Embedding finite groups into algebraic groups

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Representing groups in groups

Let H be a finite group. If $\mathbf{G}$ is an algebraic group, we can try to understand embeddings $H \to \mathbf{G}$.

- $\mathbf{G} = \mathrm{GL}_n(k)$. Then embeddings are kH-modules of dimension n.
- $\mathbf{G} = \mathrm{Sp}_{2n}(k)$. Then embeddings need to fix an alternating form.
- $\mathbf{G} = \mathrm{O}_n(k)$. Then embeddings need to fix a symmetric form.
- $\mathbf{G} = E_n, F_4, G_2$. Then what?

Definition

Let $H(q)$ be a finite group of Lie type, and let V be a module for H. If V is the restriction of a module for $\mathbf{H}$ then V is a **blueprint**. *If $\mathbf{H}$ and H fix the same subspaces of V then H is an* **accurate blueprint**.

Blueprints are where it's at

This completely answers the question for GL_n.

Proposition

Let $H(q)$ be a finite group embedded in GL_n. Let V be the corresponding n-dimensional module for H. The injection $H(q) \to \mathrm{GL}_n$ extends to a map $\mathbf{H} \to \mathrm{GL}_n$ if and only if V is a blueprint.

That's GL_n. What about the other classical groups? Assume p is odd here, to make life simpler. This embedding lies in D_n or B_n if $S^2(V)^H$ is non-zero, and lies in C_n if $\Lambda^2(V)^H$ is non-zero. If $S^2(V)^{\mathbf{H}} \neq 0$ then certainly $S^2(V)^H \neq 0$, but the converse might fail for some small q (e.g., $q = 3$ and $\mathbf{H} = \mathrm{SL}_2$). In almost all cases though $S^2(V)^H = S^2(V)^{\mathbf{H}}$ (and $\Lambda^2(V)^H = \Lambda^2(V)^{\mathbf{H}}$, and so this mostly answers the question for classical groups, in particular if the blueprint is accurate.

We are still left with the exceptional groups though.

Blueprints and maximal subgroups

Let $H = H(q_0)$ be a subgroup of $G = G(q)$. (Note that q_0 need not equal q, e.g., subfield subgroups.) If H is the restriction of an algebraic subgroup $\mathbf{H}$ in $\mathbf{G}$, then we can tell whether H is maximal in G by taking the fixed points of the maximal positive-dimensional subgroups of $\mathbf{G}$.

Hence, if we want to understand how to extend morphisms we need to understand finite subgroups of exceptional groups, and for those that are equicharacteristic Lie type groups know whether they lie inside algebraic subgroups of the same type

Therefore we will look at what is known about maximal subgroups of the finite exceptional groups of Lie type.

Of course, the maximal subgroups of finite simple groups are of interest for other reasons (I needed to know them for understanding generating a simple group by elements of specified orders).

Maximal subgroups of finite classical groups

Aschbacher in some sense classified all maximal subgroups of the finite classical groups. We will briefly see how this works.

Let M be a maximal subgroup of $\mathrm{SL}_n(q)$. If M acts reducibly then it lies inside the stabilizer of an m-space for some m, which is a parabolic subgroup. Hence we may assume that M acts irreducibly.

If M acts irreducibly but not absolutely irreducibly then it lies inside $\mathrm{GL}_{n/d}(q^d)$ for some d dividing n. Hence we may assume that M acts absolutely irreducibly.

We take the Fitting subgroup. If this is non-trivial, either M normalizes a p-subgroup, so is in a parabolic, or a p'-subgroup, and this is a semisimple subgroup or the extraspecial type maximal subgroups.

Maximal subgroups of finite classical groups

We take the Bender subgroup $E(M)$, the largest normal subgroup that is a product of simple groups. One can see that either $E(M)$ is simple or we lie inside a normalizer of another subspace decomposition.

Hence M is an almost simple group. If M is a group of Lie type in defining characteristic then M is the intersection of $\mathrm{GL}_n(q)$ with an algebraic version of M.

Thus we see that M is either the intersection of $\mathrm{GL}_n(q)$ with an algebraic subgroup of GL_n, so M is a blueprint, or it is an almost simple (modulo the center) group acting absolutely irreducibly on the natural module, and this simple group is either alternating, sporadic or Lie type in non-defining characteristic.

What about exceptional groups instead?

If that is the situation with classical groups, what is the situation with exceptional groups? The ideal case we can hope for is the same distinction, that a maximal subgroup is a blueprint for a positive-dimensional subgroup of G or that it is almost simple acting absolutely irreducibly on a minimal module.

Unfortunately this isn't true. Let's work through some of the proof to see what's wrong. Subspace stabilizers are algebraic subgroups, true, but they need not be positive dimensional, so that's the first problem.

