Minimally active elements in matrix groups

David A. Craven

University of Birmingham

Algebra and Geometry Seminar,
University of Newcastle,
7th February, 2017

Primitive permutation groups containing a p-cycle

Let G be a transitive subgroup of the symmetric group S_n, and suppose that G contains a p-cycle $(1, 2, \ldots, p)$ for some prime p. There are many such subgroups of S_n if $p \mid n$, for example the wreath product $S_p \wr S_{n/p}$.

Suppose in addition that G acts primitively on $\{1, \ldots, n\}$. In this case the list is very short:

1. $G = A_n, S_n$, $p \leq n$;
2. $n = p = (q^d - 1)/(q - 1)$, and $\mathrm{PSL}_d(q) \leq G \leq S_p$;
3. $n = p$, $G \leq \mathrm{AGL}_1(p) \leq S_p$;
4. $n = p + 1 = 2^m$, $G \leq \mathrm{ASL}_m(2)$;
5. $n = p + 1$, $\mathrm{PSL}_2(p) \leq G \leq S_{p+1}$;
6. $n = p + 2 = 2^m + 1$ and $\mathrm{PSL}_2(2^m) \leq G \leq S_{p+2}$;
7. $n = p, p + 1$ and G is $\mathrm{PSL}_2(11)$, M_{11} M_{12}, M_{23}, M_{24}.

More primitive permutation groups

This is consistent with an old theorem of Jordan.

Theorem

Let G be a primitive permutation group of degree n containing a p-cycle. If $n \geq p+3$ then $G = A_n$ or $G = S_n$.

Any permutation group gives rise to a matrix group over any ring.

If the ring is a field of characteristic p, then a p-cycle u becomes a matrix that is conjugate to a triangular matrix with Jordan normal form consisting of a block of size p and $n-p$ blocks of size 1.

If V is a subquotient of this permutation module then u again acts on V as a triangular matrix (up to conjugacy), again with exactly one non-trivial Jordan block (if u acts non-trivially on V) but this block need not have size p in this case.

Groups generated by transvections

A transvection u is a matrix conjugate to the matrix $1+\varepsilon_{1,2}$, where $\varepsilon_{i,j}$ is the matrix with a single 1 in the (i,j)-position and 0s everywhere else.

If k is a field of characteristic p then u is triangular, with Jordan normal form consisting of a block of size 2 and all other blocks of size 1.

If $G \leq \mathrm{GL}(V)$ is an irreducible matrix group generated by a class of transvections then G is one of the following (Kantor, 1979):

1. $\mathrm{SL}(W)$, $\mathrm{Sp}(W)$, $\mathrm{SU}(W)$, or $\mathrm{SO}^{\pm}(W)$ if $p=2$, where $W \otimes k = V$;
2. $S_{n+1} \leq \mathrm{SL}_n(2)$ or $S_{n+2} \leq \mathrm{SL}_n(2)$ for n even;
3. $\mathbb{F}_4 \leq k$, $3 \cdot A_6 \leq \mathrm{SL}_3(k)$ or $3 \cdot \mathrm{P}\Omega_6^-(3) \leq \mathrm{SL}_6(k)$;
4. $p=2$, T is a toral subgroup and $G = T \rtimes S_n$;
5. $\mathbb{F}_9 \leq k$ and $G = \mathrm{SL}_2(5) \leq \mathrm{SL}_2(k)$.

Therefore if p is odd and $\dim(V) > 2$ then G is a classical group in its natural representation, with something similar if $p=2$ and $\dim(V) > 6$.

Fusion systems

In recent work with Bob Oliver and Jason Semeraro, we considered irreducible matrix groups G over the field $\mathbb{F}_p$ with the following properties:

1. G has a Sylow p-subgroup of order p, generated by u;
2. u is conjugate to all of its powers, or equivalently $N_G(\langle u\rangle)/C_G(\langle u\rangle)$ has order $p-1$;
3. u has Jordan normal form with exactly one non-trivial Jordan block.

We determined all such matrix groups and proved the following:

Theorem

If $G \leq \mathrm{GL}(V)$ has the properties above, and $\dim(V) \geq p+3$, then G is A_n or S_n acting on the deleted permutation module, or a subgroup of the normalizer of a torus $T \rtimes S_n$.

Both of these cases appeared in the previous theorem on transvections.

Monodromy of Kloosterman sheaves

This is a Kloosterman sum:

$$\mathrm{Kl}_{2,p}(a) = \frac{-1}{\sqrt{p}} \sum_{x \in \mathbb{F}_p^\times} \exp\left(\frac{ax + x^{-1}}{p}\right)$$

where $a \in \mathbb{F}_p^\times$.
This is a hyper-Kloosterman sum:

$$\mathrm{Kl}_{n,q}(a) = \frac{-1}{\sqrt{q^{(n-1)/2}}} \sum_{x_1,\dots,x_n \in \mathbb{F}_q^\times, \prod x_i = a} \exp\left(\frac{\mathrm{tr}(x_1 + \cdots + x_n)}{p}\right)$$

where $a \in \mathbb{F}_q^\times$.

