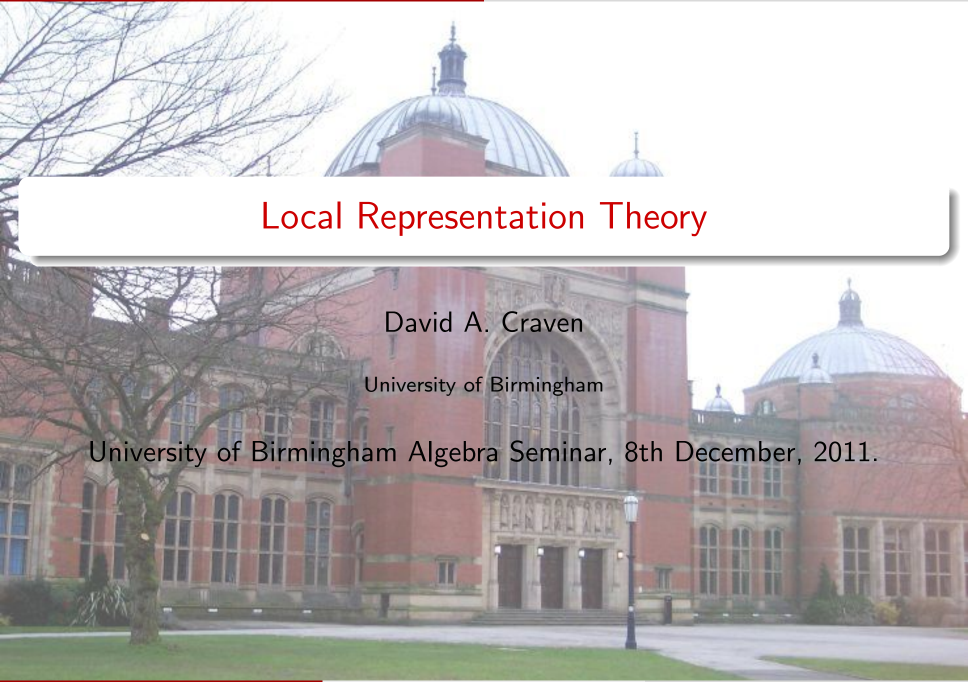

# Local Representation Theory


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# Notation and Conventions

Throughout this talk,

- $G$ is a finite group,
- $\ell$ is a prime,
- $k$ is a field of characteristic $\ell$,
- $B$ is a block of $kG$, with defect group $D$ and Brauer correspondent $b$;
- $P$ is a Sylow $\ell$-subgroup of $G$.

I will (try to) use red for definitions and green for technical bits that can be ignored.

About the first half of this talk is joint work with Raphaël Rouquier.

# Representation Theory is Local

Many features of the modular representation of a finite group are conjectural, some not even conjectural. Broadly, they fall into three categories:

- finiteness conditions
- numerical conditions
- structural conditions

As an example of the first, we have Donovan's conjecture.

As examples of the second, we have the Alperin–McKay conjecture, Alperin's weight conjecture, and Brauer's height-zero conjecture.

As an example of the third, we have Broué's conjecture.

# Representation Theory is Local

Some of the conjectures before (Alperin–McKay, Alperin's weight, Broué) relate the structure of a block $B$ of $kG$ to the structure of its Brauer correspondent $b$, a block of $k\mathrm{N}_G(D)$, where $D$ is a defect group of $B$. Write $\ell(B)$ for the number of simple $B$-modules.

Alperin's weight conjecture gives a precise conjecture about $\ell(B)$ in terms of local information (normalizers of $p$-subgroups). If $D$ is abelian, the conjecture reduces to

$$\ell(B) = \ell(b).$$

Broué's conjecture gives a structural understanding of Alperin's weight conjecture.

**Conjecture (Broué, 1988)**

*Let $G$ be a finite group, and let $B$ be a $\ell$-block of $G$ with abelian defect group $D$. If $b$ is the Brauer correspondent of $B$ in $\mathrm{N}_G(D)$, then $B$ and $b$ are derived equivalent.*

# When Is Broué's Conjecture Known?

Broué's conjecture is known for quite a few groups:

- $G$ soluble
- $A_n$, $S_n$ (Chuang–Rouquier, Marcus)
- $\mathrm{GL}_n(q)$, $\ell \nmid q$ (Chuang–Rouquier)
- $D$ cyclic, $C_2 \times C_2$ (Rouquier, Erdmann, Rickard)
- $G$ finite, $\ell = 2$, $B$ principal
- $G$ finite, $\ell = 3$, $|P| < 81$, $B$ principal (Koshitani, Kunugi, Miyachi, Okuyama, Waki)
- $\mathrm{SL}_2(q)$, $\ell \mid q$ (Chuang, Kessar, Okuyama)
- various low-rank Lie type groups $L(q)$ with $\ell \nmid q$ and sporadic groups. (Okuyama, Holloway, etc.)

