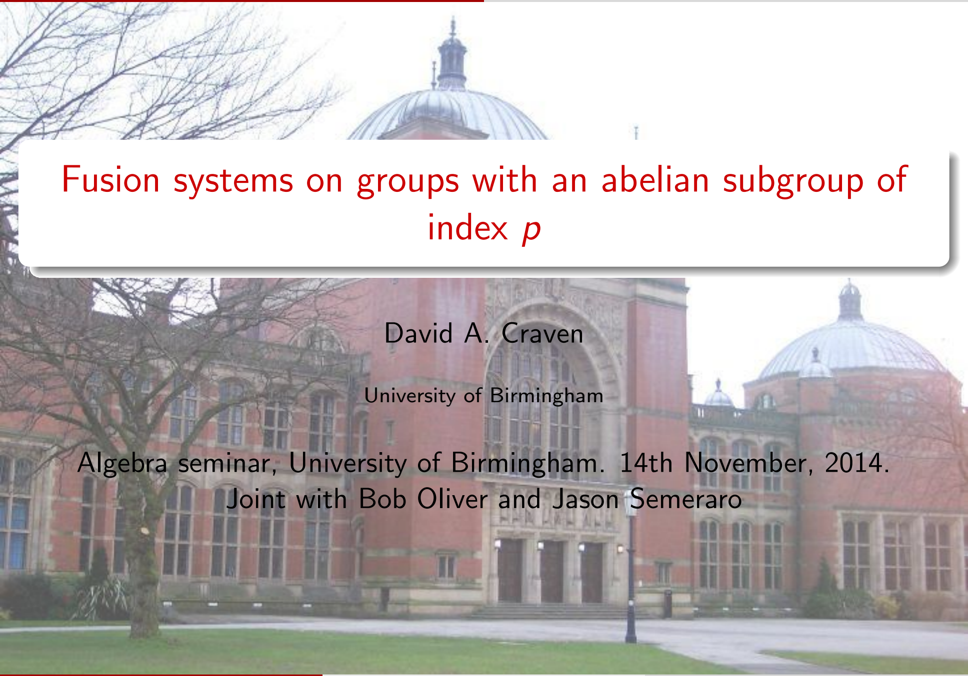

Fusion systems on groups with an abelian subgroup of index p

David A. Craven

University of Birmingham

Algebra seminar, University of Birmingham. 14th November, 2014.
Joint with Bob Oliver and Jason Semeraro

Fusion systems

A quick review about fusion systems. Let G be a finite group with Sylow p-subgroup S. The fusion system $\mathcal{F}_S(G)$ is a category whose objects are all subgroups of S, and whose morphisms are all injective maps $c_g : P \to Q$ induced by conjugation by elements $g \in G$.

There is a general definition of a fusion system though, which takes place on any p-group. If S is a p-group then a fusion system $\mathcal{F}$ on S is a category whose objects are all subgroups of S, and whose morphisms satisfy the following:

- $\mathcal{F}_S(S) \subseteq \mathcal{F}$;
- if $\phi : P \to Q$ in $\mathcal{F}$ is an isomorphism (i.e., $|P| = |Q|$) then $\phi^{-1} : Q \to P$ lies in $\mathcal{F}$;
- if $\phi : P \to Q$ is a morphism in $\mathcal{F}$ then the isomorphism $\phi : P \to P\phi$ lies in $\mathcal{F}$.

Saturated fusion systems

There are many fusion systems on a given finite p-group, too many to have any kind of structure. We need to introduce two more axioms, based on finite groups, to make our fusion systems behave well.

If S is a Sylow p-subgroup of G, then $\mathrm{Out}_G(S)$ is a p'-group, or equivalently, $\mathrm{Inn}(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_G(S)$.

This leads to the statement about a saturated fusion system, namely that $\mathrm{Inn}(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$. This is part of a more general statement, which is related to induction and extremal subgroups.

A subgroup P of S is extremal (in S) if $\mathrm{N}_S(P)$ is a Sylow p-subgroup of $\mathrm{N}_G(P)$. Every subgroup of S is G-conjugate to an extremal subgroup of S.

