Units and the Unit Conjecture

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London Algebra Colloquium, 4th November, 2010

A Toy Example

Let K be a field of characteristic $p \geq 0$ and let $R = K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be the Laurent polynomial ring with n variables.

Question: Are there any zero divisors of R?

Answer: Clearly no.

Thinking of R as KG, where G is the group $\mathbb{Z}^n$, we see that there are no zero divisors in the group rings of finitely generated abelian groups with no elements of finite order.

The Zero Divisor Conjecture

If G is a group and x is an element of order n in G, then $x^n = 1$, and so the element $x - 1$ is a zero divisor. Hence if we want that there are no zero divisors in KG, as in the case of the abelian group $\mathbb{Z}^n$, then we need G to be torsion free (i.e., have no non-trivial elements of finite order).

Conjecture (Zero divisor conjecture)

If G is a torsion-free group and K is a field, then KG has no zero divisors.

Zero Divisors in Group Rings

The zero divisor conjecture has been solved for increasingly large classes.

Theorem (Bovdi, 1960)

Poly-$\mathbb{Z}$ groups satisfy the zero divisor conjecture.

Theorem (Formanek, 1973)

Torsion-free supersoluble groups satisfy the zero divisor conjecture.

Theorem (Farkas–Snider, 1976, and Cliff, 1980)

Torsion-free, virtually polycyclic groups satisfy the zero divisor conjecture.

Theorem (Kropholler–Linnell–Moody, 1988)

Torsion-free, virtually soluble groups satisfy the zero divisor conjecture.

A Toy Example Again

Let K be a field of characteristic $p \geq 0$ and let $R = K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be the Laurent polynomial ring with n variables.
What are the invertible elements of R?

These are simply the monomials $\lambda x_{i_1} x_{i_2} \dots x_{i_m}$, with $\lambda \in K \setminus \{0\}$.

Thinking of R as KG, where G is the group $\mathbb{Z}^n$, we see that the units of $R = KG$ are simply λg, where $\lambda \in K$ and $g \in G$.

The Unit Conjecture

Let G be a finite 2-group, and let K be the field $\mathbb{F}_2$. If ζ lies in the Jacobson radical of the group algebra KG, then $1+\zeta$ is a unit. Since the Jacobson radical has codimension 1, this means that half of the elements of KG are units.

More generally, if G is a group and x has order p in G, then $\hat{x}^2 - p\hat{x} = 0$ (where $\hat{x} = 1 + x + \cdots + x^{p-1}$) and so often there exists $a \in K$ such that $(\hat{x} - a)(\hat{x} - 1/a) = 1$. (For when no such a exists, there are similar constructions.) Hence if we want

$$U(KG) = \{\lambda g : \lambda \in K,\ g \in G\},$$

as in the case of the abelian group $\mathbb{Z}^n$, then we need G to be torsion free. We always have that λg is a unit: these are called **trivial units**.

Conjecture (Unit conjecture, Kaplansky, 1969)

If G is a torsion-free group and K is a field, then all units of KG are trivial.

A Minimal Counterexample to the Unit Conjecture

- If there is a zero divisor in KG then there is a non-trivial unit in KG. Hence the unit conjecture for G implies the zero divisor conjecture for G.
- The unit conjecture is known for torsion-free abelian groups by the previous argument, and actually for torsion-free nilpotent groups.
- More generally, if G is a torsion-free group, $N \trianglelefteq G$, and $G/N \cong \mathbb{Z}$, then the unit conjecture is true for G if and only if it is true for N.
- Let G be a torsion-free supersoluble group. Then G has a normal subgroup N such that G/N is either $\mathbb{Z}$ or D_∞, the infinite dihedral group. By induction on the Hirsch length, the first case cannot occur, so any counterexample to the unit conjecture of minimal Hirsch length has an infinite dihedral quotient and a finite abelianization. **From now on we assume that G is such a group.**

A Splitting Theorem for Supersoluble Groups

Suppose that G has an infinite dihedral quotient with kernel N, generated by Nx and Ny.

