Minimally active modules for finite groups.

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What is a minimally active module?

Let G be a finite group, let p be a prime, and suppose that the Sylow p-subgroup T of G has order p. Let $k = \mathbb{F}_p$.

Definition

A kG-module M is minimally active if the indecomposable summands of the restriction of M to T contain at most one non-trivial module.

Equivalently:

1. As a kT-module, $\dim(\mathrm{soc}(M) \cap \mathrm{rad}(M)) \leq 1$. ($C_M(T) \cap [M, T]$ has order p.)
2. The sizes of the Jordan blocks of an element x of order p are $a, 1^i$ for some $a \geq 1$ and $i \geq 0$.

Some easy properties

1. Submodules and quotients of minimally active modules are minimally active.
2. If M is minimally active and $M = M_1 \oplus M_2$, then $O^{p'}(G)$ acts trivially on at least one of the M_i.
3. Restrictions of minimally active modules are minimally active.
4. If M is indecomposable, minimally active and $\dim(M) > p$ then M is a trivial-source module (i.e., a summand of a permutation module).
5. If M is indecomposable, minimally active and $\dim(M) > p + 1$ then M is simple.

But David: why should I care?

I had better have a good reason to introduce these modules and occupy an hour of your life with them. If you think fusion systems are a good thing then I have a good reason. If you don't then I don't, really.

A fusion system was introduced last week by Sejong, I hope. I won't do a formal introduction, but I'll just give a working definition.

Let S be a finite p-group. We construct a category $\mathcal{F}$ with objects all subgroups of S, and as morphisms some injective group homomorphisms between them, that

- contain all conjugation maps induced by elements of S,
- have the map $\phi : P \to \mathrm{im}(\phi)$ whenever there is a map $\phi : P \to Q$, and
- have the inverses $\phi^{-1} : Q \to P$ whenever ϕ is a bijection.

Saturated fusion systems

As a fundamental example, we take a finite group G, a fixed Sylow p-subgroup S of G, and let $\mathcal{F}_S(G)$ be the fusion system whose morphisms are all conjugation maps $c_g : P \to Q$ where $g \in G$ satisfies $P^g \leq Q$ for $P, Q \leq S$.

If we want to use fusion systems, we need another axiom, called saturation. This is the bit I won't formally define, but it concerns being able to extend maps $\phi : P \to Q$ to certain overgroups of P, so that one may use induction on the index $|S : P|$.

Even though I haven't defined it, I will now assume that all fusion systems are saturated. A saturated fusion system that doesn't come from a finite group is called exotic.

The search for exotic systems

Exotic fusion systems seem to offer a glimpse into what finite simple groups that don't exist should look like. More or less all exotic fusion systems are simple (or built up from exotic simple fusion systems), and many fusion systems of simple groups (at least for p small, as we shall see) are themselves simple.

The 'simplest' exotic systems were found by Ruiz and Viruel, and are on the extraspecial group 7_+^{1+2} of exponent 7. Others have been found, for example by Solomon–Benson on Sylow 2-subgroups of $\mathrm{Spin}_7(r)$ for r odd (the only known simple exotic systems at the prime 2) and by on certain 3-groups of maximal class by Díaz–Ruiz–Viruel. Another set of exotic fusion systems were constructed by Clelland and Parker, using modules for $\mathrm{GL}_2(p)$.

What the Ruiz–Viruel and the Clelland–Parker examples have in common is that the Sylow p-subgroup S in both cases possesses an abelian subgroup A of index p.

Minimal examples?

If S is abelian, then Alperin's fusion theorem, which states that every map in $\mathcal{F}$ is a product of (restrictions of) automorphisms of subgroups that contain their own centralizer, proves that every map in S is a restriction of an automorphism of S.

In other words, if H is a p'-group of automorphisms of S, then we can construct the group $S \rtimes H$, and $\mathcal{F}_S(S \rtimes H)$ is a saturated fusion system on S, and all saturated fusion systems on S arise in such a way.

If however, the abelian subgroup has index p, then we have lots of examples where this is not the case, for example $G = S_{p^2}$, where the Sylow p-subgroup is $C_p \wr C_p$, or $\mathrm{GL}_p(q)$ for $p \mid (q-1)$, or the Monster at $p = 13$, and so on.

It therefore seems like a good idea to 'classify' (in a suitable sense) all saturated fusion systems on p-groups with an abelian subgroup of index p.

What do you mean, 'classify'?

I don't want to try to classify every saturated fusion system on such groups, because in particular it would require classifying all p'-subgroups of $\mathrm{GL}_n(p)$.

The theory of **tame** and **reduced** fusion systems was started was Andersen, Oliver and Ventura and 2012. The central tenet is as follows:

A reduced fusion system is tame if an only if all saturated fusion systems reducing to it are realizable as fusion systems of finite groups.

OK, great. So what is a reduced fusion system? What is a tame fusion system for that matter?

