Perverse Equivalences, Deligne–Lusztig theory and Broué's Conjecture

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Notation and Conventions

Throughout this talk,

- G is a finite group,
- ℓ is a prime,
- k is a field of characteristic ℓ,
- B is a block of kG, with defect group D and Brauer correspondent b;
- P is a Sylow ℓ-subgroup of G.

I will (try to) use red for definitions and green for technical bits that can be ignored.

About the first half of this talk is joint work with Raphaël Rouquier.

Representation Theory is Local

Many features of the modular representation of a finite group are conjectural, some not even conjectural. Broadly, they fall into three categories:

- finiteness conditions
- numerical conditions
- structural conditions

As an example of the first, we have Donovan's conjecture.

As examples of the second, we have the Alperin–McKay conjecture, Alperin's weight conjecture, and Brauer's height-zero conjecture.

As an example of the third, we have Broué's conjecture.

Representation Theory is Local

Some of the conjectures before (Alperin–McKay, Alperin's weight, Broué) relate the structure of a block B of kG to the structure of its Brauer correspondent b, a block of $k\mathrm{N}_G(D)$, where D is a defect group of B. Write $\ell(B)$ for the number of simple B-modules.

Alperin's weight conjecture gives a precise conjecture about $\ell(B)$ in terms of local information (normalizers of p-subgroups). If D is abelian, the conjecture reduces to

$$\ell(B) = \ell(b).$$

Broué's conjecture gives a structural understanding of Alperin's weight conjecture.

Conjecture (Broué, 1990)

Let G be a finite group, and let B be a ℓ-block of G with abelian defect group D. If b is the Brauer correspondent of B in $N_G(D)$, then B and b are derived equivalent.

When Is Broué's Conjecture Known?

Broué's conjecture is known for quite a few groups:

- G soluble
- A_n, S_n (Chuang–Rouquier, Marcus)
- $\mathrm{GL}_n(q)$, $\ell \nmid q$ (Chuang–Rouquier)
- D cyclic, $C_2 \times C_2$ (Rouquier, Erdmann, Rickard)
- G finite, $\ell = 2$, B principal
- G finite, $\ell = 3$, $|P| < 81$, B principal (Koshitani, Kunugi, Miyachi, Okuyama, Waki)
- $\mathrm{SL}_2(q)$, $\ell \mid q$ (Chuang, Kessar, Okuyama)
- various low-rank Lie type groups $L(q)$ with $\ell \nmid q$ and sporadic groups. (Okuyama, Holloway, etc.)

Principal Blocks Are Good

In representation theory, one standard method of proof is to reduce a conjecture to finite simple groups and then use their classification.

In general, there is no (known) reduction of Broué's conjecture to simple groups, but for principal blocks there is.

Theorem

Let G be a finite group. If P is abelian, then there are normal subgroups $H \leq L$ of G such that

- $\ell \nmid |H|$,
- $\ell \nmid |G : L|$, *and*
- L/H *is a direct product of simple groups and an abelian* ℓ*-group.*

For **principal** blocks, we may assume that $H = 1$. A derived equivalence for L (compatible with automorphisms of the simple components) passes up to G. Thus if Broué's conjecture for principal blocks holds for all simple groups (with automorphisms), it holds for all groups.

How Do You Find Derived Equivalences?

There are four main methods to prove that B and b are derived equivalent.

1. Okuyama deformations: using many steps, deform the Green correspondents of the simple modules for B into the simple modules for b. This works well for small groups.
2. Rickard's Theorem: randomly find complexes in the derived category of b related to the Green correspondents of the simple modules for B, and if they 'look' like simple modules (i.e., Homs and Exts behave nicely) then there is a derived equivalence $B \to b$.
3. More structure: if B and b are more closely related (say Morita equivalent) then they are derived equivalent. More generally, find another block B' for some other group, an equivalence $B \to B'$, and a (previously known) equivalence $B' \to b$.
4. Perverse equivalence: build a derived equivalence up step by step in an algorithmic way.

What is a Perverse Equivalence?

Let A and B be finite-dimensional algebras, $\mathcal{A} = \textit{mod-}A$, $\mathcal{B} = \textit{mod-}B$.

An equivalence $F : D^b(\mathcal{A}) \to D^b(\mathcal{B})$ is perverse if there exist

- orderings on the simple modules $S_1, S_2, \ldots, S_r$, T_1, $T_2, \ldots, T_r$, and
- a function $\pi : \{1, \ldots, r\} \to \mathbb{Z}$

such that, if $\mathcal{A}_i$ denotes the Serre subcategory generated by $S_1, \ldots, S_i$, and $D^b_i(\mathcal{A})$ denotes the subcategory of $D^b(\mathcal{A})$ with support modules in $\mathcal{A}_i$, then

- F induces equivalences $D^b_i(\mathcal{A}) \to D^b_i(\mathcal{B})$, and
- $F[\pi(i)]$ induces an equivalence $\mathcal{A}_i/\mathcal{A}_{i-1} \to \mathcal{B}_i/\mathcal{B}_{i-1}$.