This helps study the number of representations of an integer by an integral, positive-definite quadratic form in four variables, and more generally the spectral theory of automorphic forms. A Kloosterman sheaf over $\bar{\mathbb{Q}}_\ell$ is an ℓ-adic sheaf which has a (hyper-)Kloosterman sum as its trace function.

In 1988, Katz determined the monodromy of Kloosterman sheaves, the

Unifying these examples

Let G be a finite group and let V be a kG-module for k a field of characteristic $p \mid |G|$. A p-element u of G acts minimally actively on V if, in its Jordan normal form, u acts with at most one non-trivial block. Equivalently, u is minimally active if $C_V(u) \cap [V, u]$ has dimension at most 1. (It has dimension 0 if u acts trivially on V.)

All of the examples above have u acting minimally actively. Thus we can ask the following question:

Question

If u acts minimally actively (and non-trivially) on a simple module V, and $\dim(V) \geq o(u) + 3$, is (G, V) one of a small list of examples? If G is generated by conjugates of u is this list globally bounded?

Maybe '$o(u) + 3$' needs to be replaced by some larger bound. The list should include alternating groups, classical groups, and normalizers of split tori, and maybe others?

A basic restriction

Let G be a finite group, V be a simple kG-module and u be a p-element acting minimally actively on V. Let H denote the normal closure of u in G.

We can only really say anything about H, since if $H \leq G \leq \mathrm{GL}(V)$ then of course G contains a minimally active element. Let $\alpha(u)$ denote the minimal number of conjugates of u needed to generate H.

Lemma

If the action of H on V fixes no 1-spaces and u acts minimally actively on V, then $\dim(V) \leq \alpha(u) \cdot (o(u) - 1)$.

Note that $C_V(u)$ has codimension at most $o(u) - 1$. Therefore, the centralizer $C_V(\langle u_1, \ldots, u_i \rangle)$ has codimension at most $i \cdot (o(u) - 1)$, where the u_j are conjugates of u. Finally, $C_V(H) = 0$ and $u_1, \ldots, u_{\alpha(u)}$ generate H.

For all finite simple groups G, if $u \in G$ then almost certainly $\alpha(u) = 2$. (If $u^2 = 1$ then $\alpha(u) = 3$ normally works.)

What do irreducible matrix groups look like?

In order to answer a question like this, one needs to understand the structure of an irreducible subgroup of $\mathrm{GL}(V)$.

This is given by Aschbacher's theorem. Broadly speaking, $G \leq \mathrm{GL}(V)$ stabilizes one of a few obvious geometric structures, or the image of G under the map $\mathrm{GL}(V) \to \mathrm{PGL}(V)$ is almost simple.

Thus we have two sub-questions to solve to answer our question:

The first is 'which of these geometric subgroups of $\mathrm{GL}(V)$ contain minimally active elements, and what sort of groups do they generate'?

The second is 'can one classify all pairs (G, V) such that $G/Z(G)$ is almost simple, V is a kG-module, and G contains an element u acting minimally actively on V'?

Answering the second question

The second question requires the classification of finite simple groups, which exists. Recall that if G is a (non-abelian) finite simple group then G is one of

1. an alternating group A_n;
2. a group of Lie type $G(q)$;
3. one of twenty-six sporadic groups.

Since groups of Lie type depend on a field, we have the cases where $p \mid q$ and where $p \nmid q$, which have very different theories.

Until further notice, let G be a group such that $G/Z(G)$ is almost simple (and with $Z(G)$ cyclic and $G' = G$) and V be a simple kG-module with $u \in G$ acting minimally actively on V.

Sporadic groups

Here, if $u \in G$ has prime-power order then $\alpha(u)$ is known explicitly, as there are only 26 groups to check.

In fact, this has already been done if G is simple, by Di Martino, Pellegrini and Zalesski in 2014. They did not check almost simple groups, but there are no examples there.

Theorem

Suppose that G is sporadic. There are seventeen possibilities for (G, V), and in these G is either a Mathieu group, a Conway group, J_1, J_2, J_3, Suz or Ru. In all cases, p is one of $2, 7, 11, 13, 17, 19, 23, 29$.

In all cases, $\dim(V) \leq o(u) + 1$, and in all but three cases (all for $p = 2$) the Sylow p-subgroup of G is of order p.