Definition

A saturated fusion system $\mathcal{F}$ on S is **reduced** *if it has no normal subgroups, no normal subsystems on S itself, and no non-trivial morphisms $\mathcal{F} \to \mathcal{F}_T(T)$ for some $T > 1$ (i.e. $\mathrm{O}_p(\mathcal{F}) = 1$ and $\mathrm{O}^p(\mathcal{F}) = \mathrm{O}^{p'}(\mathcal{F}) = \mathcal{F}$).*

Have I thrown away too much?

No. In particular, all simple fusion systems and semisimple fusion systems are reduced. Hence if we are interested in simple fusion systems, the larger class of reduced fusion systems is still fine for us.

I seem to have a lot more room on this slide, so I can say a few words about tameness. A fusion system $\mathcal{F}$ is **tame** if there exists a finite group G with Sylow p-subgroup S, and firstly $\mathcal{F} = \mathcal{F}_S(G)$, and secondly the map

$$\kappa_G : \mathrm{Out}(G) \to \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}_S^c(G))$$

is split surjective, where $\mathcal{L}_S^c(G)$ is the centric linking system. Since I am interested in simple fusion systems, and we will not be checking whether any of these things are tame, this isn't really important for us today.

More than one abelian maximal subgroup

It seems reasonable to split the cases up into where there is more than one abelian subgroup of index p, and where there isn't.

In the case where S has more than one abelian subgroup of index p, Bob Oliver has already done this case. This will clearly contain the examples on $S = p_+^{1+2}$, and won't contain the examples on $C_p \wr C_p$.

Theorem (Oliver)

In this case, if S possesses a reduced fusion system then $S = p_+^{1+2}$.

So from now on we will assume that S contains a unique abelian subgroup of index p.

The action on A

Let $G = \mathrm{Aut}_{\mathcal{F}}(A)$. We will assume that A is elementary abelian in the rest of this talk. Thus A becomes an $\mathbb{F}_p G$-module. What kind of structure does this module and the group G have?

Firstly, since A is fully normalized and has index p in S, G has a Sylow p-subgroup T of order p. We get the following conditions on A and G:

1. $|\mathrm{Aut}_G(T)| = p - 1$;
2. A is minimally active;
3. A has no trivial quotients, i.e., $[G, A] = A$.
4. $\mathrm{C}_A(G) \leq [T, A]$, which is slightly weaker than A having no trivial submodules;
5. And a technical condition on the action of $\mathrm{Aut}_G(T)$. Always satisfied if $\dim(A) \leq p$.

Understanding G

Suppose that A is a (faithful) simple minimally active module of dimension n, yielding an embedding of G into $\mathrm{GL}_n(q)$. Suppose that $G/Z(G)$ is not an almost simple subgroup of $\mathrm{PGL}_n(q)$. This means that G falls into one of a few geometrically defined classes of maximal subgroups, e.g., parabolic subgroups, direct products of GL_ms, wreath products, etc.

As A is simple, this gets rid of things like parabolics and products of groups. If A is not absolutely irreducible then the action of T on A would have multiple non-trivial Jordan blocks, and the same if A were writeable as $X \otimes Y$ for X, Y of dimension at least 2. Thus A is not in extension type subgroups or wreath products.

We continue like this until $G \leq C_{p-1} \wr S_n$ is a collection of monomial matrices, a couple of central products inside extraspecial type maximal subgroups, or is almost simple (modulo the centre). Thus we want to understand minimally active modules for almost simple groups.

$\mathrm{GL}_2(p)$

Since T has order p, if G is Lie type in defining characteristic then G is of type $\mathrm{PSL}_2(p)$. For $\mathrm{GL}_2(p)$ there are simple modules of dimension $1, \dots, p$, and each of these is minimally active, and in addition $|\mathrm{Aut}_G(T)| = p - 1$. These yield the Clelland–Parker examples.

However, there are more modules for $\mathrm{GL}_2(p)$. If A is a module of dimension $i > 1$, then A has extensions with two other modules N_1 and N_2, of dimensions $p + 1 - i$ and $p - 1 - i$. This yields indecomposable modules of dimension $p + 1$ and $p - 1$, both minimally active also. Apart from a couple of modules with 1-dimensional socle, these are all minimally active modules for G.

The indecomposable modules of dimension $p - 1$ yield new, exotic fusion systems, whereas almost all of those of dimension $p + 1$ fail the technical condition that I sort of told you about, which is satisfied whenever $\dim(A) \leq p$. (This comes back later.)

Alternating and sporadic groups

For alternating and sporadic groups, there is a useful result that we can apply that will make our lives much easier.

Proposition

If a simple group G is either of alternating or sporadic type, and $p > 3$ divides $|G|$, then G is generated by two elements of order p.

This is important: if A is a minimally active module then the socle of the action of an element x of order p has codimension at most $p-1$ (since the non-trivial block has dimension at most p). If $G = \langle x, y \rangle$ then the intersection of $C_A(x)$ and $C_A(y)$ has codimension at most $2p-2$.

Thus if A is simple then $\dim(A) \leq 2p-2$. Since $\dim(A) > p+1$ meant that A is simple (I told you that before: were you not listening?) this means that $\dim(A) < 2p-2$ whenever G is generated by two elements of order p.