If P is extremal, then $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_G(P)$.

Extremal subgroups

Let P be a subgroup of S and Q be an extremal conjugate of P. If $P^g = Q$, then $\mathrm{N}_S(P)^g \leq \mathrm{N}_G(Q)$. In general it will not be contained in $\mathrm{N}_S(Q)$, however. But since $\mathrm{N}_S(Q) \in \mathrm{Syl}_p(\mathrm{N}_G(Q))$, there exists $x \in \mathrm{N}_G(Q)$ such that $\mathrm{N}_G(P)^{gx} \leq \mathrm{N}_S(Q)$, and also $P^{gx} = Q$.

If P is conjugate to an extremal Q, then there exists g such that $P^g = Q$ and $\mathrm{N}_S(P)^g \leq \mathrm{N}_S(Q)$.

More generally, if $c_g : P \to Q$ is a conjugation map, then this induces a map $c_g : \mathrm{Aut}(P) \to \mathrm{Aut}(Q)$, so we can construct the subgroup $A_g = \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(Q)^{c_g^{-1}}$. Let N_g denote the preimage of A_g in $\mathrm{N}_S(P)$. N_g is the largest possible subgroup of $\mathrm{N}_S(P)$ that we could conjugate by g and stay within S.

We have to replace g by gx, this time for some $x \in \mathrm{C}_G(Q)$ (so c_g doesn't change) but then $N_g^{gx} \leq S$.

The saturation axiom

A fusion system will be saturated if every $\mathcal{F}$-conjugacy class of subgroups contains an 'extremal' member. What does that mean? We have seen the two conditions needed for E to be extremal:

1. $\mathrm{Aut}_S(E)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(E)$;
2. if $\phi : P \to E$ is an isomorphism in $\mathcal{F}$, then ϕ is the restriction of a map $\bar{\phi} : N_\phi \to \mathrm{N}_S(E)$, where N_ϕ is the preimage in $\mathrm{N}_S(P)$ of the subgroup A_ϕ defined before, namely

$$A_\phi = \mathrm{Aut}_S(P) \cap \mathrm{Aut}_S(E)^{\phi^{-1}}.$$

A fusion system is saturated if every $\mathcal{F}$-conjugacy class contains an extremal member.

Alperin's fusion theorem

So what is the point of extremal members? The Sylow subgroup condition tries to make sure we don't have too many p-elements for automorphisms, which would definitely break coming from a finite group. The extension axiom allows us to do induction. We will illustrate this by proving a theorem.

Theorem (Alperin's fusion theorem)

If $\mathcal{F}$ is a saturated fusion system on a finite p-group S, then $\mathcal{F}$ is generated as a category by $\mathrm{Aut}_{\mathcal{F}}(E)$ for E an extremal, centric, radical subgroups (and allowing for restrictions). (Note that S is one of these.)

Thus we are saying that every map in a fusion system can be written as a chain of (restrictions of) automorphisms of extremal, centric radical subgroups (whatever they are).

Proof of Alperin's theorem

Let's prove it. Let $\phi : P \to Q$ be an isomorphism in a saturated fusion system $\mathcal{F}$. Choose an extremal conjugate E of P (and also Q), and choose maps $\alpha : P \to E$ and $\beta : Q \to E$ such that $N_\alpha = \mathrm{N}_S(P)$ and $N_\beta = \mathrm{N}_S(Q)$. Remember we can do this!

Since E is extremal, we get maps $\bar{\alpha} : \mathrm{N}_S(P) \to \mathrm{N}_S(E)$ and $\bar{\beta} : \mathrm{N}_S(Q) \to \mathrm{N}_S(E)$. **By induction on** $|S : P|$, we can assume that $\bar{\alpha}$ and $\bar{\beta}$ are good, so therefore are α and β. So ϕ is good if and only if $\alpha^{-1}\phi\beta \in \mathrm{Aut}_{\mathcal{F}}(E)$ is. So we can assume $\phi \in \mathrm{Aut}_{\mathcal{F}}(E)$.

