules $1_G = M_0, M_1, M_2, \ldots, M_{p-1}$, of dimensions $1, 2, 3, \ldots, p$. [The module M_2 is the natural module, and the module M_i is given by $S^i(M_1)$, the ith symmetric power.]

The point is, since the tensor product of two algebraic modules is algebraic, and clearly M is algebraic if and only if M^{σ} is, then to prove that $\mathrm{SL}_2(p^a)$ has the SMA property (for the prime p), it suffices to show that the fundamental modules are algebraic. In fact, we need to understand the general tensor products of fundamental modules, since other simple modules appear in the tensor powers of the fundamental modules. We begin by analyzing the tensor product of two arbitrarily-chosen M_i.

Proposition 5.2 Suppose that $0 \leqslant j \leqslant i$ are integers with $i + j < p$. Then

$$M_i \otimes M_j = M_{i-j} \oplus M_{i-j+2} \oplus \cdots \oplus M_{i+j}.$$

Proposition 5.3 Suppose that $0 \leqslant i \leqslant j < p - 1$ with $i + j \geqslant p$. Write $a = p - (i + j)$. Then there exists indecomposable modules W_α with $0 \leqslant \alpha \leqslant p - 2$, such that

$$V_i \otimes V_j = V_{j-i} \oplus V_{j-i+2} \oplus \cdots \oplus V_{2p-(j+i+4)} \oplus \begin{cases} V_{p-1} \oplus W_1 \oplus W_3 \oplus \cdots \oplus W_a & a \text{ is odd} \\ W_0 \oplus W_2 \oplus \cdots \oplus W_a & a \text{ is even} \end{cases}.$$

Furthermore, each W_α is self-dual, uniserial of length 3, and has socle (and head) isomorphic with $V_{p-\alpha-2}$ and heart (radical modulo socle) isomorphic with $V_\alpha \otimes V_1^\sigma$.

Proposition 5.4 Suppose that $0 \leqslant i \leqslant p - 1$. Write $a = i - 1$. Then we have

$$V_i \otimes V_{p-1} = \begin{cases} V_{p-1} \oplus W_1 \oplus W_3 \oplus \cdots \oplus W_a & a \text{ is odd} \\ W_0 \oplus W_2 \oplus \cdots \oplus W_a & a \text{ is even} \end{cases}.$$

This means that if we can understand the decompositions of the modules $W_i \otimes V_j$, and that these tensor products lead to no more indecomposable modules, then we have shown that all modules are algebraic. However, it is slightly more complicated than that.

Theorem 5.5 Suppose that $0 \leqslant i \leqslant p - 2$. Then

$$W_i \otimes V_{p-1} = \begin{cases} (V_{p-1} \oplus W_1 \oplus W_3 \oplus \cdots \oplus W_{i-1}) \otimes V_1^\sigma \oplus A_0 \oplus A_2 \oplus \cdots \oplus A_{p-i-3} & i \text{ is even} \\ (W_0 \oplus W_2 \oplus \cdots \oplus W_{i-1}) \otimes V_1^\sigma \oplus 2V_{p-1} \oplus A_1 \oplus A_3 \oplus \cdots \oplus A_{p-i-3}) & i \text{ is odd} \end{cases},$$

where A_i is a module with the same Brauer character as $2 \cdot W_i$. Furthermore, A_i is self-dual, has socle length 3, has socle and top both isomorphic with $2 \cdot V_{p-i-2}$, and has heart isomorphic with $2 \cdot V_i \otimes V_1^\sigma$.

Unfortunately, the last part of this result remains conjectural.

Conjecture 5.6 $A_i = 2 \cdot W_i$.

Putting all of this together, we get the following theorem.

Theorem 5.7 Assume Conjecture 5.6. Then every tensor product $V_i \otimes V_j \otimes V_k$ of three fundamental modules can be written in the form

$$A \oplus B \otimes V_1^\sigma,$$

where A and B are direct sums of tensor products of two fundamental modules.

This theorem can be easily used to establish the general SMA property for $\mathrm{SL}_2(q)$ in defining characteristic. However, Conjecture 5.6 has so far resisted all attempts at resolution.

5.2 $p = 2$

Here the situation becomes more complicated. Firstly, it depends on the congruence class of q modulo 8. Roughly speaking, for $q \equiv 1, 3, 5 \mod 8$, the group does have the SMA property, but for $q \equiv 7 \mod 8$, it does not. The results are more complicated here, so we restrict ourselves to the case where $q \equiv 7 \mod 8$, where things are more interesting. To examine this case, we need so-called endo-trivial modules, which we will deal with in the next section.