An example: $\mathrm{SL}_2(q)$ for $p \nmid q$

Let's consider an example: let G be the group $\mathrm{SL}_2(q)$ for q even, and suppose that p is odd. Let u be a p-element of G.

The order of u divides either $q-1$ or $q+1$. Since $\alpha(u) = 2$ this means that if u acts minimally actively on V then $\dim(V) \leq 2o(u)$.

On the other hand, the dimensions of simple kG-modules are (a subset of) $1, q-1, q, q+1$. Thus $\dim(V) \geq q-1$ if V is non-trivial.

Suppose that $o(u) \neq q \pm 1$. As $q \pm 1$ is odd, this means that $o(u) \leq (q+1)/3$. We therefore have (by $\alpha(u) \cdot o(u) > \dim(V)$)

$$2(q+1)/3 \geq \alpha(u) \cdot o(u) > \dim(V) \geq q-1,$$

which means $q \leq 4$. This means that $G = \mathrm{SL}_2(4)$ and we can just work with this group. Otherwise $q \pm 1$ is a prime power, so is either 9 or a Fermat or Mersenne prime.

Groups of Lie type in general

This is indicative of the general approach. For most groups of Lie type we do not know the complete collection of character degrees, but we do know the small ones. Also, if u is a p-element then $o(u)$ does divide a cyclotomic polynomial in q (more or less).

The result is also the same: we get some families (e.g., $\mathrm{SL}_2(q)$ for $q-1$ a Mersenne prime), together with a few small groups (e.g., $\mathrm{SL}_2(4)$).

The conclusion is that if u acts minimally actively on a (non-trivial) simple module V then V is either a Weil module or one of a small list of other cases (e.g., $\mathrm{Sp}_4(4)$ for $p=17$, $\dim(V)=18$).

The only violations of $\dim(V)\leq o(u)+2$ for V minimally active are for $3_1\cdot \mathrm{PSU}_3(4).2=G_{34}$, with $o(u)=2$ and $\dim(V)=6$, and $2\cdot\Omega_8^+(2)=W(E_8)$, $o(u)=3,5$ with $\dim(V)=8$. These are both complex reflection groups.

A possible classification

There are dozens of examples, but this seems to be true.

Conjecture

Suppose that u acts minimally actively on the simple module V, and that V is primitive and tensor indecomposable. One of the following holds:

1. *G is a classical group (including B_2 and D_3) or G_2 acting on the natural module, (or its symmetric powers), or $L(11)$ for A_2 and $p = 2$;*
2. *G is of type B_3 acting on the spin representation;*
3. *G is a subgroup of a complex reflection group acting on the reflection representation (or a composition factor of it if it is not simple);*
4. *G has a self-centralizing cyclic Sylow p-subgroup and $\dim(V) \leq o(u) + 1$ (important subcase: V is a Weil module for a classical group in cross characteristic);*
5. *G is one of six examples, in dimensions 4, 6 and 9, and $p = 2$ or $p = 3$.*

Permutation modules again

If M is a permutation module for a primitive permutation group $G \leq S_n$ with a p-cycle then by the old theorem of Jordan mentioned at the start $n \leq p + 2$. Thus the simple part of the permutation module has dimension at most $o(u) + 1$, and of course, since $n \leq p + 2$ and G is generated by conjugates of u (so that $G \leq A_n$) we have that the Sylow p-subgroup is cyclic and self-centralizing.

Thus those examples fit into the previous theorem/conjecture, as we would expect.

Out of the other maximal subgroups of classical groups, which stabilize geometric structures in some way, there is only the extraspecial type subgroups that I need to check, to see that they fit into this framework. If this holds, then the previous theorem/conjecture should hold for all finite groups G, not just the simple ones.

An application to monodromy of Kloosterman sheaves

Last year, Corentin Perret-Gentil studied Kloosterman sheaves over $\mathcal{O} = \mathbb{Z}[\zeta_{4p}]$ rather than $\bar{\mathbb{Q}}_\ell$, meaning he looked at arithmetic and geometric monodromy over finite fields.

Given n, for sufficiently large primes he showed that the integral monodromy groups coincided with those produced by Katz, but because of a lack of knowledge of simple groups containing minimally active elements couldn't get explicit bounds.

Using these results, I speculate that one can easily prove the following:

Conjecture

Let $\mathcal{K}l_n$ be a Kloosterman sheaf over $\mathcal{O}$, and suppose that $\ell > n+1$. Then

$$G_{\mathrm{geom}}(\mathcal{K}l_n) = G_{\mathrm{arith}}(\mathcal{K}l_n) = \begin{cases} \mathrm{SL}_n(\mathcal{O}) & \textit{if } n \textit{ is odd,} \\ \mathrm{Sp}_n(\mathcal{O}) & \textit{if } n \textit{ is even.} \end{cases}$$