The Structure of Blocks with V_4 Defect Group

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The following is joint work with Charles Eaton, Radha Kessar and Marcus Linkelmann.

In this talk I will outline the progress made on a conjecture of Karin Erdmann over 25 years ago. This conjecture has essentially been proved by the team described above, in work stretching back over eighteen months. However, the past month or so at MSRI has seen the culmination of this work into an almost complete proof, with most portions of the proof available in manuscript form.

For the purposes of this talk, G is a finite group, K is an algebraically closed field of characteristic p, where $p \mid |G|$, and all modules are finite-dimensional.

1 Facts from Block Theory

We need a few preliminaries before we can talk about blocks at all; then we will define a defect group of a block.

Recall from the previous talk that a *vertex* of an indecomposable module M is a minimal subgroup Q such that M is a summand of $M\downarrow_Q\uparrow^G$. For example, a module has vertex the trivial subgroup if and only if M is a summand of KG, the group algebra; i.e., M has trivial vertex if and only if it is projective (a summand of a free module).

Theorem 1.1 (Green) All vertices of an indecomposable module are p-subgroups of G and a G-conjugate.

A *source* of an indecomposable module M is an indecomposable module S for the vertex Q such that M is a summand of $S\uparrow^G$. Since M is a summand of $M\downarrow_Q\uparrow^G$, such a module exists.

Theorem 1.2 (Green) All sources of an indecomposable module M are conjugate by an element of $\mathrm{N}_G(Q)$, where $Q = \mathrm{vx}\, M$.

Now let X denote the set of all simple KG-modules. We may turn X into a graph, by connecting M and N if and only if $\mathrm{Ext}^1_{KG}(M,N)$ is non-zero. The connected components of X are called *blocks*. A *defect group* of a block is the maximum of the vertices of the simple modules in it. (Such a subgroup exists.)

If P is a projective simple module, and M is any other module, then

$$\mathrm{Ext}^1_{KG}(P,M) = \mathrm{Ext}^1_{KG}(M,P) = 1,$$

and so $\{P\}$ is a connected component of the graph. This is a block whose defect group is the trivial subgroup. If the defect group is cyclic, then there is a nice theory about the block. The next 'easiest' p-group is $C_2 \times C_2$. Already(until recently) quite a lot about the structure of these blocks was not known.

We are now in a position to state the conjecture of Erdmann's.

Conjecture 1.3 (Erdmann, 1982) Let M be a simple module of a block with defect group V_4, and let S denote a source of M. Then $\dim S \leqslant 2$.

2 The Proof of Erdmann's Conjecture

Recently, Lluis Puig has produced a manuscript in which he produces a proof of a major conjecture of his, on endopermutation modules in nilpotent blocks. While this is too difficult to explain here, this resolves one particular case of Ermann's conjecture. This is enough for us.

Theorem 2.1 (CEKL, 2008) Suppose that Puig's conjecture on nilpotent blocks is true. Then Erdmann's conjecture is true.

The problem with Puig's proof is that people are still as yet unconvinced about its veracity. Our current strategy aims to remove the dependence on Puig's proof by ealing with the cases that are contained within the remit of Puig's conjecture.

Theorem 2.2 (CEKL, 2008) Suppose that G is a finite group such that neither $E_6(q) \rtimes P$ nor $E_7(q) \rtimes P$ are involevd in G, where $q \equiv 1 \mod 4$ and P is a V_4 group of outer automorphisms of $E_6(q)$ or $E_7(q)$. Then G satisfies Erdmann's conjecture.

These remaining configurations of groups are hoped to be tackled using Deligne–Lusztig theory.

Erdmann's conjecture has a variety of other consequences which, taken together with known results, give a reasonably complete description of the structure of blocks whose defect group is V_4.

3 Reductions

We would like to reduce the conjecture to finite simple groups, or perhaps to groups closely associated with them.

Theorem 3.1 (EKL, 2006) Erdmann's conjecture is true for all groups if and only if it is true for all groups $H \rtimes P$, where H is an odd central extension of a simple group, and P is either trivial, of order 2, or a V_4 group.

It is the addition of this 2-group of automorphisms above the quasisimple group that gives us the most headaches. Puig's conjecture implies that the case where P is non-trivial is always true, but as we are trying to make this proof independent of Puig's proof, we need that 2-group to remain.

Applying the classification of the finite simple groups, we reduce to a list of possibilities for H. The following is easy.

Proposition 3.2 Let H be a sporadic or alternating group. Then Erdmann's conjecture is true.

The next result uses theorems on the reality of elements, as described in Tara Bonda's talk.

Proposition 3.3 (C, 2008) Let H be a symplectic or orthogonal group, or the Steinberg group ${}^3D_4(q)$. Then Erdmann's conjecture is true.

Radha Kessar used some of the theory of special linear groups to deal with the linear and unitary cases.

Theorem 3.4 (K, 2008) Let H be a linear or unitary group. Then Erdmann's conjecture is true.

Work of Charles Eaton and I sorted most of the exceptional groups.

Theorem 3.5 (CE, 2008) Let H be an exceptional group, except for $E_6(q)$ or $E_7(q)$ when P is a V_4-group. Then Erdmann's conjecture is true.

The remaining two cases are being dealt with now.