Units in Group Rings

Joint with Peter Pappas

David A. Craven

University of Oxford

Westfälische Wilhelms-Universität Münster, 13th December, 2010

Conjectures in Topology and Algebra

Let R be a commutative (regular) ring and let G be a group with no elements of finite order (i.e., torsion free).

There are many conjectures in both algebra and topology regarding G and the group ring RG. For example:

Topology	Algebra
Farrell–Jones conjecture	Zero divisor conjecture
Baum–Connes conjecture	Kaplansky conjecture
Novikov conjecture	Unit conjecture

Topological Conjectures

(We stick to the case where G is torsion free for simplicity.)

Conjecture (Farrell–Jones)

The assembly map

$$H_n(BG; \mathbf{K}(R)) \to K_n(RG)$$

is an isomorphism for $n \in \mathbb{Z}$.

For low dimensions this is much easier to state, as $K_n(RG) = 0$ for $n \leq -1$, $K_0(R) \cong K_0(RG)$ and $G_{\mathrm{ab}} \otimes_{\mathbb{Z}} K_0(R) \oplus K_1(R) \cong K_1(RG)$.

Specializing to the case $R = \mathbb{Z}$ and $n = 1$, we get the following statement.

Conjecture

Let G be a torsion-free group. Then $\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm G\} = 0$.

A Toy Example

Let K be a field of characteristic $p \geq 0$ and let $S = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the Laurent polynomial ring with n variables.

The ring S is naturally isomorphic to the group ring KG, where $G \cong \mathbb{Z}^n$.

S has the following properties:

- S has no zero divisors.
- The only idempotents in S are 0 and 1.
- If $x \in S$ is invertible then $x = \lambda \cdot g$ for $\lambda \in K$ and $g \in G$.
- (Much harder) $\mathrm{Wh}(G) = 0$.

Removing Torsion

Let G be a group, and let $g \in G$ have finite order n. We may assume that $n = p$, a prime. Let K be a field, and let

$$\hat{g} = 1 + g + g^2 + \cdots + g^{p-1}.$$

Notice that $(\hat{g})^2 = p \cdot \hat{g}$.

- As $\hat{g}(\hat{g} - p \cdot 1) = 0$, KG has zero divisors.
- If $p \neq \operatorname{char} K$ then $1/p \cdot \hat{g}$ is an idempotent.
- If $|K| > 3$ then choose $a \neq 0, 1, 1/p$ (assuming $p \neq \operatorname{char} K$), write $b = a/(pa - 1)$, and note that

$$(1 - a\hat{g})(1 - b\hat{g}) = 1 - (a + b)\hat{g} + abp\hat{g} = 1.$$

(Other constructions of units exist for other cases.)

The Algebraic Conjectures

We have seen that torsion causes the group ring to be badly behaved.

Conjecture (Zero divisor conjecture)

If G is a torsion-free group and K is a field, then KG has no zero divisors.

Conjecture (Kaplansky conjecture)

If G is a torsion-free group and K is a field, then the only idempotents of KG are 0 and 1.

Conjecture (Unit conjecture)

If G is a torsion-free group and K is a field, then any unit of KG is of the form $\lambda \cdot g$, where $\lambda \in K \setminus \{0\}$ and $g \in G$.

(By embedding an integral domain R in its field of fractions, we get corresponding conjectures for any integral domain R.)
As we have seen, all of these conjectures hold for G abelian.

Some Classes of Group

Let G be a group.
A series for G is a filtration

$$1 = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_n = G$$

of G.

- If each G_i/G_{i-1} can be chosen to be abelian, then G is soluble.
- If each G_i/G_{i-1} can be chosen to be cyclic, then G is polycyclic.
- If, in addition, $G_i \trianglelefteq G$, then G is supersoluble.
- If each G_i/G_{i-1} can be chosen to be central in G/G_{i-1} then G is nilpotent.

We see that

$$\{\text{nilpotent}\} \subset \{\text{supersoluble}\} \subset \{\text{polycyclic}\} \subset \{\text{soluble}\}$$

A group is virtually (-) if it has a subgroup of finite index with (-).