# Principal Blocks Are Good

In representation theory, one standard method of proof is to reduce a conjecture to finite simple groups and then use their classification.

In general, there is no (known) reduction of Broué's conjecture to simple groups, but for principal blocks there is.

**Theorem**

*Let $G$ be a finite group. If $P$ is abelian, then there are normal subgroups $H \leq L$ of $G$ such that*

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- $L/H$ *is a direct product of simple groups and an abelian* $\ell$*-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for $L$ (compatible with automorphisms of the simple components) passes up to $G$. Thus if Broué's conjecture for principal blocks holds for all simple groups (with automorphisms), it holds for all groups.

# How Do You Find Derived Equivalences?

There are four main methods to prove that $B$ and $b$ are derived equivalent.

1. Okuyama deformations: using many steps, deform the Green correspondents of the simple modules for $B$ into the simple modules for $b$. This works well for small groups.
2. Rickard's Theorem: randomly find complexes in the derived category of $b$ related to the Green correspondents of the simple modules for $B$, and if they 'look' like simple modules (i.e., Homs and Exts behave nicely) then there is a derived equivalence $B \to b$.
3. More structure: if $B$ and $b$ are more closely related (say Morita equivalent) then they are derived equivalent. More generally, find another block $B'$ for some other group, an equivalence $B \to B'$, and a (previously known) equivalence $B' \to b$.
4. Perverse equivalence: build a derived equivalence up step by step in an algorithmic way.

# What is a Perverse Equivalence?

Let $A$ and $B$ be finite-dimensional algebras, $\mathcal{A} = \textit{mod-}A$, $\mathcal{B} = \textit{mod-}B$.

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \dots, S_r$, $T_1, T_2, \dots, T_r$, and
- a function $\pi : \{1, \dots, r\} \to \mathbb{Z}$

such that, for all $i$, the cohomology of $F(S_i)$ only involves $T_j$ for $j < i$, except for one copy of $T_i$ in degree $-\pi(i)$, and $T_j$ can only appear in degrees less than $-\pi(j)$.

# Benefits of a Perverse Equivalence

A perverse equivalence is ‘better’ than a general derived equivalence.

- Has an underlying geometric interpretation for Lie-type groups, coming from the cohomology of Deligne–Lusztig varieties.
- The $\pi$-function has even been determined for these groups, at least conjecturally. See later!
- There is an algorithm that gives us a perverse equivalence from $B_0(kN)$ to **some** algebra, so only need to check that the target is $B_0(kG)$. (This is simply checking that the Green correspondents are the last terms in the complexes.) This algorithm is very useful!

# An Example: $M_{11}$, $\ell = 3$

| $\pi$ | Ord. Char. | $S_1$ | $S_3$ | $S_7$ | $S_2$ | $S_4$ | $S_6$ | $S_5$ |
|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | | | | | | |
| 2 | 10 | | 1 | | | | | |
| 3 | 10 | | | 1 | | | | |
| 4 | 16 | 1 | 1 | | 1 | | | |
| 5 | 11 | 1 | | | 1 | 1 | | |
| 6 | 44 | | | 1 | 1 | 1 | 1 | |
| 7 | 55 | 1 | 1 | | 1 | 1 | 1 | 1 |
| | 10 | | | | | | | 1 |
| | 16 | 1 | | | | 1 | | 1 |

The cohomology of the complexes gives the rows of the decomposition matrix.