Theorem (Splitting theorem for units)

Let σ be an element of KG, and suppose that there is τ such that $\sigma\tau \in KN$. Then

$$\sigma = \eta^{-1}(\alpha_1 + \beta_1\gamma_1)(\alpha_2 + \beta_2\gamma_2)\ldots(\alpha_n + \beta_n\gamma_n),$$

with $\gamma_i \in \{x, y\}$, and $\alpha_i, \beta_i, \eta \in KN$. (This element lies in the localized group ring $(KN)^{-1}(KG)$.)

The linear terms after the η^{-1} are called the split form of σ. One may assume that α_i and β_i are left-coprime.

What is this η?

Write s for the split of σ, so that $\sigma = \eta^{-1}s$. For example, if $s = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)$, then

$$s = \alpha_1\alpha_2 + \alpha_1\beta_2 y + \beta_1\alpha_2^x x + \beta_1\beta_2^x xy.$$

In order for $\eta^{-1}s$ to be in KG, we must have that η divides each coefficient in front of the words in x and y.

Proposition

If $\eta = 1$ in the split of σ, then σ is not a unit.

Proof.

If $\eta = 1$ in the split, then s has an inverse, so by rebracketing, $\alpha_1 + \beta_1 x$ has an inverse. But this lies inside $\langle N, x\rangle$, a subgroup of G of smaller Hirsch length. $\square$

No Units of Length 2

The length of an element g in $D_\infty = \langle x, y \rangle$ is the standard length in a generating set. The length of an element of G is the length of its image in $G/N \cong D_\infty$. The length of an element of KG is the maximum of the lengths of the elements in the support.

Theorem

There are no non-trivial units of length 1 *or* 2*.*

Proof.

Elements of length 1 lie in $\langle N, x \rangle$ or $\langle N, y \rangle$, so by induction are not units. Let σ be a unit of length 2, and split σ as $\sigma = \eta^{-1} s$. We have

$$s = \alpha_1 \alpha_2 + \alpha_1 \beta_2 y + \beta_1 \alpha_2^x x + \beta_1 \beta_2^x xy,$$

so any left divisor of η must left-divide each of $\alpha_1\alpha_2$, $\alpha_1\beta_2$, $\beta_1\alpha_2^x$ and $\beta_1\beta_2^x$. This is not possible as α_i and β_i are left-coprime. □

This Doesn't Work for Length 3

The length-3 elements are as follows:

Word	Coefficient
xyx	$\beta_1\beta_2^x\beta_3^{yx}$
yx	$\alpha_1\beta_2\beta_3^y$
xy	$\beta_1\beta_2^x\alpha_3^{yx}$
y	$\alpha_1\beta_2\alpha_3^y$
x	$\alpha_1\alpha_2\beta_3 + \beta_1\alpha_2^x\alpha_3^x$
1	$\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2^x\beta_3^x x^2$

We will construct an example later where there can be a common divisor of all of these coefficients, while the α_i and β_i are coprime, so the previous technique does not extend.

At the moment we cannot do length-3 elements in full generality, and so we need to specialize. We will consider the 'most likely', 'most accessible', candidate for a counterexample to the unit conjecture.

A Non-UP Group

A group G is a unique-product group if, whenever X and Y are non-empty, finite subsets of G, there exists $z \in G$ such that z is expressible **uniquely** as a product $x \cdot y$, where $x \in X$ and $y \in Y$.

If G is a unique-product group, then Strojnowski proved that actually there are two such elements z, whenever $|X|, |Y| \geq 2$. Hence if G is a unique-product group, G satisfies the unit conjecture.

Are all torsion-free groups unique-product groups? **NO**. It was proved by Rips and Segev that there are torsion-free, non-UP groups. An easier example, Γ, was considered by Promislow.
Using a computer, Promislow searched randomly in Γ, and found a subset X (with $|X| = 14$) such that $X \cdot X$ had no unique product.
This was the first real use of the computer in this field.

