All Roads Lead to Rome: the proof of the road colouring conjecture

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[Added later: In 2009, A. N. Trahtman published The road coloring problem in Israel J. Math.] This solved a long-standing conjecture, which we will describe (most of) now.

Let $\Gamma$ be a digraph, and suppose that all vertices of $\Gamma$ have the same out-degree. Is there a colouring $\{\alpha_1, \ldots, \alpha_n\}$ of the edges, and a word $s$, such that there is a vertex $v_0$ with $vs = v_0$ for all $v \in V$? In other words, is there a colouring and a set of instructions such that wherever you start you end up at the same vertex? Such a word is called a synchronizing word. Not all digraphs have a synchronizing word: for example, take the square with an arrow in each direction. Any word $s$ such that $vs = v$ has even length, and if $us = v$ for some vertex $u$ adjacent to $v$ then $u$ has odd length. More generally, the following condition is necessary.

The gcd of the lengths of all cycles in $\Gamma$ is 1.

We also want that every vertex is reachable from every other vertex. Call such a digraph an AGW graph, named after Adler, Goodwyn and Weiss.

**Theorem 1 (Trahtman, 2007)** Every AGW graph has a colouring such that it possesses a synchronizing word.

We won’t prove this, but we will get quite close.

**Definition 2** Let $\Gamma$ be an AGW graph, with a colouring. We introduce an equivalence relation on $V$ by saying that $u \equiv v$ if for all words $s$, there is a word $t$ such that $u(st) = v(st)$.

A pair $(u, v)$ of vertices is called synchronizing if there is a word $s$ such that $us = vs$, and if no such word exists the pair is called deadlock. A synchronizing pair $(u, v)$ is called stable if for any word $s$, the pair $(us, vs)$ is also synchronizing.
A theorem of Kari states that $\Gamma$ is synchronizing (i.e., has a synchronizing word) if there is a colouring with a stable pair. (Go by induction. Take the quotient digraph by $\equiv$, and colour the resulting smaller digraph, then lift this to a recolouring of $\Gamma$.)

The set of all outgoing edges from a vertex is called a bunch if they all have the same target.

As a remark, if $v \in V$ has two incoming bunches, from $u_1$ and $u_2$, then for any colour $\alpha$, $u_1\alpha = v = u_2\alpha$, so $(u_1,u_2)$ is a stable pair (write any word $s$ as $s = \alpha s'$, so then $u_is = u_i\alpha s' = vs'$, and $(u_1s, u_2s)$ is synchronizing).

Suppose that $v_1, \ldots, v_d$ are vertices and that the out-degree of each vertex is $d$. Let $A$ denote the adjacency matrix of $\Gamma$. Let $u$ be a left eigenvector with positive integer components, having no common divisors. The $i$th component of $u$ is called the weight of $v_i$, and is denoted $w(v_i)$. Define $w(D)$ to be the sum of the weights of the vertices of $D \subseteq V$.

We claim that $w(V) = n$, where $n$ is the out-degree of $v_i$.

Consider all subsets $D$ of $V$ such that $|Ds| = 1$ for some $s$, and $w(D)$ is maximal. These are called $F$-maximal subsets.

We claim that if $U$ is $F$-maximal, so if $Us^{-1}$, where $s$ is a word. It suffices to assume that $s$ is a single colour $\alpha_i$. Since $wA = nw$, we have

$$\sum_{i=1}^{n} w(U\alpha_i^{-1}) = nw(U).$$

Therefore, either one of the $U\alpha_i^{-1}$ has greater weight than $U$, or all $U\alpha_i^{-1}$ have the same weight. If $U$ is $F$-maximal, the former possibility cannot occur, and so $U\alpha_i^{-1}$ is also $F$-maximal. This will play a role in the proof to follow.

**Theorem 3 (Friedman, 1990)** There exists a partition of $\Gamma$ into $F$-maximal sets.

**Proof:** Let $T_0$ be an $F$-maximal set, and let $s$ be a word such that $|T_0s| = 1$, $T_1s = v_0$. If $T_0 = V$, then we are done (and proved the theorem!).

If $T_0 \neq V$, we claim we can extend $s$ backwards to $s'$, so that $s'$ sends $T_0$ to one vertex and some other $F$-maximal set to another vertex. Consider $s^{-1}$, which maps $V$ to subsets of $V$, which form a partition of $V$. Let $v \notin T_0$, and let $r$ map $v_0$ to $v$. Let $s' = srs$.