Since E is extremal and N_ϕ always contains $\mathrm{C}_S(E)$, if $\mathrm{C}_S(E) \not\leq E$ then we can extend ϕ to $E\mathrm{C}_S(E)$, and by induction on $|S : E|$ we are good again. Thus we can assume that $\mathrm{C}_S(E) \leq E$, i.e., E is centric.

If $N_\phi > E$ then we are good by induction. But that is equivalent to saying that $A_\phi > \mathrm{Inn}(E)$. Since A_ϕ is the intersection of two Sylow p-subgroups (it is $\mathrm{Aut}_S(E) \cap \mathrm{Aut}_S(E)^{\phi^{-1}}$) we see that if $\mathrm{Inn}(E) < O_p(\mathrm{Aut}_{\mathcal{F}}(E))$ then we can extend ϕ and we are done. Thus E is radical. And we are done.

So why fusion systems?

So why bother with fusion systems? Five possible reasons, with various degrees of usefulness and truth.

1. The fusion system is related to a topological object called the (p-completed) classifying space, $BG_p^\wedge$, so topologists like it.
2. There are exotic fusion systems, i.e., fusion systems that do not come from finite groups. More later!
3. It potentially might help simplify the proof of the classification of the finite simple groups.
4. Every block of a finite group has a fusion system attached, and it is believed that these are all group fusion systems. This could help understand particularly block cohomology and Alperin's weight conjecture.
5. Some new theorems about finite groups can be proved, for example the statement that if A and B are subgroups of S that are strongly closed in S, so is AB.

The search for exotic systems

Exotic fusion systems seem to offer a glimpse into what finite simple groups that don't exist should look like. More or less all exotic fusion systems are simple, and many fusion systems of simple groups (at least for p small, as we shall see) are themselves simple.

The 'simplest' exotic systems were found by Ruiz and Viruel, and are on the extraspecial group 7_+^{1+2} of exponent 7. Others have been found, for example by Solomon on Sylow 2-subgroups of $\mathrm{Spin}_7(r)$ for r odd (the only known simple exotic systems) and by on certain 3-groups of maximal class by Díaz–Ruiz–Viruel. Another set of exotic fusion systems were constructed by Clelland and Parker, using modules for $\mathrm{GL}_2(p)$.

What the Ruiz–Viruel and the Clelland–Parker examples have in common is that the Sylow p-subgroup S in both cases possesses an abelian subgroup A of index p.

Minimal examples?

If S is abelian, then Alperin's fusion theorem, which states that every map in $\mathcal{F}$ is a product of (restrictions of) automorphisms of subgroups that contain their own centralizer, proves that every map in S is a restriction of an automorphism of S.

In other words, if H is a p'-group of automorphisms of S, then we can construct the group $S \rtimes H$, and $\mathcal{F}_S(S \rtimes H)$ is a saturated fusion system on S, and all saturated fusion systems on S arise in such a way.

If however, the abelian subgroup is maximal, then we have lots of examples where this is not the case, for example $G = S_{p^2}$, where the Sylow p-subgroup is $C_p \wr C_p$, or $\mathrm{GL}_p(q)$ for $p \mid (q-1)$, or the Monster at $p = 13$, and so on.

It therefore seems like a good idea to 'classify' (in a suitable sense) all saturated fusion systems on p-groups with an abelian subgroup of index p.

The reduction

Let S be a finite p-group and let A be an abelian subgroup of S of index p. Suppose that $\mathcal{F}$ is a saturated fusion system on S. Suppose that A is elementary abelian. Since $\mathrm{Aut}_{\mathcal{F}}(A) \leq \mathrm{GL}_n(p)$ for $|A| = p^n$, we have that $G = \mathrm{Aut}_{\mathcal{F}}(A)$ possesses an $\mathbb{F}_pG$-module of dimension n. Furthermore, since $\mathrm{Aut}_S(A) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(A))$ (A is extremal) we have that a Sylow p-subgroup of G has index p. Let x be an element of order p.

The analysis of possible fusion system structures, in particular with certain extra conditions on the structure of $\mathcal{F}$ to make it reasonable to classify them, gives us a few key facts about this $\mathbb{F}_p(G)$-module. Let U denote a Sylow p-subgroup of G, of order p.