If M has a centre then M is contained in a p-local subgroup, but these are not so easy to understand any more.

So, everything looks pretty bad then.

What about exceptional groups instead?

Despite this, we can get the following theorem.

Theorem

Let M be a maximal subgroup of a finite exceptional group of Lie type. One of the following holds:

1. *M is the fixed points of a Frobenius map of a positive-dimensional subgroup of the corresponding algebraic group.*
2. *M is an* **exotic p-local subgroup**
3. *M is the subgroup $(\mathrm{Alt}_5 \times \mathrm{Alt}_6) \cdot 2^2$ and $G = E_8$, $p > 5$.*
4. *M is almost simple.*

The subgroup in part 3 was discovered by Borovik, who proved this theorem, as did Liebeck–Seitz. The exotic p-local subgroups are known.

The exotic p-locals

Here are the exotic p-local subgroups. These are all maximal in the cases below.

- $2^3.\mathrm{SL}_3(2) < G_2(p)$, $p \geq 3$,
- $3^3.\mathrm{SL}_3(3) < F_4(p)$, $p \geq 5$,
- $3^{3+3}, \mathrm{SL}_3(3) < E_6^\epsilon(p)$, $p \equiv \epsilon \bmod 3$, $p \geq 5$
- $5^3.\mathrm{SL}_3(5) < E_8(p^a)$, $p \neq 2, 5$, $a \in \{1, 2\}$, $p^2 \equiv (-1)^{3-a} \bmod 5$
- $2^{5+10}.\mathrm{SL}_5(2) < E_8(p)$, $p \geq 3$.

They exist for other primes as well, but are not maximal.

Maximal subgroups of exceptional algebraic groups

The maximal subgroups M of positive dimension in exceptional algebraic groups have been completely classified by Liebeck and Seitz. They are maximal parabolics, maximal-rank subgroups, $(2^2 \times D_4).\mathrm{Sym}_3 < E_7$ (p odd), $A_1 \times \mathrm{Sym}_5 < E_8$, ($p > 5$), or M^0 is one of a short list:

G	M^0
G_2	A_1 ($p \geq 7$)
F_4	A_1 ($p \geq 13$), G_2 ($p = 7$), A_1G_2 ($p \geq 3$)
E_6	A_2 ($p \geq 5$), G_2 ($p \neq 7$), C_4 ($p \geq 3$), F_4, A_2G_2
E_7	A_1 ($p \geq 17$), A_1 ($p \geq 19$), A_2 ($p \geq 5$), A_1A_1 ($p \geq 5$), A_1G_2 ($p \geq 3$), A_1F_4, G_2C_3
E_8	A_1 ($p \geq 23$), A_1 ($p \geq 29$), A_1 ($p \geq 31$), B_2 ($p \geq 5$), A_1A_2 ($p \geq 5$), $A_1G_2G_2$ ($p \geq 3$), G_2F_4

Almost simple subgroups

So we are now looking at classifying the maximal almost simple subgroups. Unlike the classical case, where there are infinitely many cases so probably no reasonable answer, here there should just be a list. This has already been done for $G_2(q)$, ${}^2B_2(q^2)$, ${}^2G_2(q^2)$ and ${}^2F_4(q^2)$ (and ${}^3D_4(q^3)$ if you think of that as an exceptional group).

This leaves $F_4(q)$, $E_6(q)$, ${}^2E_6(q^2)$, $E_7(q)$ and $E_8(q)$.

A trifurcation

We want to focus on subgroups of Lie type in the same characteristic as the ambient algebraic group, and we make the following distinction. Suppose that the rank of the algebraic group is n.

- A (finite) subgroup is **large rank** if it has untwisted rank more than $n/2$.
- A (finite) subgroup is **medium rank** if it has untwisted rank between 2 and $n/2$, except for ${}^2B_2(q^2)$ and ${}^2G_2(q^2)$.
- A (finite) subgroup is **small rank** if it is one of $\mathrm{SL}_2(q)$, ${}^2B_2(q^2)$ and ${}^2G_2(q^2)$.

The results about embedding $H(q)$ into an algebraic group G depend on whether H has large, medium or small rank, at least until now.

Large-rank subgroups

Here we know the most, since there are not really many possible ways that (for instance) E_6 can be embedded in E_8.

Theorem (Liebeck–Saxl–Testerman, 1996)

Let $q > 2$. If $H(q)$ is a large-rank subgroup of an exceptional algebraic group G, then the inclusion map extends to a morphism of algebraic groups.

If $q = 2$ then something similar was proved by Liebeck and Seitz.