Lie type in non-defining characterstic

For groups of Lie type in non-defining characteristic, it looks as if, for $p > 5$ dividing $|G|$, they are also generated by two elements of order p. However, we are some way from proving this statement, so we cannot use it.

We need another way to bound the dimension of a minimally active module.

Proposition

If $T \in \mathrm{Syl}_p(G)$ and $\mathrm{C}_G(T)$ is abelian, then the dimension of any minimally active module is at most $2p - 1$.

This follows from the theory of canonical characters, which implies that there are at most $(p-1)\chi(1)$ trivial summands in the restriction of a minimally active module to T, where $\chi \in \mathrm{Irr}(C_G(T))$.

Now, if we could only find a way to make the centralizer $C_G(T)$ abelian.

Induction to the rescue

Obviously the centralizer isn't abelian in all cases. But we can set up an induction using the following result.

Proposition

Suppose that $G = G(q^{\delta_G})$ is a group of Lie type. If $T \in \mathrm{Syl}_p(G)$ has order p then either $C_G(T)$ is abelian (p is regular semisimple) or there exists $H = H(q^{\delta_H})$ a subgroup of G such that $T \leq H$, $C_H(T)$ is abelian and $\mathrm{Aut}_G(T) = \mathrm{Aut}_H(T)$.

As an example, if $G = \mathrm{GL}_n(q)$ and $p \mid \Phi_d(q)$, then $H = \mathrm{GL}_d(q)$ or $H = \mathrm{GL}_{d+1}(q)$ will work.

Thus we now simply have to construct all modules for groups of Lie type of dimension at most $2p-1$, where $p \mid \Phi_d(q) \mid (q^d - 1)$, and where $|\mathrm{Aut}_G(T)|$ is of order at most $4dt$ where t is the maximal size of a graph automorphism (this follows from knowledge of normalizers of Φ_d-tori in Lie type groups).

All the modules

So $p \mid \Phi_d(q^t) \mid (q^{td} - 1)$ and $|\mathrm{Aut}_G(T)| \leq 4dt$.

The twin statements $p - 1 \geq 4dt$ and $p \leq q^{td} - 1$ already put strong conditions on p, q and d. Throw in Landazuri–Seitz lower bounds on dimensions of modules for groups of Lie type, e.g., $\dim(A) \leq q^{(n-1)t} - 1$ for $\mathrm{GL}_n(q^t)$ and we get a finite, and small, list of possibilities.

Assume G is not alternating or $\mathrm{PSL}_2(p)$. We have one of:

1. $G = \mathrm{SL}_2(8) : 3 = {}^2G_2(3)$ or $G = 6 \cdot \mathrm{PSL}_3(4)$ and $p = 7$;
2. $G = \mathrm{PSU}_3(3).2 = G_2(2)$ or $G = 6_1 \cdot \mathrm{PSU}_3(4).2_2 = G_{34}$ and $p = 7$;
3. $G = \mathrm{PSU}_3(4) : 4$ and $p = 13$;
4. $G = \mathrm{PSU}_4(2) = \mathrm{PSp}_4(3)$ and $p = 5$;
5. $G = \mathrm{PSU}_5(2).2$ and $p = 11$;
6. $G = \mathrm{Sp}_4(4).4$ and $p = 17$;
7. $G = \mathrm{Sp}_6(2)$ and $p = 5, 7$ or $G = 2 \cdot \mathrm{Sp}_6(2)$ and $p = 7$;
8. $G = 2 \cdot \Omega_8^+(2)$ and $p = 7$;
9. $G = G_2(3).2$ or $G = {}^2B_2(8) : 3$ and $p = 13$.

Do all of these give exotic fusion systems?

No.

The ones of dimension at most p do, and there are a lot of those, but if the dimension is more than p then this technical condition on the action of $\mathrm{Aut}_G(T)$ on the socle of A and on T needs to be satisfied. This fails for (for example) $6 \cdot \mathrm{Suz}$ and $p = 11$, where there is a 12-dimensional module, but is satisfied by the group $\mathrm{GL}_2(p)$, where A is a $(p + 1)$-dimensional indecomposable module whose top is the natural module.

The complete list of groups and modules is now known, except for groups G lying inside $C_{p-1} \wr S_n$.

What is this technical condition?

If we have a non-trivial G-action, we can consider $Z_0 = C_A(T) \cap [T, A]$. This 1-dimensional subspace is an $N_G(T)$-module. We also have the conjugation action of $N_G(T)$ on T. This yields an action of $N_G(T)$ on

$$Z_0 \times T \cong \mathbb{Z}_p \times \mathbb{Z}_p.$$

If $\dim(A) > p$, then instead of $N_G(T)$ we need to take the centralizer in this of $C_A(T)/Z_0$, which is of course much smaller. Write μ for the image in $Z_0 \times T$.

Write Δ_i for the twisted diagonal subgroup $\{(x^i, x) \mid x \in \mathbb{Z}_p\}$. If μ contains Δ_0 or Δ_{-1} then we get an exotic fusion system. If it contains both (in the case where $\dim(A) \le p$) we can potentially build others. Depending on the image of μ, either $P = C_p \times C_p$ or $Q = p_+^{1+2}$ or both are essential.