Note that mod-B is determined, up to equivalence, by A, π, and the ordering of the S_i.

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such that, for all i, the cohomology of $F(S_i)$ only involves T_j for $j < i$, except for one copy of T_i in degree $-\pi(i)$, and T_j can only appear in degrees less than $-\pi(j)$.

Benefits of a Perverse Equivalence

The perverse equivalence is 'better' than a general derived equivalence.

- Has an underlying geometric interpretation (for Lie-type groups).
- The π-function 'comes from' Lusztig's A-function. For $\ell \mid \Phi_d(q)$, if $d = 1$ or $d = 2$, π is $2A/d$, but for $d \geq 3$ this does not work (see later!).
- There is an algorithm that gives us a perverse equivalence from $B_0(kN)$ to **some** algebra, so only need to check that the target is $B_0(kG)$. (This is simply checking that the Green correspondents are the last terms in the complexes.)

This algorithm is very useful!

An Example: M_{11}, $\ell = 3$

π	Ord. Char.	S_1	S_3	S_7	S_2	S_4	S_6	S_5
0	1	1						
2	10		1					
3	10			1				
4	16	1	1		1			
5	11	1			1	1		
6	44			1	1	1	1	
7	55	1	1		1	1	1	1
	10							1
	16	1				1		1

The cohomology of the complexes gives the rows of the decomposition matrix.

An Example: $PSL_4(q)$, $\ell = 3$, $3 \mid (q+1)$, $P = C_3 \times C_3$

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
3	$q(q^2+q+1)$	1	1			
4	$q^2(q^2+1)$		1	1		
5	$q^3(q^2+q+1)$	1	1	1	1	
6	q^6	1			1	1

$$X_2: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0.$$
$$X_5: \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(234) \oplus M_{4,1} \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0.$$
$$X_3: \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(5) \oplus M_{1,1} \to M_{1,1} \oplus M_{1,2} \to C_3 \to 0.$$
$$X_4: \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(4) \to M_{4,2} \to C_4 \to 0.$$

Which Groups Have Perverse Equivalences?

- All groups, D cyclic or $C_2 \times C_2$
- $\mathrm{PSL}_3(q)$, $\ell = 3 \mid (q-1)$, P abelian
- $\mathrm{PSL}_4(q)$, $\mathrm{PSL}_5(q)$, $\ell = 3 \mid (q+1)$, $P = C_3 \times C_3$
- $\mathrm{PSU}_3(q)$, $\ell = 3 \mid (q+1)$, P abelian
- $\mathrm{PSU}_4(q)$, $\mathrm{PSU}_5(q)$, $\ell = 3 \mid (q-1)$
- b a block of $\mathrm{PSU}_n(q)$, $\ell = 5 \mid (q+1)$, b has defect group $C_5 \times C_5$
- $\mathrm{PSp}_4(q)$, $\ell = 3 \mid (q-1)$ or $(q+1)$, $P = C_3 \times C_3$
- (almost) $\Omega_8^+(q)$, $\ell = 5 \mid (q^2+1)$, $P = C_5 \times C_5$
- (almost) ${}^3D_4(q)$, $\ell = 7 \mid (q^2+q+1)$, $P = C_7 \times C_7$
- $G_2(q)$, $\ell = 5 \mid (q+1)$, $P = C_5 \times C_5$
- S_6, A_7, A_8, $\ell = 3$ (A_6 does not)
- M_{11}, $M_{22}.2$, M_{23}, HS, $\ell = 3$ (M_{22} does not)
- $\mathrm{SL}_2(8)$, J_1, ${}^2G_2(q)$, $\ell = 2$ in two steps
- S_n, A_n, $\mathrm{GL}_n(q)$ in multiple steps

Some Remarks

- Since $\pi(-)$, the ordering and the first category determine the perverse equivalence, it is a very compact way of defining a (type of) derived equivalence.
- Computationally, this reduces finding a derived equivalence to finding the Green correspondents of the simple modules for G, a much simpler task.
- For groups of Lie type, it seems as though the complexes above do not really depend on ℓ, and only on d, where $\ell \mid \Phi_d(q)$. It might be possible to use these perverse equivalences to prove real results in this direction.

The Parameter π

Let $\ell \mid \Phi_d(q)$ and let χ be a unipotent character in the principal ℓ-block of kG.

The parameter π should be the absolute value of the minimal degree in the cohomology of the Deligne–Lusztig variety $X(w)$ suitably translated so that the trivial has degree 0 in which the given unipotent character χ appears.

In the case where $\ell \mid \Phi_1(q)$ or $\ell \mid \Phi_2(q)$, this degree is conjectured to be $2A$ and A respectively (so $2A/d$, where $\ell \mid \Phi_d(q)$), where A is the degree of the polynomial (in q) giving $\chi(1)$. This has been checked in a number of situations, and is the guess for $\pi(-)$ in the constructed perverse equivalences earlier.