Since $T_0$ is $F$-maximal, so is $T_1 = T_0r^{-1}s^{-1}$. Notice that $T_1s' = v_0$ and $T_0s' = v_0rs \neq v_0$. Hence $s'$ sends $T_0$ to $v_0rs$ and $T_1$ to $v_0$. Repeating this procedure completes the theorem.
1 Cliques

An $F$-clique is a subset $V_s \subseteq V$ (for some word $s$) such that all pairs in $V_s$ are in deadlock.

Lemma 4 Let $w$ be the weight of an $F$-maximal set. The size of an $F$-clique $A$ is $w(\Gamma)/w$, i.e., the size of a partition given in Theorem 3 above.

Proof: If $u,v$ lie in some $F$-maximal set, then they cannot belong to some $F$-clique, so $|A| \leq W(\Gamma)/w$. 

Now let $V_s$ be an $F$-clique. If $v \in V_s$, the sum of all weights $v v_s^{-1}$ is $w(\Gamma)$. Therefore

$$w(\Gamma) = \sum_{v \in V_s} w(v v_s^{-1}).$$

The weight of each $w(v v_s^{-1})$ is at most $w$, as this is the maximum weight of a set $D$ such that $|Ds| = 1$. Hence

$$w(\Gamma) = \frac{w(\Gamma)}{w} \cdot w \leq |A| \cdot w.$$

Hence $w(\Gamma)/w \leq |A|$, as claimed. \qed

Lemma 5 Let $A$ be an $F$-clique. For any $s$, $As$ is also an $F$-clique, and any vertex belongs to some $F$-clique.

Proof: Obvious. \qed

Lemma 6 Let $A$ and $B$ ($|A| > 1$) be distinct $F$-clique. Suppose that $\Gamma$ has no stable pairs. Then $|A| = |B|$, and $|A \setminus B| > 1$.

Proof: By Lemma 4 $|A| = |B|$. Let $A \setminus B = \{v\}$, and $B \setminus A = \{u\}$. As $\Gamma$ has no stable pairs, for some word $s$, $(v v_s, u u_s)$ is a deadlock. Also, any pair from $A$, $As$, $B$, $Bs$ is a deadlock. Hence $(A \cup B)s$ is a set, any pair of vertices from which is a deadlock. But $|(A \cup B)s|$ is greater than $|A|$, so must have two elements from the same $F$-maximal set, a contradiction. Thus $|A \setminus B| > 1$. \qed

2 Spanning Subgraphs

Definition 7 A subgraph $S$ of $\Gamma$ is a spanning subgraph if it contains all vertices and exactly one out-edge for each vertex. A maximal sub-rooted tree with root on a cycle from $S$ and having no common edges with cycles from $S$ is called a tree of $S$. The distance of a vertex from the root is called the level of the vertex.
The idea of this definition is that a spanning subgraph might consist of all edges of a particular colour. By studying these, we will be able to prove the theorem.

**Lemma 8** Suppose that any vertex on $\Gamma$ has no two incoming bunches. Then $\Gamma$ has a spanning subgraph such that all vertices of maximal positive level belong to one non-trivial tree.

Let’s see how to prove the main theorem from this lemma. If we have a vertex with two incoming bunches, we are done by the remarks earlier. If not, by Lemma 8, $\Gamma$ has a spanning subgraph $R$ such that vertices of maximal positive level $L$ belong to one tree of $R$.

Give the edges of $R$ the colour $\alpha$, and $C$ be all vertices from cycles of $R$. Colour all other edges arbitrarily.

Some $F$-clique $F$ has non-empty intersection with set $N$ of maximal level $L$. $N$ belongs to one tree. $|N \cap F| = 1$, because the word $\alpha^L$ maps all elements of $N$ to the root. The word $\alpha^{L-1}$ maps $F$ to another $F$-clique $F'$, of size $|F|$. Certainly $|F' \setminus C| = 1$ because if $v$ is on a tree, then $va^{L-1}$ is on a cycle (as the root is on a cycle) unless $v$ has level $L$, in which case

$$|F' \setminus C| = |N \alpha^{L-1} \cap F'| = 1$$

(as no two $va^{L-1}$ in $N \alpha^{L-1}$ are in deadlock. Therefore $|C \cap F_1| = |F_1| - 1$.

Now, let $m$ be a multiple of all lengths of all cycles in $C$. For any $v \in C$, $va^m = v$. Therefore if $F'' = F' \alpha^m$, we have that $F'' \subseteq C$ and $C \setminus F' = F' \cap F''$. Thus $F'$ and $F''$ of size $|F'| > 1$, $|F''| > 1$, have $|F'| - 1$ common vertices, contradicting Lemma 6. Thus there exists a stable couple by Lemma 6, and we are done.