The Representation Theory of Non-Existent Groups of Lie Type

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Coxeter Groups

As most of you will know, Coxeter groups are incredibly important in finite (and infinite!) group theory. They are groups generated by involutions, with presentation

$$\langle s_1, \dots, s_r \mid (s_i s_j)^{m_{ij}} = 1 \rangle,$$

where the m_{ij} are non-negative integers with $m_{ii} = 1$ (so that s_i has order 2). A long time ago, H.S.M. Coxeter classified the finite such groups in 1935, and found they fell into several families (in all cases $m_{ij} = 2$ unless otherwise stated):

- A_n: $m_{i,i+1} = 3$, $n \geq 1$ (symmetric groups)
- BC_n: $m_{1,2} = 4$ $n \geq 2$
- D_n: $m_{1,3} = m_{i,i+1} = 3$ $(i > 1)$ $n \geq 4$
- E_n: $m_{n,3} = m_{i,i+1} = 3$ $(i < n-1)$, $n = 6, 7, 8$
- F_4: $m_{1,2} = m_{3,4} = 3$, $m_{2,3} = 4$
- H_n: $m_{1,2} = 5$, $m_{i,i+1} = 3$, $1 < i < n$, $n = 3, 4$
- $I_2(n)$: $m_{1,2} = n$

Groups of Lie Type

Before Coxeter's classification, Weyl groups were classified as the reflection groups of root systems. They are those Coxeter groups for which the m_{ij} are in the set $\{2, 3, 4, 6\}$, so not H_3, H_4, or $I_2(n)$ unless $n = 6$, in which case we have $I_2(6) = G_2$.

The precise definition of a group of Lie type is difficult; they are finite analogues of algebraic groups, and the most popular of them, used in every example, is $\mathrm{GL}_n(q)$, although other common ones are $\mathrm{SU}_n(q)$, $\mathrm{Sp}_{2n}(q)$, $\mathrm{SO}^{\pm}_{2n}(q)$, and $\mathrm{SO}_{2n+1}(q)$. We also have the exceptional groups of Lie type, suggestively called $E_6(q)$, $E_7(q)$, $E_8(q)$, $F_4(q)$ and $G_2(q)$. But there are some more of them, corresponding to automorphisms of the Dynkin diagram.

These are ${}^2E_6(q)$, ${}^2B_2(2^{2n+1})$, ${}^2F_4(2^{2n+1})$ and ${}^2G_2(3^{2n+1})$. Altogether there are sixteen families of groups of Lie type, and they form (along with the alternating groups) all but 26 of the finite simple groups.

Representations of Groups of Lie type

Let $G = G(q)$ be a group of Lie type. The representation theory of $\mathrm{GL}_n(q)$ is almost as old as representation theory itself. Modern day representation theory of groups of Lie type is dominated by Harish-Chandra and Deligne–Lusztig theories.

In 1955 Sandy Green gave all irreducible characters of $\mathrm{GL}_n(q)$, but for all simple groups of Lie type, the construction of all irreducible characters needed Deligne–Lusztig theory, and was completed by Lusztig in 1986.

It hinges on a Jordan decomposition for characters, which in some sense reduces the study of irreducible characters to unipotent and semisimple characters. The unipotent characters are our focus today.

The unipotent characters of $G(q)$ are in bijection with the Weyl group of q, which for $\mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$ is the symmetric group S_n, for $\mathrm{SO}_{2n+1}(q)$ and $\mathrm{Sp}_{2n}(q)$ is BC_n, and for the other orthogonal groups is D_n.

What is a Unipotent Character?

Formally, a unipotent character of $G = \mathbf{G}^F$ is a constituent of a Deligne–Lusztig character $R^{\mathbf{G}}_{\mathbf{T}_w}(1)$ for $w \in W$.

This is the ℓ-adic cohomology of a Deligne–Lusztig variety. For $w = 1$, this is simply the permutation module G/B. In general, there are as many of these characters as there are conjugacy classes of $w \in W$.