Is the set X (or some subset of it) the support of a unit in $K\Gamma$?

The Passman group Γ

This group Γ is given by the presentation

$$\Gamma = \langle x, y \mid x^{-1}y^2x = y^{-2},\ y^{-1}x^2y = x^{-2}\rangle.$$

Write $z = xy$, $a = x^2$, $b = y^2$, $c = z^2$.
Idea 1: $H = \langle a, b, c\rangle$ is an abelian normal subgroup of G, and G/H is the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
Idea 2: $N = \langle a, b\rangle$ is an abelian normal subgroup, and G/N is the infinite dihedral group D_∞. This second quotient gives a length function on the elements of the group.

- The elements of N (of the form $a^i b^j$) are defined to be length 0.
- Length 1 elements are αx or αy, with $\alpha \in N$.
- Length 2 elements are αxy or αyx, with $\alpha \in N$.
- And so on.
- The Promislow set X has length 5.

The group ring $K\Gamma$

- We now want to consider the group ring $K\Gamma$, where K is any field.
- We extend the length function from Γ to $K\Gamma$: the length of a sum of elements of G is the maximum of the lengths of the elements.
- We want to rewrite the elements of $K\Gamma$, using the subgroup $H = \langle a, b, c\rangle$ this time. Any element may be written as $Ax + By + C + Dz$, where $A, B, C, D \in KH$.
- This rewriting allows us to construct a representation as matrices over $K\langle a, b, c\rangle$.

$$\begin{pmatrix} C & A & B & D \\ A^x a & C^x & D^x a & B^x \\ B^y b & D^y a^{-1} c^{-1} & C^y & A^y a^{-1} b c^{-1} \\ D^z c & B^z b^{-1} & A^z b^{-1} c & C^z \end{pmatrix}$$

(Here, A^x indicates the conjugate of A by x, and so on.)

Theorems on $K\Gamma$

Using the splitting theorem for units in $K\Gamma$, we can produce two important theorems.

Theorem

The length of a unit in $K\Gamma$ is equal to the length of its inverse.

Theorem

An element of $K\Gamma$ is a unit if and only if its determinant is in K.

Thus it must be really easy to check if an element of $K\Gamma$ is invertible, simply by checking its determinant. A length-3 element looks like the following:

$$\alpha_1 x + (\alpha_2 + \alpha_3 c)y + \alpha_4 + (\alpha_5 + \alpha_6 c^{-1})z.$$

(Here, $\alpha_i \in N$.)

