Algebraic Modules and the Heller Operator

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For the purposes of this talk, G is a finite group, K is an algebraically closed field of characteristic p, where $p \mid |G|$, and all modules are finite-dimensional.

1 Introduction

The tensor product of two vector spaces, V and W over K, is easy to describe: it has as basis all symbols $v \otimes w$, where v and w are basis vectors in V and W respectively. If V and W have the honour of being KG-modules, rather than being just boring vector spaces, then $V \otimes W$ gets to be a KG-module by assigning $(v \otimes w)g$ to be $vg \otimes wg$, and extending by the usual linearity.

So far, so good. Now suppose that we have our two KG-modules, which are usually referred to as M and N but since I've started with V and W I'll continue with them. What is $V \otimes W$ as a KG-module? The answer is, in general, 'we don't know'. What we mean by the question we just asked, is if V and W are known KG-modules, can we write the tensor product of V and W as a sum of *indecomposable* modules X_i? (The Krull–Schmidt theorem holds for KG-modules, so the X_i are uniquely determined.)

This question might be thought of as asking if we can factorize the sum of two integers into primes (only much harder...), so in some sense expecting to understand the terms involved in every case is asking a bit too much. However, we might be able to say something at least, and several people have done so.

Theorem 1.1 (Benson–Carlson, [3]) Let G be a finite group and M and N be absolutely indecomposable KG-modules.

(i) $K|M \otimes N$ if and only if $p \nmid \dim M$ and $M \cong N^*$, in which case $2 \cdot K$ is not a summand of $M \otimes N$.

(ii) If $p \mid \dim M$, then every summand of $M \otimes N$ has dimension a multiple of p.

Proposition 1.2 (Auslander–Carlson, [2], Proposition 4.9) Let G be a finite group and K be a field of charactersitic p. If M is an indecomposable module of dimension a multiple of p, then $M \oplus M$ is a direct summand of $M \otimes M^* \otimes M$.

What I'm going to talk about today is my pet project, which is to understand *algebraic modules*. These were first discussed by Alperin in [1], but didn't garner much attention over the years. To define them properly, I'm going to have to introduce the Green ring, although this isn't particularly difficult to comprehend. After I've defined algebraic modules, and chatted a bit about their properties, we'll look at the Heller operator, which *is* hard to understand, if it's the first time you've seen such things. We'll then see how tohe two of them work together to make my life interesting, and then we'll have a cup of tea, presumably.

2 The Green Ring and Algebraic Modules

If we're living in modular-representation-theory world, life is considerably more complicated than in ordinary-representation-theory world. One of the central theorems in ordinary representation theory is that the number of irreducible G-modules is equal to the number of conjugacy classes of G. In modular representation theory, the number of simple modules is equal to the number of conjugacy classes of elements of G whose order is not divisible by p (the order of the element, not the conjugacy class).

The other central theorem in ordinary representation theory is that every representation of G is a direct sum of irreducible representations. This doesn't happen in modular representation theory, and in general (unless the Sylow p-subgroup of G is cyclic) there are infinitely many modules that cannot be written as the direct sum of two smaller representations.

It gets worse: unless the Sylow p-subgroup of G is cyclic, dihedral, semidihedral or generalized quaternion, there are so many different indecomposable representations that we don't have a hope in hell of classifying them. It is what is known as a *wild* problem. So I find it quite interesting when you can say something about all indecomposable modules in general. I find it more interesting when it's me who's doing the saying.

The *Green ring* of a finite group can be thought of as all positive and negative linear combinations of indecomposable modules, with direct sum playing the role of addition and tensor product playing the role of multiplication.

The structure of the Green ring, while a commutative ring with a 1, is far from that of traditional commutative rings. For example, it is not an integral domain: in general, it has nilpotent elements. It is also in general infinite-dimensional. We can still, however, carry over some notions from algebraic number theory. One of those is algebraic modules.