# An Example: $\mathrm{PSL}_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

| $\pi$ | Ord. Char. | $S_1$ | $S_2$ | $S_5$ | $S_3$ | $S_4$ |
|---|---|---|---|---|---|---|
| 0 | 1 | 1 | | | | |
| 3 | $q(q^2+q+1)$ | 1 | 1 | | | |
| 4 | $q^2(q^2+1)$ | | 1 | 1 | | |
| 5 | $q^3(q^2+q+1)$ | 1 | 1 | 1 | 1 | |
| 6 | $q^6$ | 1 | | | 1 | 1 |

$$X_2 : \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0.$$
$$X_5 : \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(234) \oplus M_{4,1} \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0.$$
$$X_3 : \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(5) \oplus M_{1,1} \to M_{1,1} \oplus M_{1,2} \to C_3 \to 0.$$
$$X_4 : \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(4) \to M_{4,2} \to C_4 \to 0.$$

# Which Groups Have Perverse Equivalences?

- All groups, $D$ cyclic or $C_2 \times C_2$
- $\mathrm{PSL}_3(q)$, $\ell = 3 \mid (q-1)$, $P$ abelian
- $\mathrm{PSL}_4(q)$, $\mathrm{PSL}_5(q)$, $\ell = 3 \mid (q+1)$, $P = C_3 \times C_3$
- $\mathrm{PSU}_3(q)$, $\ell = 3 \mid (q+1)$, $P$ abelian
- $\mathrm{PSU}_4(q)$, $\mathrm{PSU}_5(q)$, $\ell = 3 \mid (q-1)$
- $b$ a block of $\mathrm{PSU}_n(q)$, $\ell = 5 \mid (q+1)$, $b$ has defect group $C_5 \times C_5$
- $\mathrm{PSp}_4(q)$, $\ell = 3 \mid (q-1)$ or $(q+1)$, $P = C_3 \times C_3$
- (almost) $\Omega_8^+(q)$, $\ell = 5 \mid (q^2+1)$, $P = C_5 \times C_5$
- (almost) ${}^3D_4(q)$, $\ell = 7 \mid (q^2+q+1)$, $P = C_7 \times C_7$
- $G_2(q)$, $\ell = 5 \mid (q+1)$, $P = C_5 \times C_5$
- $S_6$, $A_7$, $A_8$, $\ell = 3$ ($A_6$ does not)
- $M_{11}$, $M_{22}.2$, $M_{23}$, $HS$, $\ell = 3$ ($M_{22}$ does not)
- $\mathrm{SL}_2(8)$, $J_1$, ${}^2G_2(q)$, $\ell = 2$ in two steps
- $S_n$, $A_n$, $\mathrm{GL}_n(q)$ in multiple steps

# Some Remarks

- Since $\pi(-)$, the ordering and the first category determine the perverse equivalence, it is a very compact way of defining a (type of) derived equivalence.
- Computationally, this reduces finding a derived equivalence to finding the Green correspondents of the simple modules for $G$, a much simpler task.
- For groups of Lie type, it seems as though the complexes above do not really depend on $\ell$, and only on $d$, where $\ell \mid \Phi_d(q)$. It is possible to use these perverse equivalences to prove real results in this direction.

# Groups of Lie Type

Let $G = G(q)$ be a group of Lie type (e.g., $\mathrm{GL}_n(q)$, $\mathrm{SL}_n(q)$, $\mathrm{Sp}_{2n}(q)$): the order of $G$ is

$$|G| = q^N \prod_{i \in I} \Phi_i(q).$$

Suppose that $\ell \nmid q$ divides exactly one of the cyclotomic polynomials $\Phi_d(q)$ in the product. Then the Sylow $\ell$-subgroup is abelian, and contained in a $\Phi_d$-torus.

The unipotent characters of $G$ are certain irreducible characters of $G$, not depending on $q$. A unipotent block of $G$ is one containing a unipotent character, such as the principal block, which contains the trivial character.

# Geometric Broué

Broué's conjecture has a special version for unipotent blocks of groups of Lie type, called the geometric form. The derived equivalence in this case comes from the cohomology of a Deligne–Lusztig variety $Y_\zeta$, where $\zeta$ is any primitive $d$th root of unity. (The variety changes depending on $\zeta$.)

This derived equivalence is supposed to be perverse, with perversity function $\pi(-)$ being the (unique) degree in the cohomology of $Y_\zeta$ in which a given unipotent character appears.

A big open problem since the late 70s is: What is this degree?

# The Parameter $\pi$

Let $\ell \mid \Phi_d(q)$ and let $\chi$ be a unipotent character in the principal $\ell$-block of $kG$. The generic degree of $\chi$ is a polynomial in $q$, that is a product of $q$ and cyclotomic polynomials $\Phi_i(q)$. The relative degree is the generic degree of $\chi$ divided by the relative degree of the unipotent character of the $d$-cuspidal pair associated to $\chi$.