Unipotent Character Degrees

The degrees of unipotent characters are polynomials in q, a product of q, cyclotomic polynomials in q, and possibly a fraction $1/d$. The polynomials do not depend on q. For the exceptional groups, there is simply a list of them in the back of Roger Carter's book. For the classical groups, since there are infinitely many different GL_ns, we need a combinatorial procedure.

For GL_n, the character degrees are given by a combinatorial formula dependent on the associated partition, resembling that of the symmetric group. You get the unitary groups by replacing q by $-q$ (and negating the degree if necessary), and for types BC and D you have to use symbols (essentially pairs of partitions).

Lusztig computed them for the exceptional groups, and there is simply a list of them, with no obvious combinatorial description in most cases.

But Unipotent Degrees Are Just Polynomials, So...

We (well, Lusztig) can give a list of properties that the unipotent degrees satisfy (e.g., the number of them is that of the Coxeter group), and search for a collection of them for the non-crystallographic groups, i.e., $I_2(n)$, H_3 and H_4. These were constructed in the 1980s, although only published in 1993.

For $I_2(n)$, they are as follows ($\eta = e^{2\pi i/n}$, $X_i = (q-\eta^i)(q-\eta^{-i})$):

Character	Degree
$\phi_{1,0}$	1
$\phi_{1,n}$	q^n
$\phi_{2,i}$, $i < n/2$	$\frac{(1-\eta^i)(1-\eta^{-i})}{n}\frac{q\Phi_2(q^n-1)}{\Phi_1 X_i}$
$I_2(n)[i,j]$, $i < j < n-i$	$\frac{\eta^i+\eta^{-i}-\eta^j-\eta^{-j}}{n}\frac{q(q^2-1)(q^n-1)}{X_i X_j}$
$\phi'_{1,n/2}$, $\phi''_{1,n/2}$	$\frac{2}{n}\frac{q(q^n-1)}{\Phi_1\Phi_2}$

From Complexes to Prime Fields

Let's move away from ordinary representation theory, and consider representation theory over a finite field. The first problem here is to know what we mean by a character: if the representation ρ has degree a multiple of p then the trace of the matrix is 0, whereas we should have that $\phi(1)$ is the degree of the representation.

The solution is the following: let n be the p'th part of $|G|$, where G is some finite group, and choose ζ a primitive nth root of unity in $k = \bar{\mathbb{F}}_p$. We only define a Brauer character on elements g whose order is prime to p; $\rho(g)$ is a matrix of order divisible by n, so its eigenvalues are all nth roots of unity, hence powers of ζ. Instead of summing these powers, we choose a primitive nth root ζ' in $\mathbb{C}$, and sum the corresponding powers of ζ' in $\mathbb{C}$.

Decomposition

Here is why we know we have the right definition.

Theorem

The restriction of an ordinary character to the p'-elements is a non-negative integral linear combination of irreducible Brauer characters.

This allows us to produce the decomposition matrix, a matrix whose rows are labelled by the ordinary irreducibles, columns by the Brauer irreducibles, and populated by these non-negative integers. One may also construct a graph, with vertices the irreducible complex characters, and connecting two of them if an irreducible Brauer character appears in the decomposition of both of them. The connected components of this graph are called blocks.

Notice that this also distributes the Brauer characters into blocks as well.

Modular representation theory of groups of Lie type

What can we say about the modular representation theory? To give this question some substance we first need to introduce blocks. Let k be a 'large' field of characteristic p, and consider the group algebra kG. Write

$$1 = e_1 + e_2 + \cdots + e_r,$$

where the e_i are central elements such that $e_i e_j = \delta_{ij} e_i$. Any two such decompositions have a common refinement, and the e_i in the finest such decomposition are called blocks. If M is a module, then since $e_i e_j = \delta_{ij} e_i$, we see that

$$M = 1 \cdot M = e_1 \cdot M \oplus \cdots \oplus e_r \cdot M.$$

Thus if M is indecomposable, e.g., simple, $e_i \cdot M = \delta_{ij} M$ for some j. We say that M belongs to e_j. Extend this to sums of modules belonging to the same block. Then submodules and quotients of modules belonging to e_j belong to e_j. Since $e_j \cdot kG$ belongs to e_j obviously, every block has at least one simple module belonging to it.