The determinant of a length-3 element

$$
\begin{gathered}
\alpha_1\alpha_1^x\alpha_1^y\alpha_1^z - \alpha_1\alpha_1^x\alpha_4^y\alpha_4^z a - \alpha_1\alpha_2^x\alpha_3^y\alpha_1^z + \alpha_1\alpha_2^x\alpha_4^y\alpha_6^z - \alpha_1\alpha_3^x\alpha_2^y\alpha_1^z + \alpha_1\alpha_3^x\alpha_4^y\alpha_5^z - \alpha_1\alpha_5^x\alpha_1^y\alpha_5^z b + \alpha_1\alpha_5^x\alpha_2^y\alpha_4^z ab \\
-\alpha_1\alpha_6^x\alpha_1^y\alpha_6^z b + \alpha_1\alpha_6^x\alpha_3^y\alpha_4^z ab - \alpha_2\alpha_1^x\alpha_1^y\alpha_3^z + \alpha_2\alpha_1^x\alpha_6^y\alpha_4^z + \alpha_2\alpha_2^x\alpha_2^y\alpha_2^z + \alpha_2\alpha_2^x\alpha_3^y\alpha_3^z - \alpha_2\alpha_2^x\alpha_5^y\alpha_5^z a^{-1} \\
-\alpha_2\alpha_2^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_2\alpha_3^x\alpha_2^y\alpha_3^z - \alpha_2\alpha_3^x\alpha_6^y\alpha_5^z a^{-1} + \alpha_2\alpha_4^x\alpha_1^y\alpha_5^z ba^{-1} - \alpha_2\alpha_4^x\alpha_2^y\alpha_4^z b - \alpha_3\alpha_1^x\alpha_1^y\alpha_2^z + \alpha_3\alpha_1^x\alpha_5^y\alpha_4^z \\
+\alpha_3\alpha_2^x\alpha_3^y\alpha_2^z - \alpha_3\alpha_2^x\alpha_5^y\alpha_6^z a^{-1} + \alpha_3\alpha_3^x\alpha_2^y\alpha_2^z + \alpha_3\alpha_3^x\alpha_3^y\alpha_3^z - \alpha_3\alpha_3^x\alpha_5^y\alpha_5^z a^{-1} - \alpha_3\alpha_3^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_3\alpha_4^x\alpha_1^y\alpha_6^z ba^{-1} \\
-\alpha_3\alpha_4^x\alpha_3^y\alpha_4^z b - \alpha_4\alpha_2^x\alpha_4^y\alpha_2^z b^{-1} + \alpha_4\alpha_2^x\alpha_5^y\alpha_1^z a^{-1}b^{-1} - \alpha_4\alpha_3^x\alpha_4^y\alpha_3^z b^{-1} + \alpha_4\alpha_3^x\alpha_6^y\alpha_1^z a^{-1}b^{-1} - \alpha_4\alpha_4^x\alpha_1^y\alpha_1^z a^{-1} \\
+\alpha_4\alpha_4^x\alpha_4^y\alpha_4^z + \alpha_4\alpha_5^x\alpha_1^y\alpha_3^z - \alpha_4\alpha_5^x\alpha_6^y\alpha_4^z + \alpha_4\alpha_6^x\alpha_1^y\alpha_2^z - \alpha_4\alpha_6^x\alpha_5^y\alpha_4^z + \alpha_5\alpha_1^x\alpha_4^y\alpha_2^z ab^{-1} - \alpha_5\alpha_1^x\alpha_5^y\alpha_1^z b^{-1} \\
+\alpha_5\alpha_4^x\alpha_3^y\alpha_1^z - \alpha_5\alpha_4^x\alpha_4^y\alpha_6^z - \alpha_5\alpha_5^x\alpha_2^y\alpha_2^z a - \alpha_5\alpha_5^x\alpha_3^y\alpha_3^z a + \alpha_5\alpha_5^x\alpha_5^y\alpha_5^z + \alpha_5\alpha_5^x\alpha_6^y\alpha_6^z - \alpha_5\alpha_6^x\alpha_3^y\alpha_2^z a + \alpha_5\alpha_6^x\alpha_5^y\alpha_6^z \\
+\alpha_6\alpha_1^x\alpha_4^y\alpha_3^z ab^{-1} - \alpha_6\alpha_1^x\alpha_6^y\alpha_1^z b^{-1} + \alpha_6\alpha_4^x\alpha_2^y\alpha_1^z - \alpha_6\alpha_4^x\alpha_4^y\alpha_5^z - \alpha_6\alpha_5^x\alpha_2^y\alpha_3^z a + \alpha_6\alpha_5^x\alpha_6^y\alpha_5^z - \alpha_6\alpha_6^x\alpha_2^y\alpha_2^z a \\
-\alpha_6\alpha_6^x\alpha_3^y\alpha_3^z a + \alpha_6\alpha_6^x\alpha_5^y\alpha_5^z + \alpha_6\alpha_6^x\alpha_6^y\alpha_6^z + c(-\alpha_1\alpha_2^x\alpha_2^y\alpha_1^z + \alpha_1\alpha_2^x\alpha_4^y\alpha_5^z - \alpha_1\alpha_6^x\alpha_1^y\alpha_5^z b + \alpha_1\alpha_6^x\alpha_2^y\alpha_4^z ab \\