Define $\zeta = e^{2\pi i/d}$, and for $f$ a polynomial in $q$ write $\phi_d(f)$ for the number of non-zero zeroes of $f$ (with multiplicity) of argument at most that of $\zeta$ (with argument in $[0, 2\pi)$, with the exception that positive reals count for $1/2$ (as their argument is 'both' 0 and $2\pi$). Write $a(f)$ for the multiplicity of the zero at 0. Write $\pi_d(f) = (\deg f + a(f))/d + \phi_d(f)$. It should be that if $f$ denotes the relative degree of $\chi$, then $\pi_d(f)$ is the parameter $\pi$ for $\chi$.

This conjecture has been shown to hold in a variety of situations, both for the perverse equivalences and because $\pi(\chi)$ should be the degree in the cohomology (with non-compact support) of a suitably chosen Deligne–Lusztig variety $Y_\zeta$.

# The Cyclic Case

The case where the defect group is cyclic is one where we can say the most. Here the $\pi$-function and bijection are fully understood.

**Theorem**

*Suppose that $G$ is of Lie type, $B$ is a unipotent block, and $D$ is cyclic. If $G$ does not have type $E_7$ or $E_8$ (and even then in many cases) the 'combinatorial form' of Broué's conjecture is true, with $\pi(-) = \pi_d(-)$ and bijection given by the eigenvalues of a root of the Frobenius map.*

The method of proof is simple: using the $\pi$-function and bijection, we construct the Brauer tree of the block, and compare it to the known one (when it is known, i.e., not for some blocks of $E_7$ and $E_8$). Combinatorial Broué's conjecture holds if and only if the Brauer tree is correct.

Notice that this allows us to make conjectures as to the shape of the Brauer tree in the remaining cases, and this has lead some outstanding cases being resolved.

# Beyond Abelian

Broué's conjecture only deals with the case where $D$ is abelian. If $\mathrm{N}_G(P)$ controls fusion in $P$ with respect to $G$ then $\ell(B) = \ell(b)$ again, but we don't always have a derived equivalence (e.g., the principal block of $\mathrm{Sz}(8)$). Something more complicated must happen, even in this case.

We can use Alperin's weight conjecture to try to guide us. Another way of writing Alperin's weight conjecture is

$$|\mathrm{Irr}_{\text{non-proj}}(B)| = \sum_{\sigma \in \mathcal{R}/G} (-1)^{|\sigma|+1} w(B_\sigma),$$

where

- $\mathcal{R}$ is the poset of radical chains, not starting at 1.
- $B_\sigma$ is the corresponding block of $\mathrm{N}_G(\sigma)$
- $w(B_\sigma)$ is the number of non-projective irreducible $B_\sigma$-characters that are $V_\sigma$-projective
- $V_\sigma$ is the minimal element of $\sigma$.

# Algebraic Topology, Enter Stage Right

Now we compare this to results from homotopy theory, factorizing the classifying space $BG$, at least up to $p$-completion, over the normalizers of various collections of $p$-subgroups. The following theorem is only approximately true.

**Theorem (Normalizer decomposition)**

*Let $\mathcal{A}_p(G)$ denote the poset of all p-subgroups of G or any ample collection, and let $\mathcal{N}$ denote the set of all chains in $\mathcal{A}_p(G)$, up to G-conjugation. We have that the map*

$$\underrightarrow{\mathrm{Hocolim}}_{\sigma\in\mathcal{N}} B[N_G(\sigma)] \to BG$$

*is a mod-p cohomology equivalence.*

# The Future

Ordinary cohomology passes through a homotopy colimit (via Bousfield–Kan spectral sequences) to create an alternating sum, as with Alperin's weight conjecture. However, Hochschild cohomology $HH^1(-)$, which can count the number of simple modules, is not (in general) functorial with respect to homotopy colimits, so a naïve attack of Alperin's weight conjecture won't work.

**Question**

*What is the analogue of the derived category that will extend Broué's conjecture to the non-abelian case?*

Our guide to this problem should be the theory of homotopy colimits over fusion systems, associating a space at each point of the orbit category of the centric radical subgroups. This is, in a very real sense, the major problem in group representation theory.