Some Examples of Progress on the Farrell–Jones Conjecture

Theorem (Farrell–Roushon, 2000)

If G is a braid group (of type A) then $\mathrm{Wh}(G) = 0$.

Theorem (Farrell–Linnell, 2003)

If G is virtually polycyclic then $\mathrm{Wh}(G) = 0$.

Theorem (Bartels–Lück–Reich)

If G is virtually nilpotent then the Farrell–Jones conjecture is true for RG.

Progress on the Zero Divisor Conjecture

The zero divisor conjecture has been solved for a large class of groups, containing all soluble groups.

- For G supersoluble, the conjecture can be proved using algebra. (Formanek, 1973)
- For G polycyclic, the proof for characteristic 0 (Farkas–Snider, 1976) uses the fact that $H^*(G, K)$ is finite dimensional. For characteristic $p > 0$ (Cliff, 1980) similar ideas are used.
- For G soluble, the proof uses K-theory. (Kropholler–Linnell–Moody, 1988)

Progress on the Unit Conjecture

In contrast to the zero divisor conjecture, very little progress has been made on the unit conjecture. Two of the best theorems are the following:

Theorem

If G is a group and N is an abelian normal subgroup such that G/N is cyclic, then the unit conjecture holds for G.

Theorem

Let G be a group, and let N be a normal subgroup such that $G/N \cong \mathbb{Z}$. The unit conjecture is true for G if and only if it is true for N.

(This solves the problem for nilpotent groups.)

Define the Hirsch length, $h(G)$, of a polycyclic group to be the number of $\mathbb{Z}$ factors G_i/G_{i-1} in a series for G. This allows us to proceed by induction on $h(G)$.

Supersoluble Groups: An Inductive Approach

The following proposition is very useful.

Proposition

Let G be a torsion-free supersoluble group. Then G has a normal subgroup N such that G/N is either $\mathbb{Z}$ or infinite dihedral, D_∞.

(D_∞ is generated by x and y, with $x^2 = y^2 = 1$; it can be thought of as the group generated by two reflections in $\mathbb{R}^2$ with an irrational angle between their reflecting lines.)

Let G be a supersoluble group and N be as above. Notice that $h(N) = h(G) - 1$, so by induction the unit conjecture is true for N. If $G/N \cong \mathbb{Z}$ then UC is true for G. Hence $G/N \cong D_\infty$.

Hirsch Length 1 and 2

(From the previous slide: if G has a normal subgroup N such that $G/N \cong \mathbb{Z}$, or N is abelian and G/N is cyclic, then G satisfies UC. Also, G has a quotient either $\mathbb{Z}$ or D_∞.)

- If $h(G) = 1$ then $h(N) = 0$ and $N = 1$, so that $G = D_\infty$, which is not torsion free, or $\mathbb{Z}$, for which UC holds.
- If $h(G) = 2$ then $h(N) = 1$, so $N = \mathbb{Z}$ by above. If $G/N \cong \mathbb{Z}$ then done, so $G/N \cong D_\infty$. As the generators of G/N square to a non-trivial element of N, both act trivially on N: hence G has an abelian normal subgroup of index 2, so done again.
- Hence a minimal counterexample to UC must have Hirsch length 3.

Unique-Product Groups

Let G be a group. We say that G is a unique-product group (UP group) if, whenever A and B are non-empty, finite subsets of G, there is an element of

$$A \cdot B = \{a \cdot b \mid a \in A,\ b \in B\}$$

that is expressible **uniquely** as $a \cdot b$.

If G is a torsion-free group and $\sigma, \tau \in RG$ are such that $\sigma\tau = 0$, then write A, B for the supports of σ, τ. Notice that every element in AB must appear (at least) twice in order for them to cancel off in $\sigma\tau$. Hence if G is a UP group then G satisfies the zero divisor conjecture.

This won't work for the unit conjecture though, as if $\sigma\tau = 1$, 1 might be the unique product!

In 1980, Strojnowski proved that if G is a UP group, then there are actually **two** elements of AB that are uniquely expressible, assuming $|A|, |B| \geq 2$. Hence UP groups satisfy the unit conjecture.

Unique-Product Groups: Don't Get Your Hopes Up

This is great, but not all torsion-free groups are UP groups. Rips and Segev (1987) proved the existence of a non-UP group, but this was **complicated**.

