Blocks with a Klein Four Defect Group

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The following is joint work with Charles Eaton, Radha Kessar and Marcus Linkelmann.

For the purposes of this talk, G is a finite group, K is an algebraically closed field of characteristic p, where $p \mid |G|$, and all modules are finite-dimensional.

1 Facts from Block Theory

Given a field K and a finite group G, we may of course form the group algebra, KG. Since this is some ring, we can write

$$KG = B_1 \oplus B_2 \oplus \cdots \oplus B_r,$$

where the B_i are two-sided ideals that cannot be written as the sum of two smaller two-sided ideals. These are called *blocks*.

We also have a 1 in KG. Write

$$1 = e_1 + e_2 + \cdots + e_r,$$

where the e_i are central idempotents such that if $e_i = e + e'$, where e and e' are also central idempotents, then $e = 0$ or $e' = 0$. If we label things correctly, then $e_i \in B_i$, and in fact $B_i = KGe_i$. The e_i are also called blocks. Some people call the B_i the block ideals and the e_i the blocks, and some people call the B_i the blocks and the e_i the block idempotents. Confusion reigns.

Let M be an indecomposable module. Then $M \cdot 1 = M$, and so

$$M = Me_1 \oplus Me_2 \oplus \cdots \oplus Me_r.$$

Since M is meant to be indecomposable, there exists a unique i such that $Me_i = M$, and for all other j the module Me_j is the zero module. We say that M *belongs to the block* e_i (or B_i).

Let M be an indecomposable module. It turns out that if P denotes a Sylow p-subgroup of G, then M is a summand of $(M \downarrow_P) \uparrow^G$. (The proof of this is the same as Maschke's Theorem.) Therefore there are minimal p-subgroups Q such that M is a summand of $(M \downarrow_Q) \uparrow^G$. These are called *vertices* of M.

For example, a module has vertex the trivial subgroup if and only if M is a summand of KG, the group algebra; i.e., M has trivial vertex if and only if it is projective (a summand of a free module).

Theorem 1.1 (Green) All vertices of an indecomposable module are p-subgroups of G and a G-conjugate.

A *source* of an indecomposable module M is an indecomposable module S for the vertex Q such that M is a summand of $S \uparrow^G$. Since M is a summand of $M \downarrow_Q\uparrow^G$, such a module exists.

Theorem 1.2 (Green) All sources of an indecomposable module M are conjugate by an element of $\mathrm{N}_G(Q)$, where $Q = \mathrm{vx}\, M$.

There is an invariant of a block, called a *defect group*, that controls the complexity of the modules that belong to it. We will define a defect group now, using vertices. Let $\mathscr{M}$ denote the collection of all indecomposable modules belonging to some block B. To each of these modules one may associate a conjugacy class of p-subgroups $\mathcal{P}$, and thus we get a collection $\mathscr{P}$ of p-subgroups (up to conjugation). This takes on the structure of a poset, using the obvious ordering. It turns out that this poset has a unique maximal element, and this is called the *defect group* of a block.

The defect group is trivial if and only if B is (isomorphic to) a matrix algebra, and contains a unique indecomposable module, which is a projective simple module. If the defect group D of B is cyclic, then B contains only finitely many indecomposable modules, and if D is dihedral, semidihedral or quaternion then, although there are infinitely many indecomposable modules in B, they can be parameterized. For all other defect groups, the indecomposable modules belonging to them are too complicated to 'understand'.

2 Blocks with a Klein Four Defect Group

Consider the group algebra KV_4. This has one simple module, which has trivial source. The projective indecomposable module for this group algebra has the structure

$$\begin{matrix} K \\ K \oplus K. \\ K \end{matrix}$$

Consider the group algebra KA_4. This has a single block, and has three simple modules, K, S_1 and S_2, each with trivial source. The projective indecomposable modules for this group algebra have the structure

$$\begin{matrix} K \\ S_1 \oplus S_2, \\ K \end{matrix} \qquad \begin{matrix} S_1 \\ K \oplus S_2, \\ S_1 \end{matrix} \qquad \begin{matrix} S_2 \\ S_1 \oplus K. \\ S_2 \end{matrix}$$

Finally, consider the group algebra KA_5. This has two blocks: the principal block (containing the trivial module) and a block of defect zero, containing the 4-dimensional simple module. The principal block contains three simple modules, K, S_1 and S_2, with K having trivial source and S_i having 2-dimensional source. The indecomposable modules for this group algebra have the structure

$$\begin{matrix} & K & \\ S_1 & & S_2 \\ K & \oplus & K, \\ S_2 & & S_1 \\ & K & \end{matrix} \qquad \begin{matrix} S_1 \\ K \\ S_2, \\ K \\ S_1 \end{matrix} \qquad \begin{matrix} S_2 \\ K \\ S_1. \\ K \\ S_2 \end{matrix}$$

In 1982, Karin Erdmann published a paper in which it was proved that every block with V_4 defect group had either one or three simple modules (originally proved by Brauer) with the principal indecomposable modules looking as above. These facts can be determined from the module category, and so Erdmann really proved that there are three types of module category that can be associated to a block with Klein four defect group.

However, the sources of simple modules cannot be determined using the information in the module category, the so-called *Morita equivalence* class of the block. We need something sharper: this is the notion of a *Puig equivalence*. We won't discuss the technical details here, but rather mention that a Puig equivalence is a Morita equivalence that also remembers some of the representation theory of the group. For example, two Puig equivalent blocks have the same sources of the simple modules.

Conjecture 2.1 (Erdmann, 1982) Let B be a block with defect group V_4, and suppose that it is Morita equivalent to a block b from the list $\{KV_4, KA_4, B_0(KA_5)\}$. Then the sources of the simple modules are isomorphic.

