All Roads Lead to Rome: the proof of the road colouring conjecture

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[Added later: In 2009, A. N. Trahtman published *The road coloring problem* in Israel J. Math.] This solved a long-standing conjecture, which we will describe (most of) now.

Let Γ be a digraph, and suppose that all vertices of Γ have the same out-degree. Is there a colouring $\{\alpha_1, \ldots, \alpha_n\}$ of the edges, and a word s, such that there is a vertex v_0 with $vs = v_0$ for all $v \in V$? In other words, is there a colouring and a set of instructions such that wherever you start you end up at the same vertex? Such a word is called a *synchronizing word*. Not all digraphs have a synchronizing word: for example, take the square with an arrow in each direction. Any word s such that $vs = v$ has even length, and if $us = v$ for some vertex u adjacent to v then u has odd length. More generally, the following condition is necessary.

The gcd of the lengths of all cycles in Γ is 1.

We also want that every vertex is reachable from every other vertex. Call such a digraph an AGW graph, named after Adler, Goodwyn and Weiss.

Theorem 1 (Trahtman, 2007) Every AGW graph has a colouring such that it possesses a synchronizing word.

We won't prove this, but we will get quite close.

Definition 2 Let Γ be an AGW graph, with a colouring. We introduce an equivalence relation on V by saying that $u \equiv v$ if for all words s, there is a word t such that $u(st) = v(st)$.

A pair (u, v) of vertices is called *synchronizing* if there is a word s such that $us = vs$, and if no such word exists the pair is called *deadlock*. A synchronizing pair (u, v) is called *stable* if for any word s, the pair (us, vs) is also synchronizing.

A theorem of Kari states that Γ is synchronizing (i.e., has a synchonizing word) if there is a colouring with a stable pair. (Go by induction. Take the quotient digraph by $\equiv$, and colour the resulting smaller digraph, then lift this to a recolouring of Γ.)

The set of all outgoing edges from a vertex is called a *bunch* if they all have the same target.

As a remark, if $v \in V$ has two incoming bunches, from u_1 and u_2, then for any colour α, $u_1\alpha = v = u_2\alpha$, so (u_1, u_2) is a stable pair (write any word s as $s = \alpha s'$, so then $u_i s = u_i \alpha s' = v s'$, and $(u_1 s, u_2 s)$ is synchronizing).

Suppose that $v_1, \ldots, v_d$ are vertices and that the out-degree of each vertex is d. Let A denote the adjacency matrix of Γ. Let u be a left eigenvector with positive integer components, having no common divisors. The ith component of u is called the *weight* of v_i, and is denoted $w(v_i)$. Define $w(D)$ to be the dum of the weights of the vertices of $D \subseteq V$. We claim that $w(V) = n$, where n is the out-degree of v_i.

Consider all subsets D of V such that $|Ds| = 1$ for some s, and $w(D)$ is maximal. these are called *F-maximal* subsets.

We claim that if U is F-maximal, so if Us^{-1}, where s is a word. It suffices to assume that s is a single colour α_i. Since $wA = nw$, we have

$$\sum_{i=1}^{n} w(U\alpha_i^{-1}) = nw(U).$$

Therefore, either one of the $U\alpha_i^{-1}$ has greater weight than U, or all $U\alpha_i^{-1}$ have the same weight. If U is F-maximal, the former possibility cannot occur, and so $U\alpha_i^{-1}$ is also F-maximal. This will play a role in the proof to follow.

Theorem 3 (Friedman, 1990) There exists a partition of Γ into F-maximal sets.

Proof: Let T_0 be an F-maximal set, and let s be a word such that $|T_0 s| = 1$, $T_1 s = v_0$. if $T_0 = V$, then we are done (and proved the theorem!).

If $T_0 \neq V$, we claim we can extend s backwards to s', so that s' sends T_0 to one vertex and some other F-maximal set to another vertex. Consider s^{-1}, which maps V to subsets of V, which form a partition of V. Let $v \notin T_0$, and let r map v_0 to v. Let $s' = srs$.