A year later, Promislow proved that a much simpler-to-understand group was not a UP group, by making a computer multiply together lots of sets.

This group, called the Passman fours group Γ, is supersoluble, of Hirsch length 3. The set Promislow found, X, has fourteen elements in it, and $X \cdot X$ has no unique product.

Γ looks like the perfect candidate to be a counterexample to the unit conjecture.

End of Part 1

Understanding Units in Supersoluble Groups

In Part 1, we saw that, in any supersoluble counterexample to the unit conjecture, there must be a normal subgroup N with $G/N \cong D_\infty$.

The group D_∞ is a Coxeter group, so has a length function. Let x and y be reflections. Every element of D_∞ is a word in alternating x and y. Let $\ell(w)$ be the length of this word; e.g., $\ell(xyxyx) = 5$.

Define a length function on G by $\ell(g) = \ell(w)$, where w is the word in D_∞ that is the image of g.

Extend $\ell(-)$ to KG by $\ell(\sigma)$ is the maximum of $\ell(g)$, for g in the support of σ.

A Splitting Theorem for Supersoluble Groups

Suppose that G has an infinite dihedral quotient with kernel N, generated by xN and yN.

Theorem (Splitting theorem for units)

Let σ be an element of KG, and suppose that there is τ such that $\sigma\tau \in KN$. (In particular $\sigma\tau = 1$ or $\sigma\tau = 0$.) Then

$$\sigma = \eta^{-1}(\alpha_1 + \beta_1\gamma_1)(\alpha_2 + \beta_2\gamma_2)\dots(\alpha_n + \beta_n\gamma_n),$$

with $\gamma_i \in \{x, y\}$, and $\alpha_i, \beta_i, \eta \in KN$. (This element lies in the localized group ring $(KN)^{-1}(KG)$.)

The linear terms after the η^{-1} are called the split form of σ. One may assume that α_i and β_i are left-coprime.

What is this η?

Write s for the split of σ, so that $\sigma = \eta^{-1}s$. For example, if $s = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)$, then

$$s = \alpha_1\alpha_2 + \alpha_1\beta_2 y + \beta_1\alpha_2^x x + \beta_1\beta_2^x xy.$$

In order for $\eta^{-1}s$ to be in KG, we must have that η divides each coefficient in front of the words in x and y.

Proposition

If $\eta = 1$ in the split of σ, then σ is not a unit.

Proof.

If $\eta = 1$ in the split, then s has an inverse, so by rebracketing, $\alpha_1 + \beta_1 x$ has an inverse. But this lies inside $\langle N, x\rangle$, a subgroup of G of smaller Hirsch length. (Its inverse must also lie in $\langle N, x\rangle$.) $\square$

The Passman fours group Γ

We now return to our interesting group Γ.

This group Γ is given by the presentation

$$\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\, yx^2y^{-1} = x^{-2} \rangle.$$

Write $z = xy$, $a = x^2$, $b = y^2$, $c = z^2$.
Idea 1: $H = \langle a, b, c \rangle$ is an abelian normal subgroup of G, and G/H is the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Idea 2: $N = \langle a, b \rangle$ is an abelian normal subgroup, and G/N is the infinite dihedral group D_∞. This second quotient gives a length function on the elements of the group.

- The elements of N (of the form $a^i b^j$) are defined to be length 0.
- Length 1 elements are αx or αy, with $\alpha \in N$.
- Length 2 elements are αxy or αyx, with $\alpha \in N$.
- And so on.
- The Promislow set X has length 5.

No Units of Length 2

Theorem

There are no non-trivial units of length 1 *or* 2 *in* $K\Gamma$.

Proof.

Elements of length 1 lie in $\langle N, x\rangle$ or $\langle N, y\rangle$, which are groups of Hirsch length 2. Hence these are not units.