Broadly speaking, there are three types of non-trivial simple module for $\mathrm{PSL}_2(q)$ (which has roughly the same representation theory as $\mathrm{SL}_2(q)$) in characteristic 2: the two simple modules of dimension $(q-1)/2$; the modules of dimension $q \pm 1$ whose projective cover is uniserial; and the modules of dimension $q \mp 1$, which are projective. The last two types are certainly algebraic, and so the question of algebraicity rests on the two simple modules of dimension $(q-1)/2$. If $q \equiv 3 \mod 8$ then these modules are certainly algebraic; if $q \equiv 5 \mod 8$, then various arguments (should) easily prove that the two simple modules are algebraic; if $q \equiv 1 \mod 8$, then nothing much is really known, although for $q = 9, 17, 25, 41$ they are algebraic; and if $q \equiv 7 \mod 8$, then they certainly are not.

6 Endo-trivial Modules

A module M is said to be *endo-trivial* if $\mathrm{End}_K(M)$ is of the form $K \oplus P$, where P is a projective module. The concept came from p-groups: in this setting, if G is a p-group, and M a KG-module, then M is endo-trivial if $\mathrm{End}_K(M)$ is the sum of K and various copies of KG. Now, we use the adjointness of Hom and $\otimes$ to get a nicer description of endo-trivial modules. Recall that

$$\mathrm{Hom}(A \otimes B, C) = \mathrm{Hom}(A, C \otimes B^*).$$

Then

$$\mathrm{End}_K(M) = \mathrm{Hom}(M, M) = \mathrm{Hom}(K, M \otimes M^*) = M \otimes M^*.$$

Lemma 6.1 (Benson, Carlson) Suppose that M and N are KG-modules. Then K is a summand of $M \otimes N$ if and only if p does not divide $\dim M$ and $N \cong M^*$.

Thus if $\dim M$ is coprime to p, then $\mathrm{End}(M)$ always contains a summand isomorphic with K.

Lemma 6.2 Suppose that M is an endo-trivial module. Then M can be written as $M = N \oplus P$, where N is an indecomposable module and P is projective.

This indecomposable component of an endo-trivial module is often called the *cap* of the module. Write $\hat{M}$ for this cap.

Proposition 6.3 The tensor product of two endo-trivial modules is endo-trivial.

To see this, remember that if M and N are endo-trivial, then $M \otimes M^*$ and $N \otimes N^*$ are of the form K plus projective, and we can see that

$$(M \otimes N) \otimes (M \otimes N)^* = K \oplus P,$$

for some projective module P.

Notice also that if M is endo-trivial, then so is M^*, and that $M \otimes K = M$. Now, to get a group out of this,we need to remove all reference to projectives, and define, for two self-capped endo-trivial modules,

$$M \cdot N = \widehat{M \otimes N}.$$

Now we get a group $T(G)$. This group is certainly abelian, and clearly countable. In fact, it is finitely generated as well. We can ask questions about its torsion subgroup and its torsion-free rank (or Betti number).

Theorem 6.4 (Alperin, 2001) Let G be a finite p-group. Write $m_p(G)$ for the p-rank of G (rank of largest elementary abelian subgroup of G). Write $b(G)$ for the Betti number of $T(G)$. Then

(i) If $m_p(G) = 1$ then $b(G) = 0$;

(ii) if $m_p(G) = 2$ then $b(G)$ is the number of conjugacy classes of maximal elementary abelian subgroups; and

(iii) if $m_p(G) \geqslant 3$ then $b(G)$ is 1 plus the number of conjugacy classes of maximal elementary abelian subgroups of order p^2.

In particular, suppose that G is a non-cyclic abelian p-group. Then $T(G)$ is isomorphic with $\mathbb{Z}$.

Determining the torsion subgroup is more difficult.

Theorem 6.5 (Carlson, Thévenaz, 2005) Suppose that G is a finite p-group that is not cyclic, quaternion or quasidihedral. Then the torsion subgroup $T_t(G)$ of $T(G)$ is trivial.

If G is cyclic of order 2, then $T_t(G)$ is also trivial, whereas if G is cyclic of order 3 or more, then $T_t(G)$ has order 2. Finally, $T_t(\mathrm{SD}_{2^n}) = C_2$ and $T_t(Q_{2^n}) = C_2 \times C_4$.

We can obviously also see the following.

Proposition 6.6 Let M be an endo-trivial module. Then M is algebraic if and only if $\hat{M}$ lies in $T_t(G)$.

Now let G be an arbitrary finite group, and suppose that M is endo-trivial. Then $M \downarrow_H$ is endo-trivial for any subgroup H, and in particular, when H is a Sylow p-subgroup. Thus information about endo-trivial modules for p-groups can be used to piece together information for M.