+\alpha_2\alpha_2^x\alpha_2^y\alpha_3^z - \alpha_2\alpha_2^x\alpha_6^y\alpha_5^z a^{-1} - \alpha_3\alpha_1^x\alpha_1^y\alpha_3^z + \alpha_3\alpha_1^x\alpha_6^y\alpha_4^z + \alpha_3\alpha_2^x\alpha_2^y\alpha_2^z + \alpha_3\alpha_2^x\alpha_3^y\alpha_3^z - \alpha_3\alpha_2^x\alpha_5^y\alpha_5^z a^{-1} \\
-\alpha_3\alpha_2^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_3\alpha_3^x\alpha_2^y\alpha_3^z - \alpha_3\alpha_3^x\alpha_6^y\alpha_5^z a^{-1} + \alpha_3\alpha_4^x\alpha_1^y\alpha_5^z ba^{-1} - \alpha_3\alpha_4^x\alpha_2^y\alpha_4^z b - \alpha_4\alpha_2^x\alpha_4^y\alpha_3^z b^{-1} \\
+\alpha_4\alpha_2^x\alpha_6^y\alpha_1^z a^{-1}b^{-1} + \alpha_4\alpha_6^x\alpha_1^y\alpha_3^z - \alpha_4\alpha_6^x\alpha_6^y\alpha_4^z + \alpha_5\alpha_1^x\alpha_4^y\alpha_3^z ab^{-1} - \alpha_5\alpha_1^x\alpha_6^y\alpha_1^z b^{-1} + \alpha_5\alpha_4^x\alpha_2^y\alpha_1^z - \alpha_5\alpha_4^x\alpha_4^y\alpha_5^z \\
-\alpha_5\alpha_5^x\alpha_2^y\alpha_3^z a + \alpha_5\alpha_5^x\alpha_6^y\alpha_5^z - \alpha_5\alpha_6^x\alpha_2^y\alpha_2^z a - \alpha_5\alpha_6^x\alpha_3^y\alpha_3^z a + \alpha_5\alpha_6^x\alpha_5^y\alpha_5^z + \alpha_5\alpha_6^x\alpha_6^y\alpha_6^z - \alpha_6\alpha_6^x\alpha_2^y\alpha_3^z a \\
+\alpha_6\alpha_6^x\alpha_6^y\alpha_5^z) + c^{-1}(-\alpha_1\alpha_3^x\alpha_3^y\alpha_1^z + \alpha_1\alpha_3^x\alpha_4^y\alpha_6^z - \alpha_1\alpha_5^x\alpha_1^y\alpha_6^z b + \alpha_1\alpha_5^x\alpha_3^y\alpha_4^z ab - \alpha_2\alpha_1^x\alpha_1^y\alpha_2^z + \alpha_2\alpha_1^x\alpha_5^y\alpha_4^z \\
+\alpha_2\alpha_2^x\alpha_3^y\alpha_2^z - \alpha_2\alpha_2^x\alpha_5^y\alpha_6^z a^{-1} + \alpha_2\alpha_3^x\alpha_2^y\alpha_2^z + \alpha_2\alpha_3^x\alpha_3^y\alpha_3^z - \alpha_2\alpha_3^x\alpha_5^y\alpha_5^z a^{-1} - \alpha_2\alpha_3^x\alpha_6^y\alpha_6^z a^{-1} + \alpha_2\alpha_4^x\alpha_1^y\alpha_6^z ba^{-1} \\
-\alpha_2\alpha_4^x\alpha_3^y\alpha_4^z b + \alpha_3\alpha_3^x\alpha_3^y\alpha_2^z - \alpha_3\alpha_3^x\alpha_5^y\alpha_6^z a^{-1} - \alpha_4\alpha_3^x\alpha_4^y\alpha_2^z b^{-1} + \alpha_4\alpha_3^x\alpha_5^y\alpha_1^z a^{-1}b^{-1} + \alpha_4\alpha_5^x\alpha_1^y\alpha_2^z - \alpha_4\alpha_5^x\alpha_5^y\alpha_4^z \\
-\alpha_5\alpha_5^x\alpha_3^y\alpha_2^z a + \alpha_5\alpha_5^x\alpha_5^y\alpha_6^z + \alpha_6\alpha_1^x\alpha_4^y\alpha_2^z ab^{-1} - \alpha_6\alpha_1^x\alpha_5^y\alpha_1^z b^{-1} + \alpha_6\alpha_4^x\alpha_3^y\alpha_1^z - \alpha_6\alpha_4^x\alpha_4^y\alpha_6^z - \alpha_6\alpha_5^x\alpha_2^y\alpha_2^z a \\
-\alpha_6\alpha_5^x\alpha_3^y\alpha_3^z a + \alpha_6\alpha_5^x\alpha_5^y\alpha_5^z + \alpha_6\alpha_5^x\alpha_6^y\alpha_6^z - \alpha_6\alpha_6^x\alpha_3^y\alpha_2^z a + \alpha_6\alpha_6^x\alpha_5^y\alpha_6^z)
\end{gathered}
$$