Let σ be a unit of length 2, and split σ as $\sigma = \eta^{-1}s$. We have

$$s = \alpha_1\alpha_2 + \alpha_1\beta_2 y + \beta_1\alpha_2^x x + \beta_1\beta_2^x xy,$$

so any divisor of η must divide each of $\alpha_1\alpha_2$, $\alpha_1\beta_2$, $\beta_1\alpha_2^x$ and $\beta_1\beta_2^x$. This is not possible as α_i and β_i are coprime. $\square$

This Method Doesn't Work for Length 3

The length-3 elements are as follows:

Word	Coefficient
xyx	$\beta_1\beta_2^x\beta_3^{yx}$
yx	$\alpha_1\beta_2\beta_3^y$
xy	$\beta_1\beta_2^x\alpha_3^{yx}$
y	$\alpha_1\beta_2\alpha_3^y$
x	$\alpha_1\alpha_2\beta_3 + \beta_1\alpha_2^x\alpha_3^x$
1	$\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2^x\beta_3^x x^2$

Let s be the product of linear terms with $\alpha_1 = \alpha_2 = \alpha_3 = \beta_3 = 1$, $\beta_1 = -a$, $\beta_2 = 1 - a$. Let $\sigma = (a-1)^{-1}s$. This **is** an element of $K\Gamma$.

(This is not a unit of $K\Gamma$.)

The group ring $K\Gamma$

- We now want to consider the group ring $K\Gamma$, where K is any field.
- We want to rewrite the elements of $K\Gamma$, using the subgroup $H = \langle a, b, c\rangle$ this time. Any element may be written as $Ax + By + C + Dz$, where $A, B, C, D \in KH$.
- Take the permutation representation of Γ over H, and use this to embed Γ in $\mathrm{GL}_4(R)$, where $R = K[a^{\pm 1}, b^{\pm 1}, c^{\pm 1}]$.
- In fact, $K\Gamma$ embeds in $M_4(R)$, via

$$\begin{pmatrix} C & A & B & D \\ A^x a & C^x & D^x a & B^x \\ B^y b & D^y a^{-1} c^{-1} & C^y & A^y a^{-1} b c^{-1} \\ D^z c & B^z b^{-1} & A^z b^{-1} c & C^z \end{pmatrix}$$

(Here, A^x indicates the conjugate of A by x, and so on.)

Theorems on $K\Gamma$

Using the splitting theorem for units in $K\Gamma$, we can produce two important theorems.

Theorem

The length of a unit in $K\Gamma$ is equal to the length of its inverse.

Theorem

An element of $K\Gamma$ is a unit if and only if its determinant is in K.

Thus it must be really easy to check if an element of $K\Gamma$ is invertible, simply by checking its determinant. A length-3 element looks like the following:

$$\alpha_1 x + (\alpha_2 + \alpha_3 c)y + \alpha_4 + (\alpha_5 + \alpha_6 c^{-1})z.$$

(Here, $\alpha_i \in KN$.)