In 1996, Linckelmann proved the following result, which gives some more context to the conjecture.

Theorem 2.2 (Linckelmann, 1996) Erdmann's conjecture is true if and only if there are exactly three Puig equivalence types of block with defect group V_4.

To give yet another formulation of this conjecture, we look at the Green correspondence. Recall that if M is an indecomposable module with vertex Q, then the restriction $M\downarrow_{\mathrm{N}_G(Q)}$ splits up as one (indecomposable) summand with vertex Q and other summands with vertex contained properly within Q (up to conjugacy). In 1982, Erdmann proved the following.

[Here we need a definition of the functor Ω.]

Theorem 2.3 (Erdmann, 1982) Let B be a block with defect group P, and let $H = \mathrm{N}_G(P)$. Let f denote the Green correspondence from B to the corresponding block b of H. Let S_0, S_1 and S_2 denote (where applicable) the one or three simple modules in B.

(i) If B is Morita equivalent to KV_4, then there is an integer $i \in \mathbb{Z}$ such that $\Omega^i(f(S_0))$ is simple.

(ii) If B is Morita equivalent to KA_4, then there is an integer $i \in \mathbb{Z}$ such that $\Omega^i(f(S_j))$ is simple for $j = 0, 1, 2$ (and the simple modules are non-isomorphic).

(iii) If B is Morita equivalent to $B_0(KA_5)$, then there is an integer $i \in \mathbb{Z}$ such that $\Omega^i(f(S_0))$ is simple, and S_1 and S_2 are periodic.

Erdmann's conjecture is equivalent to the statement that in all parts of the theorem above, the integer i is 0.

There is another equivalent formulation, which is how I became interested in this conjecture: recall that a module is algebraic if it satisfies a polynomial with integer coefficients, where addition and multiplication are given by direct sum and tensor product.

Proposition 2.4 (C, 2007) Erdmann's conjecture is true if and only if all simple modules in blocks with Klein four defect group are algebrai.

Thus Erdmann's conjecture determines the possible sources of the simple modules, the Green correspondence in such blocks, and the Puig equivalence classes of block.

3 The Proof of Erdmann's Conjecture

The first step is what makes it possible to attack this conjecture.

Theorem 3.1 (EKL, 2006) Erdmann's conjecture is true for all groups if and only if it is true for all groups HP, where H is an odd central extension of a simple group, and P is isomorphic with a Klein four group, which acts as automorphisms on $H/\mathrm{Z}(H)$.

Thus we reduce the entire conjecture to checking it for each finite quasisimple group, possibly with automorphisms on top. The idea is that P is the defect group of a block B with defect group V_4 that we are examining. This will be our situation from now on.

One of the main tools in attacking this conjecture is to notice that it is true for real blocks. To see this, note that a source is only defined up to conjugacy; however, if it is of the form $\Omega^i(K)$, as we know it is, then it is unique up to isomorphism. The dual of $\Omega^i(K)$ is $\Omega^{-i}(K)$; so if a block is self-dual, then a simple module in it is self-dual, and therefore its source is self-dual. Thus the source is K, and so the conjecture is true for this block.

The first theorem is easy.

Theorem 3.2 (E (2006), and C (2007)) Suppose that H is sporadic or alternating. Then B is real, and hence Erdmann's conjecture is true.

The next theorem is also easy, but it requires some knowledge of the groups involved.

Theorem 3.3 (C, 2008) Suppose that H is symplectic, orthogonal or of type ${}^3D_4(q)$. Then B is real, and hence Erdmann's conjecture is true.

To prove this, we actually show that *all* 2-blocks of the quasisimple group H are real.

Charles Eaton and I sorted out the exceptional groups, with two exceptions (!).

Theorem 3.4 (CE, 2008) Let H be an exceptional group, except for $E_6(q)$ or $E_7(q)$ when $P \cap H = 1$. Then Erdmann's conjecture is true.

Radha Kessar is responsible for the next two theorems, which finish off all finite quasisimple groups.

Theorem 3.5 (K, 2007/8) Let H be linear or unitary. Then Erdmann's conjecture is true.

Theorem 3.6 (K, 2008) Let H be of type $E_6(q)$ or $E_7(q)$, where $P \cap H = 1$. Then Erdmann's conjecture is true.

This last result is a nice application of Deligne–Lusztig theory, and so we will briefly describe some of the highlights, in the case where H is of type $E_6(q)$.

- There is a unique V_4 subgroup of $\mathrm{Out}(H/\mathrm{Z}(H))$ up to conjugation, and so we may assume that P consists of the graph automorphism, field automorphism, and their product.
- Since the graph automorphism inverts the centre of $3.E_6(q)$, and P is meant to centralize $\mathrm{Z}(H)$, we see that we may assume that $\mathrm{Z}(H) = 1$.
- The adjoint group is actually $E_6(q).3$, and we can take this group instead. (This makes the Deligne–Lusztig theory easier.) The dual group is the simply connected group $L = 3.E_6(q)$.
- By a theorem of Feit and Zuckerman, all semisimple elements of L are real. (Remember that the semisimple elements of L label the Lusztig series.)
- Let B be a block of defect zero in KH, and let M denote the unique simple module. Then M comes from a Lusztig series with (real) semisimple label s.
- Since M has trivial vertex, the centralizer of s is a torus. Hence the Lusztig series contains a single element, namely the module M.
- Finally, as s is real, the dual of M also lies in the series defined on s, and so M is self-dual, as required.

4 After the Proof

There are several avenues of possible further development.

- The Feit conjecture for V_4 vertex.
- Blocks with quaternion group Q_8.
- Studying dihedral blocks.
- Blocks with defect group $C_3 \times C_3$.
- 2-blocks with abelian defect group.