Unipotent blocks

A unipotent block is a block of kG that has unipotent characters belonging to it. If p is large enough (for example, not dividing the order of the Weyl group and not dividing q) then the number of irreducible Brauer characters belonging to the block is the same as the number of unipotent characters in the block.

Since the unipotent characters are independent of q, it seems reasonable to ask that the unipotent blocks are independent of q. For this to make sense, we have to choose which q and p we want to ask this question for.

Choosing our p and q

Let $G = G(q)$ be a group of Lie type: the order of G is

$$|G| = q^N \prod_{d \in I} \Phi_d(q)^{a_d}.$$

If $p \mid |G|$ then either $p \mid q$, which leads to one theory (here there are usually only two blocks), or $p \nmid q$, in which case $p \mid \Phi_d(q)$ for some d. We are mostly interested in the case where there is no other d' such that $p \mid \Phi_{d'}(q)$; in this case, the Sylow p-subgroup P is abelian, homocyclic, of rank a_d. In particular, if $a_d = 1$ then P is cyclic.

We will always assume that p divides exactly one $\Phi_d(q)$ from now on. The unipotent blocks of kG do not depend on q **or** p, as long as the d involved is the same.

The Cyclic Case: Brauer Trees

Recall the graph constructed before: the vertices are the irreducible complex characters, and connecting two of them if an irreducible Brauer character appears in the decomposition of both of them. Identify two irreducible ordinary characters if their restriction to the p-regular elements is the same, and attach an exceptionality to that vertex of the multiplicity.

Theorem

If the Sylow p-subgroup P of any finite group is cyclic, then the graph above is a forest, all decomposition numbers are 0 *or* 1, *and there is at most one exceptional node.*

In this case, the graph of a block is called the Brauer tree. The question of whether every tree is a Brauer tree of some block of a finite group was answered negatively by Feit in 1984, where he reduced the problem to the finite simple groups. The Brauer trees of alternating and classical groups are lines. Those of sporadic groups are 'irrelevant' only exist for $p < 72$.

Exceptional Groups

This leaves the exceptional groups of Lie type, for which all but E_7 and E_8 were done in the early 1990s. Apart from two unipotent blocks for now (for $G = E_8$, one for $d = 15$ and one for $d = 18$) these have been finished off by Dudas, Rouquier and me.

So we are close to classifying all decomposition matrices for all finite groups (apart from small primes) whenever the Sylow p-subgroup is cyclic.

Can we do the same thing for $I_2(n)$, H_3 and H_4?

Combinatorics of Brauer Trees

There are several pieces of information you can use to assemble the Brauer tree of a block.

- Parity: the sum of degrees of adjacent vertices is divisible by p, hence each vertex is either $+$-type or $-$-type, depending on the congruence of $\chi(1)$ modulo p.
- Degree: the degree of an irreducible character (vertex) is the sum of the irreducible Brauer characters contained in it (edges incident to it). Degrees of characters are positive, so this places constraints.
- The sum of two adjacent characters is a projective character. Inducing a projective character yields a projective character. This can be used when Lie type groups are nested (e.g., $I_2(5) \leq H_3$).

These still work for the non-crystallographic groups, and so we can construct the Brauer trees of their blocks.

Brauer Trees for Pretend Lie Type Groups

I have done this for $I_2(n)$, H_3 and H_4, and the Brauer trees constructed are consistent both with the methods outlined above, and also the statement of the combinatorial version of Broué's conjecture, which uses geometric/combinatorial methods to construct the Brauer tree (but not prove that it is correct).

The combinatorial Broué conjecture makes sense for these 'groups', even though they don't exist, and you arrive at the same answer. Hence the Brauer trees constructed are correct, even if they are meaningless.

Injecting meaning into them is one of the new challenges in Lie theory.