The determinant of a length-3 element

$$
\begin{gathered}
\alpha_1\alpha_1^x\alpha_1^y\alpha_1^z - \alpha_1\alpha_1^x\alpha_4^y\alpha_4^z a - \alpha_1\alpha_2^x\alpha_3^y\alpha_1^z + \alpha_1\alpha_2^x\alpha_4^y\alpha_6^z - \alpha_1\alpha_3^x\alpha_2^y\alpha_1^z + \alpha_1\alpha_3^x\alpha_4^y\alpha_5^z - \alpha_1\alpha_5^x\alpha_1^y\alpha_5^z b + \alpha_1\alpha_5^x\alpha_2^y\alpha_4^z ab \\
-\alpha_1\alpha_6^x\alpha_1^y\alpha_6^z b + \alpha_1\alpha_6^x\alpha_3^y\alpha_4^z ab - \alpha_2\alpha_1^x\alpha_1^y\alpha_3^z + \alpha_2\alpha_1^x\alpha_6^y\alpha_4^z + \alpha_2\alpha_2^x\alpha_2^y\alpha_2^z + \alpha_2\alpha_2^x\alpha_3^y\alpha_3^z - \alpha_2\alpha_2^x\alpha_5^y\alpha_5^z a^{-1} \\
-\alpha_2\alpha_2^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_2\alpha_3^x\alpha_2^y\alpha_3^z - \alpha_2\alpha_3^x\alpha_6^y\alpha_5^z a^{-1} + \alpha_2\alpha_4^x\alpha_1^y\alpha_5^z ba^{-1} - \alpha_2\alpha_4^x\alpha_2^y\alpha_4^z b - \alpha_3\alpha_1^x\alpha_1^y\alpha_2^z + \alpha_3\alpha_1^x\alpha_5^y\alpha_4^z \\
+\alpha_3\alpha_2^x\alpha_3^y\alpha_2^z - \alpha_3\alpha_2^x\alpha_5^y\alpha_6^z a^{-1} + \alpha_3\alpha_3^x\alpha_2^y\alpha_2^z + \alpha_3\alpha_3^x\alpha_3^y\alpha_3^z - \alpha_3\alpha_3^x\alpha_5^y\alpha_5^z a^{-1} - \alpha_3\alpha_3^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_3\alpha_4^x\alpha_1^y\alpha_6^z ba^{-1} \\
-\alpha_3\alpha_4^x\alpha_3^y\alpha_4^z b - \alpha_4\alpha_2^x\alpha_4^y\alpha_2^z b^{-1} + \alpha_4\alpha_2^x\alpha_5^y\alpha_1^z a^{-1}b^{-1} - \alpha_4\alpha_3^x\alpha_4^y\alpha_3^z b^{-1} + \alpha_4\alpha_3^x\alpha_6^y\alpha_1^z a^{-1}b^{-1} - \alpha_4\alpha_4^x\alpha_1^y\alpha_1^z a^{-1} \\
+\alpha_4\alpha_4^x\alpha_4^y\alpha_4^z + \alpha_4\alpha_5^x\alpha_1^y\alpha_3^z - \alpha_4\alpha_5^x\alpha_6^y\alpha_4^z + \alpha_4\alpha_6^x\alpha_1^y\alpha_2^z - \alpha_4\alpha_6^x\alpha_5^y\alpha_4^z + \alpha_5\alpha_1^x\alpha_4^y\alpha_2^z ab^{-1} - \alpha_5\alpha_1^x\alpha_5^y\alpha_1^z b^{-1} \\
+\alpha_5\alpha_4^x\alpha_3^y\alpha_1^z - \alpha_5\alpha_4^x\alpha_4^y\alpha_6^z - \alpha_5\alpha_5^x\alpha_2^y\alpha_2^z a - \alpha_5\alpha_5^x\alpha_3^y\alpha_3^z a + \alpha_5\alpha_5^x\alpha_5^y\alpha_5^z + \alpha_5\alpha_5^x\alpha_6^y\alpha_6^z - \alpha_5\alpha_6^x\alpha_3^y\alpha_2^z a + \alpha_5\alpha_6^x\alpha_5^y\alpha_6^z \\
+\alpha_6\alpha_1^x\alpha_4^y\alpha_3^z ab^{-1} - \alpha_6\alpha_1^x\alpha_6^y\alpha_1^z b^{-1} + \alpha_6\alpha_4^x\alpha_2^y\alpha_1^z - \alpha_6\alpha_4^x\alpha_4^y\alpha_5^z - \alpha_6\alpha_5^x\alpha_2^y\alpha_3^z a + \alpha_6\alpha_5^x\alpha_6^y\alpha_5^z - \alpha_6\alpha_6^x\alpha_2^y\alpha_2^z a \\
-\alpha_6\alpha_6^x\alpha_3^y\alpha_3^z a + \alpha_6\alpha_6^x\alpha_5^y\alpha_5^z + \alpha_6\alpha_6^x\alpha_6^y\alpha_6^z + c(-\alpha_1\alpha_2^x\alpha_2^y\alpha_1^z + \alpha_1\alpha_2^x\alpha_4^y\alpha_5^z - \alpha_1\alpha_6^x\alpha_1^y\alpha_5^z b + \alpha_1\alpha_6^x\alpha_2^y\alpha_4^z ab \\