1. $|\mathrm{Aut}_G(U)| = p - 1$
2. The action of x on A has a single non-trivial Jordan block.
3. A has no trivial quotients, i.e., $[G, A] = A$.
4. $C_G(A) \leq [U, A]$, which is slightly weaker than A having no trivial submodules.

Can these modules exist?

If $G = S_n$ for $n \leq p < 2n$ then the Sylow p-subgroup of G has order p, and if M denotes the non-trivial factor in the permutation module, then M satisfies the second, third and fourth properties. Clearly G satisfies the first, so we get an example.

Let G have a Sylow p-subgroup U of order p, and let A be an $\mathbb{F}_p G$-module. We say that A is inactive if U acts on A with only one non-trivial indecomposable summand, and A is completely inactive if $\mathrm{Aut}_G(U)$ has order $p-1$. We want to understand completely inactive modules.

The first thing to notice is that inactivity is inherited by restriction to subgroups, duality, submodules and quotients. Thus if G has no non-trivial simple (completely) inactive modules, and no self-extensions of the trivial module (e.g., if G is simple), then it has no (completely) inactive modules. And neither does any group containing G.

Understanding G

Suppose that A is a (faithful) simple inactive module of dimension n, yielding an embedding of G into $\mathrm{GL}_n(q)$. Suppose that $G/Z(G)$ is not an almost simple subgroup of $\mathrm{PGL}_n(q)$. This means that G falls into one of a few geometrically defined classes of maximal subgroups, e.g., parabolic subgroups, direct products of GL_ms, wreath products, etc.

As A is simple, this gets rid of things like parabolics and products of groups. If A is not absolutely irreducible then the action of U on A would have multiple non-trivial Jordan blocks, and the same if A were writeable as $X \otimes Y$ for X, Y of dimension at least 2. Thus A is not in extension type subgroups or wreath products.

We continue like this until $G \leq C_{d-1} \wr S_n$ is a collection of monomial matrices, a couple of central products inside extraspecial type maximal subgroups, or is almost simple (modulo the centre). Thus we want to understand completely inactive modules for almost simple groups.

$\mathrm{GL}_2(p)$

Since U has order p, if G is Lie type in defining characteristic then G is of type $\mathrm{PSL}_2(p)$. For $\mathrm{GL}_2(p)$ there are simple modules of dimension $1, \dots, p$, and each of these is completely inactive. These yield the Clelland–Parker examples.

However, there are more modules for $\mathrm{GL}_2(p)$. If M is a module of dimension $i > 1$, then M has extensions with two other modules N_1 and N_2, of dimensions $p + 1 - i$ and $p - 1 - i$. This yields indecomposable modules of dimension $p + 1$ and $p - 1$, both completely inactive also. Apart from a couple of modules with 1-dimensional socle, these are all completely inactive modules for G.

The indecomposable modules of dimension $p - 1$ yield new, exotic fusion systems, whereas those of dimension $p + 1$ fail a technical condition that I haven't told you about, which is satisfied whenever $\dim A \leq p$.

Alternating and sporadic groups

For alternating and sporadic groups, there is a useful result that we can apply that will make our lives much easier.

Proposition

If a simple group G is either of alternating or sporadic type, and $p > 3$ divides $|G|$, then G is generated by two elements of order p.

This is important: if M is an inactive module then the socle of the action of an element x of order p has codimension at most $p - 1$ (since the non-trivial block has dimension at most p). If $G = \langle x, y \rangle$ then the intersection of $C_M(x)$ and $C_M(y)$ has codimension at most $2p - 2$.

Thus if M is simple then $\dim M \leq 2p - 2$. If $\dim M \geq 2p$ then M has at least two trivial submodules, and a fact about groups with cyclic Sylow p-subgroup is that if A is simple and B is indecomposable, then $\mathrm{Hom}(A, B)$ is at most 1-dimensional.

For groups of Lie type in non-defining characteristic, it looks as if, for $p > 5$ dividing $|G|$, they are also generated by two elements of order p. However, we are some way from proving this statement, so we cannot use it.