What happens for $d > 2$?

The Coxeter Case

Olivier Dudas has worked on the case where d is the Coxeter number (i.e., the largest d such that $\Phi_d(q)$ divides $|G|$). In this case, it was found that, rather than being $2\deg(\chi)/d$, it was normally!! $(\deg(\chi)+a)/d$, where a is the power of q in $\deg(\chi)$. Strange things happen whenever $(q-1)$ divides the degree of χ (so χ is a cuspidal character).

This behaviour is similar to, but not exactly the same as, the case $d = 1, 2$.

In work with Raphaël Rouquier, we had found the π-function for ${}^3D_4(q)$, $\ell \mid \Phi_3(q)$, and $\Omega_8^+(q)$, $\ell \mid \Phi_4(q)$. These were close to, but not equal to, $2A/d$.

Towards a General Conjecture

I started with the following assumptions:

1. $\pi(\chi)$ is always a positive integer if χ is non-trivial.
2. $\pi(\chi)$ is dependent only on χ, and not on the group.
3. $\pi(\mathrm{St}) = 2\deg(\mathrm{St})/d$, whenever St lies in the principal block.
4. There is a function $B_d(-)$, defined on all polynomials that are products of qs and $\Phi_r(q)$, such that $\pi(\chi) = B_d(\chi)/d$, and $B_d(fg) = B_d(f) + B_d(g)$.

Assumptions 3 and 4 imply that $B_d(q) = 2$ for all d. I also had a working assumption

5. If St lies in the principal block, it has maximal π-value.

Using these assumptions, I started to calculate B_d.

The Conjecture

[After much guesswork...]

Definition

Let d and $r \geq 2$ be integers, and define $\phi_d(r)$ to be the number of positive integers less than r/d and prime to r.
Define $B_d(q) = 2$, $B_d(\Phi_1) = 1 + d/2$, and $B_d(\Phi_r) = \phi(r) + d\phi_d(r)$.

Conjecture

The minimal degree of a unipotent character χ in the principal ℓ-block is $B_d(\chi)/d$.

Notice that if $d = 1, 2$ then $B_d(\chi) = 2\deg(\chi)$, and if d is the Coxeter number then, if $(q-1)$ does not divide χ, we have $B_d(\chi) = \deg(\chi) + a$.

Evidence for the Conjecture, I

There are explicit calculations of the cohomology of $X(w)$ in a variety of cases (all cyclic Sylow Φ_d-subgroups), performed by Olivier Dudas, and in each case the resuts match the conjecture here. The groups and primes are:

1. GU_4, $d = 4$
2. GU_6, $d = 6$
3. E_6, $d = 9$
4. E_6, ${}^2E_6$, $d = 12$
5. E_7, $d = 14$
6. E_8, $d = 24$

Evidence for the Conjecture, II

If both this conjecture, and the geometric version of Broué's conjecture, are true, then there should be perverse equivalences for the principal ℓ-blocks with $B_d(-)/d$ as the parameter π. These equivalences have been found for the following groups:

- $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_4(2)$, $\mathrm{PSL}_5(2)$, $d = 3$
- $\mathrm{PSp}_6(2)$, $\Omega_8^+(2)$, $\Omega_8^-(2)$, $G_2(3)$, $d = 3$
- $\mathrm{PSL}_5(2)$, $\mathrm{PSp}_4(2)$, $\mathrm{PSp}_6(2)$, $d = 4$
- $\mathrm{PSL}_6(2)$, ${}^3D_4(2)$, $d = 3$
- $\Omega_8^+(2)$, $d = 4$
- Principal blocks of $\mathrm{GL}_n(q)$ when P is cyclic

Evidence for the Conjecture, III

If this conjecture is true, then $B_d(\chi)/d$ should be an integer, whenever χ is a unipotent character lying in the principal ℓ-block of G.
In fact, we have more.

Theorem

Let χ and ψ be unipotent characters lying in the same ℓ-block of G. Then $B_d(\chi) \equiv B_d(\psi) \bmod d$.

This theorem suggests that $B_d(-)$ could be of interest outside the principal block, although as of yet there is no significance to it. Could it be related to the cohomology $H^i_c(Y(\dot{w}), K)$? Does it give a perverse equivalence?

Non-Principal Blocks?

As a merest hint that there is something there, consider the non-principal unipotent block of $\mathrm{PSL}_5(q)$, $\ell \mid \Phi_3(q)$.

This has three characters, with degrees $q\Phi_2\Phi_4$ (B_d-value 8), $q^2\Phi_5$ (B_d-value 11) and q^{10} (B_d-value 20). Taking $\lfloor B_d(-)/d \rfloor$ yields 2, 3, 6.

We have a perverse equivalence, with the same ordering on the normalizer as for the principal ℓ-block, as follows.

π	χ_i	S_1	S_2	S_3
2	$q\Phi_2\Phi_4$	1	0	0
3	$q^2\Phi_5$	1	1	0
6	q^{10}	0	1	1