+\alpha_2\alpha_2^x\alpha_2^y\alpha_3^z - \alpha_2\alpha_2^x\alpha_6^y\alpha_5^z a^{-1} - \alpha_3\alpha_1^x\alpha_1^y\alpha_3^z + \alpha_3\alpha_1^x\alpha_6^y\alpha_4^z + \alpha_3\alpha_2^x\alpha_2^y\alpha_2^z + \alpha_3\alpha_2^x\alpha_3^y\alpha_3^z - \alpha_3\alpha_2^x\alpha_5^y\alpha_5^z a^{-1} \\
-\alpha_3\alpha_2^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_3\alpha_3^x\alpha_2^y\alpha_3^z - \alpha_3\alpha_3^x\alpha_6^y\alpha_5^z a^{-1} + \alpha_3\alpha_4^x\alpha_1^y\alpha_5^z ba^{-1} - \alpha_3\alpha_4^x\alpha_2^y\alpha_4^z b - \alpha_4\alpha_2^x\alpha_4^y\alpha_3^z b^{-1} \\
+\alpha_4\alpha_2^x\alpha_6^y\alpha_1^z a^{-1}b^{-1} + \alpha_4\alpha_6^x\alpha_1^y\alpha_3^z - \alpha_4\alpha_6^x\alpha_6^y\alpha_4^z + \alpha_5\alpha_1^x\alpha_4^y\alpha_3^z ab^{-1} - \alpha_5\alpha_1^x\alpha_6^y\alpha_1^z b^{-1} + \alpha_5\alpha_4^x\alpha_2^y\alpha_1^z - \alpha_5\alpha_4^x\alpha_4^y\alpha_5^z \\
-\alpha_5\alpha_5^x\alpha_2^y\alpha_3^z a + \alpha_5\alpha_5^x\alpha_6^y\alpha_5^z - \alpha_5\alpha_6^x\alpha_2^y\alpha_2^z a - \alpha_5\alpha_6^x\alpha_3^y\alpha_3^z a + \alpha_5\alpha_6^x\alpha_5^y\alpha_5^z + \alpha_5\alpha_6^x\alpha_6^y\alpha_6^z - \alpha_6\alpha_6^x\alpha_2^y\alpha_3^z a \\
+\alpha_6\alpha_6^x\alpha_6^y\alpha_5^z) + c^{-1}(-\alpha_1\alpha_3^x\alpha_3^y\alpha_1^z + \alpha_1\alpha_3^x\alpha_4^y\alpha_6^z - \alpha_1\alpha_5^x\alpha_1^y\alpha_6^z b + \alpha_1\alpha_5^x\alpha_3^y\alpha_4^z ab - \alpha_2\alpha_1^x\alpha_1^y\alpha_2^z + \alpha_2\alpha_1^x\alpha_5^y\alpha_4^z \\
+\alpha_2\alpha_2^x\alpha_3^y\alpha_2^z - \alpha_2\alpha_2^x\alpha_5^y\alpha_6^z a^{-1} + \alpha_2\alpha_3^x\alpha_2^y\alpha_2^z + \alpha_2\alpha_3^x\alpha_3^y\alpha_3^z - \alpha_2\alpha_3^x\alpha_5^y\alpha_5^z a^{-1} - \alpha_2\alpha_3^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_2\alpha_4^x\alpha_1^y\alpha_6^z ba^{-1} \\
-\alpha_2\alpha_4^x\alpha_3^y\alpha_4^z b + \alpha_3\alpha_3^x\alpha_3^y\alpha_2^z - \alpha_3\alpha_3^x\alpha_5^y\alpha_6^z a^{-1} - \alpha_4\alpha_3^x\alpha_4^y\alpha_2^z b^{-1} + \alpha_4\alpha_3^x\alpha_5^y\alpha_1^z a^{-1}b^{-1} + \alpha_4\alpha_5^x\alpha_1^y\alpha_2^z - \alpha_4\alpha_5^x\alpha_5^y\alpha_4^z \\
-\alpha_5\alpha_5^x\alpha_3^y\alpha_2^z a + \alpha_5\alpha_5^x\alpha_5^y\alpha_6^z + \alpha_6\alpha_1^x\alpha_4^y\alpha_2^z ab^{-1} - \alpha_6\alpha_1^x\alpha_5^y\alpha_1^z b^{-1} + \alpha_6\alpha_4^x\alpha_3^y\alpha_1^z - \alpha_6\alpha_4^x\alpha_4^y\alpha_6^z - \alpha_6\alpha_5^x\alpha_2^y\alpha_2^z a \\
-\alpha_6\alpha_5^x\alpha_3^y\alpha_3^z a + \alpha_6\alpha_5^x\alpha_5^y\alpha_5^z + \alpha_6\alpha_5^x\alpha_6^y\alpha_6^z - \alpha_6\alpha_6^x\alpha_3^y\alpha_2^z a + \alpha_6\alpha_6^x\alpha_5^y\alpha_6^z)
\end{gathered}
$$