You don't want to see the length-4 determinant.

A Concrete Example for Length 3

The length-3 elements are as follows:

Word	Coefficient
xyx	$\beta_1\beta_2^x\beta_3^{yx}$
yx	$\alpha_1\beta_2\beta_3^y$
xy	$\beta_1\beta_2^x\alpha_3^{yx}$
y	$\alpha_1\beta_2\alpha_3^y$
x	$\alpha_1\alpha_2\beta_3+\beta_1\alpha_2^x\alpha_3^x$
1	$\alpha_1\alpha_2\alpha_3+\beta_1\alpha_2^x\beta_3^x x^2$

Let $G=\Gamma$. Let s be the product of linear terms with $\alpha_1=\alpha_2=\alpha_3=\beta_3=1$, $\beta_1=-a$, $\beta_2=1-a$. Let $\sigma=(a-1)^{-1}s$. This **is** an element of $K\Gamma$.

(This is not a unit of $K\Gamma$.)

Assaulting Length 3

(For this proof sketch we think of equality as equality up to a unit.)

- Examining the coefficients of the words in the table, we see that any prime p dividing η must divide β_2 and β_2^x.
- Next prove that if α and β are elements of KN, and that $\alpha\alpha^y - \beta\beta^y b$ is a unit, then either $\alpha = 0$ or $\beta = 0$.
- Write $D_2 = \alpha_2\alpha_2^y - \beta_2\beta_2^y b$. Prove that the gcd of D_2 and β_2 is the same as that of β_2 and α_2^y. Write D_2' for $D_2/(\alpha_2^y, \beta_2)$.
- Prove that $D_2' = (\alpha_2^y, \beta_2)^y$, which yields a factorization

$$D_2 = (\alpha_2^y, \beta_2)(\alpha_2, \beta_2^y).$$

- Therefore, writing $A = (\alpha_2, \beta_2^y)$, we have $D_2 = AA^y$, and

$$[\alpha_2/A][\alpha_2^y/A^y] - [\beta_2/A^y][\beta_2^y/A]b = D_2/AA^y$$

  is a unit. Write $\alpha = \alpha_2/A$ and $\beta = \beta_2/A^y$, which lie in KN; then $\alpha\alpha^y - \beta\beta^y b$ is a unit! Thus $\alpha_2 = 0$ or $\beta_2 = 0$.
- This easily yields a contradiction.

Length 3 and the Promislow Set

Theorem

There are no length-3 units in $K\Gamma$, for any field K.

Let X be the fourteen-element subset of Γ, such that Promislow proved that $X \cdot X$ has no unique product.

The set X does not have length 3 (it has length 5, as mentioned earlier), so the previous theorem does not apply.

However, there is an outer automorphism that centralizes y and swaps x and z. This sends X to a subset of Γ that does have length 3, so we can apply our theorem.

Corollary

The Promislow set (and any subset of it) cannot be the support of a unit in $K\Gamma$, for any field K.