We need another way to bound the dimension of an inactive module.

Proposition

If $U \in \mathrm{Syl}_p(G)$ and $\mathrm{C}_G(U)$ is abelian, then the dimension of any inactive module is at most $2p - 1$.

This follows from the theory of canonical characters, which implies that there are at most $(p-1)\chi(1)$ trivial summands in the restriction of an inactive module to U, where $\chi \in \mathrm{Irr}(C_G(U))$.

Now, if we could only find a way to make the centralizer $C_G(U)$ abelian.

Induction to the rescue

Obviously the centralizer isn't abelian in all cases. But we can set up an induction using the following result.

Proposition

Suppose that $G = G(q^{\delta_G})$ is a group of Lie type. If $U \in \mathrm{Syl}_p(G)$ has order p then either $C_G(U)$ is abelian (p is regular semisimple) or there exists $H = H(q^{\delta_H})$ a subgroup of G such that $U \leq H$, $C_H(U)$ is abelian and $\mathrm{Aut}_G(U) = \mathrm{Aut}_H(U)$.

As an example, if $G = \mathrm{GL}_n(q)$ and $p \mid \Phi_d(q)$, then $H = \mathrm{GL}_d(q)$ or $H = \mathrm{GL}_{d+1}(q)$ will work.

Thus we now simply have to construct all modules for groups of Lie type of dimension at most $2p-1$, where $p \mid \Phi_d(q) \mid q^d - 1$, and where $|\mathrm{Aut}_G(U)|$ is of order at most $4dt$ where t is the maximal size of a graph automorphism (this follows from knowledge of normalizers of Φ_d-tori in Lie type groups).

All the modules

So $p \mid \Phi_d(q^t) \mid q^{td} - 1$ and $|\mathrm{Aut}_G(U)| \leq 4dt$.

The twin statements $p - 1 \geq 4dt$ and $p \leq q^{td} - 1$ already put strong conditions on p, q and d. Throw in Landazuri–Seitz lower bounds on dimensions of modules for groups of Lie type, e.g., $\dim M \leq q^{(n-1)t} - 1$ for $\mathrm{GL}_n(q^t)$ and we get a finite, and small, list of possibilities.

Assume G is not alternating or $\mathrm{PSL}_2(p)$. We have one of:

1. $G = \mathrm{SL}_2(8) : 3 = {}^2G_2(3)$ or $G = 6 \cdot \mathrm{PSL}_3(4)$ and $p = 7$;
2. $G = \mathrm{PSU}_3(3).2 = G_2(2)$ or $G = 6_1 \cdot \mathrm{PSU}_3(4).2_2 = G_{34}$ and $p = 7$;
3. $G = \mathrm{PSU}_3(4) : 4$ and $p = 13$;
4. $G = \mathrm{PSU}_4(2) = \mathrm{PSp}_4(3)$ and $p = 5$;
5. $G = \mathrm{PSU}_5(2).2$ and $p = 11$;
6. $G = \mathrm{Sp}_4(4).4$ and $p = 17$;
7. $G = \mathrm{Sp}_6(2)$ and $p = 5, 7$ or $G = 2 \cdot \mathrm{Sp}_6(2)$ and $p = 7$;
8. $G = 2 \cdot \Omega_8^+(2)$ and $p = 7$;
9. $G = G_2(3).2$ or $G = {}^2B_2(8) : 3$ and $p = 13$.

Do all of these give exotic fusion systems?

No.

The ones of dimension at most p do, and there are a lot of those, but if the dimension is more than p then there is another technical condition on the action of $\mathrm{Aut}_G(U)$ on the socle of A and on U that needs to be satisfied. This fails for (for example) $6 \cdot \mathrm{Suz}$ and $p = 11$, where there is a 12-dimensional module, but is satisfied by the group $(3 \times 2_+^{1+6}) : S_8$, which has an 8-dimensional simple module in characteristic 7.

The complete list of groups, primes and modules that yield exotic fusion systems is now known, and like the group $(3 \times 2_+^{1+6}) : S_8$ above, there are new exotic simple fusion systems on there.