Since T_0 is F-maximal, so is $T_1 = T_0 r^{-1} s^{-1}$. Notice that $T_1 s' = v_0$ and $T_0 s' = v_0 rs \neq v_0$. Hence s' sends T_0 to $v_0 rs$ and T_1 to v_0. Repeating this procedure completes the theorem. $\square$

1 Cliques

An F-*clique* is a subset $Vs \subseteq V$ (for some word s) such that all pairs in Vs are in deadlock.

Lemma 4 Let w be the weight of an F-maximal set. The size of an F-clique A is $w(\Gamma)/w$, i.e., the size of a partition given in Theorem 3 above.

Proof: If u, v lie in some F-maximal set, then they cannot belong to some F-clique, so $|A| \leqslant W(\Gamma)/w$.

Now let Vs be an F-clique. If $v \in Vs$, the sum of all weights vs^{-1} is $w(\Gamma)$. Therefore

$$w(\Gamma) = \sum_{v \in Vs} w(vs^{-1}).$$

The weight of each $w(vs^{-1})$ is at most w, as this is the maximum weight of a set D such that $|Ds| = 1$. Hence

$$w(\Gamma) = \frac{w(\Gamma)}{w} \cdot w \leqslant |A| \cdot w.$$

Hence $w(\Gamma)/w \leqslant |A|$, as claimed. $\square$

Lemma 5 Let A be an F-clique. For any s, As is also an F-clique, and any vertex belongs to some F-clique.

Proof: Obvious. $\square$

Lemma 6 Let A and B ($|A| > 1$) be distinct F-clique. Suppose that Γ has no stable pairs., Then $|A| = |B|$, and $|A \setminus B| > 1$.

Proof: By Lemma 4 $|A| = |B|$. Let $A \setminus B = \{v\}$, and $B \setminus A = \{u\}$. As Γ has no stable pairs, for some word s, (vs, us) is a deadlock. Also, any pair from A, As, B, Bs is a deadlock. Hence $(A \cup B)s$ is a set, any pair of vertices from which is a deadlock. But $|(A \cup B)s|$ is greater than $|A|$, so must have two elements from the same F-maximal set, a contradiction. Thus $|A \setminus B| > 1$. $\square$

2 Spanning Subgraphs

Definition 7 A subgraph S of Γ is a *spanning subgraph* if it contains all vertices and exactly one out-edge for each vertex. A maximal sub-rooted tree with root on a cycle from S and having no common edges with cycles from S is called a *tree* of S. The distance of a vertex from the root is called the *level* of the vertex.

The idea of this definition is that a spanning subgraph might consist of all edges of a particular colour. By studying these, we will be able to prove the theorem.

Lemma 8 Suppose that any vertex on Γ has no two incoming bunches. Then Γ has a spanning subgraph such that all vertices of maximal positive level belong to one non-trivial tree.

Let's see how to prove the main theorem from this lemma. If we have a vertex with two incoming bunches, we are done by the remarks earlier. If not, by Lemma 8, Γ has a spanning subgraph R such that vertices of maximal positive level L belong to one tree of R.

Give the edges of R the colour α, and C be all vertices from cycles of R. Colour all other edges arbitrarily.

Some F-clique F has non-empty intersection with set N of maximal level L. N belongs to one tree. $|N \cap F| = 1$, because the word α^L maps all elements of N to the root. The word α^{L-1} maps F to another F-clique F', of size $|F|$. Certainly $|F' \setminus C| = 1$ because if v is on a tree, then $v\alpha^{L-1}$ is on a cycle (as the root is on a cycle) unless v has level L, in which case

$$|F' \setminus C| = |N\alpha^{L-1} \cap F'| = 1$$

(as no two $v\alpha^{L-1}$ in $N\alpha^{L-1}$ are in deadlock. Therefore $|C \cap F_1| = |F_1| - 1$.

Now, let m be a multiple of all lengths of all cycles in C. For any $v \in C$, $v\alpha^m = v$. Therefore if $F'' = F'\alpha^m$, we have that $F'' \subseteq C$ and $C \cap F' = F' \cap F''$. Thus F' and F'' of size $|F'| > 1$, $|F''| > 1$, have $|F'| - 1$ common vertices, contradicting Lemma 6. Thus there exists a stable couple by Lemma 6, and we are done.