A module is said to be *algebraic* if it satisfies some polynomial equation in the Green ring, with co-efficients in $\mathbb{Z}$.

Proposition 2.1 Let M be a KG-module. Then the following are equivalent:

(i) M is algebraic; and

(ii) there are only finitely many different indecomposable summands of the (infinite-dimensional) graded module
$$T(M) = M \oplus M^{\otimes 2} \oplus M^{\otimes 3} \oplus \cdots .$$

The second equivalent condition is often the easiest to use in actually deciding if modules are algebraic or not. In particular, it is very easy to use this condition to prove the following.

Lemma 2.2 Let M and N be KG-modules.

(i) M and N are algebraic if and only if $M \oplus N$ is algebraic.

(ii) If M and N are algebraic, then so is $M \otimes N$.

Hence the algebraic modules form a subring of the Green ring.

We pause to briefly give some examples of algebraic modules.

Example 2.3 All permutation modules are algebraic, as are all projective modules.

A module is called *endo-trivial* if $M \otimes M^*$ is the direct sum of a projective module and the trivial module K. These crop up all over representation theory, and have recently been classified, in a sequence of long papers. The most complicated case is proving the general non-existence of torsion endo-trivial module, by which we mean $M^{\otimes n} = K \oplus P$ for some projective module P and some $n > 0$. It is easy to see that an endo-trivial is torsion if and only if it is algebraic, and so the methods of algebraic modules might be applicable to shorten this proof somewhat, or even offer an alternative proof.

Enough about algebraic modules: we mentioned the Heller operator in the introduction, and the title of the talk for that matter, and so we should deal with it.

3 The Heller Operator

Recall that if M is a non-projective indecomposable module then $\Omega(M)$ is the kernel of the projection from the projective cover of M onto M. This is defined inductively for $\Omega^i(M)$, and $\Omega^{-i}(M) = \Omega^i(M^*)^*$.

Lemma 3.1 Let M_1 and M_2 be KG-modules.

(i) $\Omega(M_1 \oplus M_2) = \Omega(M_1) \oplus \Omega(M_2)$.

(ii) $\Omega(M_1 \otimes M_2) = \Omega^0(\Omega(M_1) \otimes M_2)$.

If $\Omega^i(M) = M$ for some non-zero i, then M is called (Ω-)*periodic*, and its period is the smallest positive i for which this statement holds. The relation $M \sim N$ if and only if $\Omega^i(M) = N$ for some i partitions the set of all non-projective indecomposable modules into equivalence classes, either finite or infinite, which will be called *Heller strings*. A finite Heller string is called a *loop*, and an infinite Heller string is a *line*. Given a non-projective indecomposable KG-module M, whether M belongs to a Heller loop or a Heller line is pivotal with regards its algebraicity.

It is about time to tell you what I've done in this vein.

Theorem 3.2 Either all modules on a Heller loop are algebraic, or none of them are.

Theorem 3.3 At most one module on a Heller line is algebraic.

Recall (!) that if the p-rank of a group is at least 2, then all modules of dimension not a multiple of p are non-periodic. Then this theorem says that algebraic modules of dimension coprime with p are very rare indeed. In the dihedral groups, which are the only groups that we know much about what's going on (see the remark about wild problems above...), we find that the work was done before I arrived on the scene.

Theorem 3.4 (Archer) There are no non-trivial odd-dimensional indecomposable algebraic modules for dihedral 2-groups.

Another thing I'm doing at the moment is working out what's happening to the even-dimensional indecomposable modules. If you want to know about them, you'll have to come back next year.

References

[1] Jonathan Alperin, *On modules for the linear fractional groups*, International Symposium on the Theory of Finite Groups, 1974, Tokyo (1976), 157–163.

[2] Maurice Auslander and Jon Carlson, *Almost-split sequences and group rings*, J. Algebra **103** (1986), 122–140.

[3] David Benson and Jon Carlson, *Nilpotent elements in the Green ring*, J. Algebra **104** (1986), 329–350.