You don't want to see the length-4 determinant.

Assaulting Length 3

(For this proof sketch we think of equality as equality up to a unit.)

- Examining the coefficients of the words in the table, we see that any prime p dividing η must divide β_2 and β_2^x.
- Next prove that if α and β are elements of KN, and that $\alpha\alpha^y - \beta\beta^y b$ is a unit, then either $\alpha = 0$ or $\beta = 0$.
- Write $D_2 = \alpha_2\alpha_2^y - \beta_2\beta_2^y b$. Prove that the gcd of D_2 and β_2 is the same as that of β_2 and α_2^y. Write D_2' for $D_2/(\alpha_2^y, \beta_2)$.
- Prove that $D_2' = (\alpha_2^y, \beta_2)^y$, which yields a factorization
$$D_2 = (\alpha_2^y, \beta_2)(\alpha_2, \beta_2^y).$$
- Therefore, writing $A = (\alpha_2, \beta_2^y)$, we have $D_2 = AA^y$, and
$$[\alpha_2/A][\alpha_2^y/A^y] - [\beta_2/A^y][\beta_2^y/A]b = D_2/AA^y$$
is a unit. Write $\alpha = \alpha_2/A$ and $\beta = \beta_2/A^y$, which lie in KN; then $\alpha\alpha^y - \beta\beta^y b$ is a unit! Thus $\alpha_2 = 0$ or $\beta_2 = 0$.
- This easily yields a contradiction.

Length 3 and the Promislow Set

Theorem

There are no length-3 units in $K\Gamma$, for any field K.

Let X be the fourteen-element subset of Γ, such that Promislow proved that $X \cdot X$ has no unique product.

The set X does not have length 3 (it has length 5, as mentioned earlier), so the previous theorem does not apply.

However, there is an outer automorphism that centralizes y and swaps x and z. This sends X to a subset of Γ that does have length 3, so we can apply our theorem.

Corollary

The Promislow set (or any subset of it) cannot be the support of a unit in $K\Gamma$, for any field K.

Length ≥ 4

There is a qualitative difference between the three cases of length ≤ 2, length 3, and length ≥ 4.

- If the length is less than 3, there are no solutions to a prime dividing the coefficients.
- If the length is 3, there is a unique solution to a prime dividing the coefficients.
- If the length is at least 4 then there are multiple solutions to a prime dividing the coefficients.

Here we have proved that there are no units in the first two cases. The length-3 case is, in some sense, the base case of the induction. In the future this should allow us to attack the general case, either proving the result if it is true, or restricting the structure of a unit sufficiently to be able to find it if it is false.

All Supersoluble Groups?

We assume here that the unit conjecture is true for Γ; if it is false there is nothing to do!

In the general case, N is no longer commutative, and the arguments above, that relied on KN being a Laurent polynomial ring, have to be replaced by non-commutative analogues. However, if one does this, the coefficients for the words are identical, and so with lots of non-commutative algebra, it might be possible to extend at least some of the proof to this case.

There are, however, no ideas as to how to apply more topological methods (e.g., cohomology or K-theory as in the zero divisor conjecture) to this problem, and so beyond supersoluble we have no idea what to do!

Non-Torsion Free?

The Farrell–Jones conjecture exists for all groups, not just torsion-free ones. (The difference is that the only torsion-free virtually cyclic group is $\mathbb{Z}$.)

For torsion-free groups, the Farrell–Jones conjecture implies the Kaplansky conjecture in characteristic 0, for example.

Perhaps it is possible to use ideas from the topological conjectures, focusing on virtually cyclic subgroups for example, to produce a generalization of the Kaplansky conjecture, and also the zero divisor conjecture and unit conjecture, to all groups.