Proposition 6.7 Suppose that $q \equiv 3 \mod 4$, and let M denote one of the two simple modules for $G = \mathrm{PSL}_2(q)$ of dimension $(q-1)/2$. Then M is endo-trivial.

This is easy to see from the composition factors of $M \otimes M^*$. In particular, suppose firstly that $q \equiv 3 \mod 8$. Then $(q-1)/2 \equiv 1 \mod 4$. Now if q has this form, then G contains Sylow 2-subgroups isomorphic with V_4, and so of order 4. In particular,

$$\dim M = 1 + 4n$$

for some n. Since we said earlier that V_4 has no torsion endo-trivial modules, for M to be endo-trivial, the cap of its restriction must be K, which the dimensions allow it to be.

Now suppose that $q \equiv 7 \mod 8$. This time, $(q-1)/2 \equiv 3 \mod 4$, and so there is no possiblity for the restriction of M to a Sylow to be of the form $K \oplus P$, where P is projective. Now the Sylow 2-subgroup of $\mathrm{PSL}_2(q)$ is dihedral, and so again there are no torsion endo-trivial modules, so M is in this case torsion-free endo-trivial, and so in particular not algebraic.

7 Larger Matrix Groups

Since $\mathrm{PSL}_2(7) \cong \mathrm{GL}_3(2)$, and we said before that the 3-dimensional module for $\mathrm{PSL}_2(7)$ in characteristic 2 is not algebraic, we find that the natural module for $\mathrm{GL}_3(2)$ is not algebraic.

Theorem 7.1 Let M denote the natural module of $\mathrm{GL}_3(p^n)$ over a field of characteristic p. Then M is not algebraic.

This well-known result can be modified so that we can get a similar result for $\mathrm{SL}_3(p^n)$.

Theorem 7.2 Write M for the 3-dimensional natural module for $\mathrm{SL}_3(p)$. Suppose that G is a finite group, and that N is a simple KG-module. Finally, suppose that H is a subgroup of G isomorphic with $\mathrm{SL}_3(2)$, and that $N \downarrow_H$ contains M as a summand. Then N is not algebraic.

With this tool, we can assault the various other groups of Lie type.

7.1 Special Linear Groups

Firstly, let $G = \mathrm{PSL}_n(q)$, and write $q = p^a$, where p is a prime. In this case, things are reasonably well-understood. Since we have described the situation for $n = 2$, we will assume in what follows that $n \geqslant 3$. Firstly, we easily see the following.

Proposition 7.3 Let $n \geqslant 3$, and let M denote the natural module for G over $\mathrm{GF}(q)$. Then M is not algebraic.

This shows that no groups containing $\mathrm{PSL}_n(q)$, for any n, and any power q of p, have the SMA property over a splitting field of characteristic p.

In non-defining characteristic, things get more unclear, since there is no natural module to consider. Indeed, we can even have a group G that contains a subgroup H, where G has the SMA property yet H does not, as the example

$$\mathrm{PSL}_2(7) \leqslant \mathrm{PSL}_2(49)$$

in characteristic 2 shows.

7.2 Symplectic Groups

Write Q_m for the $m \times m$ matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and R_m for the matrix

$$\begin{pmatrix} 0 & Q_m \\ -Q_m & 0 \end{pmatrix}.$$

then the symplectic group $\mathrm{Sp}_{2n}(q)$ is the group of all $2n$-dimensional matrices A that satisfy $A^T R_m A = R_m$. We will find a nicely-embedded subgroup of $\mathrm{Sp}_{2n}(q)$, for $n \geqslant 3$, that is isomorphic with $\mathrm{SL}_3(q)$, hence showing that the natural module for $\mathrm{Sp}_{2n}(q)$ is not algebraic.

Proposition 7.4 Suppose that $G = Sp_{2n}(q)$, where $n \geqslant 3$, and write $q = p^a$, where p is a prime. Write M for the natural module for G over $\mathrm{GF}(q)$. Then M is not algebraic.

This does leave the case of $\mathrm{Sp}_4(q)$, where we have no subgroup isomorphic with $\mathrm{SL}_3(p)$, even badly-embedded ones. Since $\mathrm{Sp}_4(2)'$ is isomorphic with $\mathrm{PSL}_2(9)$, we see that this group has the SMA property, but this offers little in the way of help to the other symplectic groups of even characteristic type, and certainly not to the case where p is odd. Even the isomorphism between $\mathrm{PSp}_4(3)$ and $\mathrm{SU}_4(2)$ does not help, since our understanding in the low-rank unitary groups is also inadequate.

The Sylow 2-subgroup of $\mathrm{Sp}_4(2^n)$ has an elementary abelian subgroup of order 2^{2n}, if that helps, and the natural module, restricted to this centre, is faithful and (I think) has whole vertex, and so is not Ω-periodic.