Theorem (Liebeck–Seitz, 2005)

If $H(2)$ is a large-rank subgroup of an exceptional algebraic group G then $H(2)$ is an accurate blueprint for some $\mathbf{H} < \mathbf{G}$, except for $\mathrm{GL}_4(2)$ inside F_4.

Medium-rank subgroups

In this case not everything has been done, but there was still a strong theorem of the same form as above.

Theorem (Liebeck–Seitz, 1998)

Let $H(q)$ be a medium-rank subgroup, and assume that $q > 9$ unless H is of type A_2, which case $q > 9$ and $q \neq 16$. If $H(q)$ is contained in an exceptional algebraic group G then $H(q)$ is an accurate blueprint for some $\mathbf{H} < \mathbf{G}$.

So this is the first case where not everything is known. This is the case we are mainly going to focus on in this talk.

Small-rank subgroups

We should of course complete the case of the rank-1 subgroups. Define

$$u(G) = \begin{cases} 12 & G = G_2, \\ 68 & G = F_4, \\ 124 & G = E_6, \\ 388 & G = E_7 \\ 1312 & G = E_8. \end{cases}$$

and $t(G) = u(G) \cdot \gcd(2, p-1)$.

Theorem (Liebeck–Seitz 1998, Lawther)

Let $H(q)$ be a small-rank subgroup contained in an exceptional algebraic group G. If $q > t(G)$ then $H(q)$ is an accurate blueprint for some $\mathbf{H} < \mathbf{G}$.

Why are F_4, E_6 and E_7 different?

With F_4, E_6 and E_7, they each have a faithful module of dimension smaller than that of the group, and hence the stabilizer of a line in this module must be a positive-dimensional subgroup. If we can prove that a subgroup H stabilizes a line, then we must have that H lies inside a positive-dimensional subgroup.

Hence from now on, we exclude the case of E_8. That particular trick will work with E_8, but the size of the module is too large to get a good handle on it. However, there are some methods that will still work there, and it might be possible to produce analogues of some of our results.

An example: $\mathrm{Sp}_4(2^n)$

Suppose that we want to show that if $H(q) = \mathrm{Sp}_4(2^n)$ lies inside G, one of F_4, E_6 and E_7, then it has a trivial submodule in its action on the minimal module, and hence is contained in a line stabilizer.

- The simple modules for H have dimension 4^i for $i \geq 0$. Since the minimal module for G has dimension at most 56, the dimensions of simple modules are 1, 4 and 16.
- H has a single conjugacy class of elements x of order 5, which are hence rational (i.e., conjugate to all their powers). The trace of x on the modules of dimension 1, 4 and 16 are 1, -1 and 1 respectively.
- There is a single conjugacy class of rational elements of order 5 in G, with character value 1, 2 and 6 as $G = F_4, E_6, E_7$.
- There are no extensions between 1s and 16s, so there must be more 4s than 1s in the composition factors of the minimal module, else H fixes a line or hyperplane. This yields a contradiction.

A theorem

This sort of thing can be used to attack lots of cases, but there are still obdurate cases that are not amenable to these ideas. There we have to try harder.

Trying harder, we can prove the following theorem.

Theorem (C.–Magaard–Parker)

If G is one of F_4, E_6 or E_7, and $H(q)$ is a group of Lie type in the same characteristic as G not equal to $\mathrm{SL}_2(q)$, then any image of H in G is contained in a positive-dimensional subgroup of G.

SL_2

Notice that we haven't dealt with SL_2 yet. This is what we had

$$u(G) = \begin{cases} 12 & G = G_2, \\ 68 & G = F_4, \\ 124 & G = E_6, \\ 388 & G = E_7 \\ 1312 & G = E_8. \end{cases}$$

and $t(G) = u(G) \cdot \gcd(2, p-1)$.

If we are just interested in maximal subgroups, we can reduce F_4 to $q = 9, 27, 81$ (this is for a very good reason) or $q = 13$, and for E_6 to $q = 11$. For E_7 some cases have been dealt with, but this is work in progress.

Coming back to morphisms

So that was a long aside to answer the question of can we extend finite groups to a morphism of algebraic groups.

The plan is as follows. If $H = H(q)$ is embedded in $\mathbf{G}$ then H is contained in a maximal algebraic subgroup by the above. By induction we can understand the semsimple subgroups, so we need to understand the parabolics. Non-abelian cohomology theory can be used to go from having $H < \mathbf{P} = \mathbf{LU}$ to having $H < \mathbf{L}$ (or not, if this is false), and we can use induction again.

Since the maximal subgroups haven't been finally classified yet, this procedure cannot start. But in many cases we can see that the minimal module being a blueprint for $H(q)$ really is enough for H to be contained in $\mathbf{H} < \mathbf{G}$.