On ZJ-Theorems for Fusion Systems

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1 Introduction

In this short note, we unify the currently known ZJ-theorem type results from [4] and [6] for saturated fusion systems. Recall that a finite group G is said to be *H-free* if there are no subquotients of G isomorphic with H. By [1, Proposition C], any saturated fusion system $\mathcal{F}$ possessing a normal, centric subgroup Q arises from a unique p-constrained finite group with no normal p'-subgroup, and we denote this subgroup by $L_Q^{\mathcal{F}}$. A (saturated) fusion system $\mathcal{F}$ on a p-group P is said to be *H-free* if, for every $\mathcal{F}$-centric subgroup $Q \leqslant P$, the model L_Q of $\mathrm{N}_{\mathcal{F}}(Q)$ is H-free. Finally, let $Qd(p)$ be the semi-direct product $(C_p \times C_p) \rtimes \mathrm{SL}_2(p)$, with the $\mathrm{SL}_2(p)$ acting in the natural way. The theorem here that we will prove is the following.

Theorem A Every $Qd(p)$-free fusion system arises from a finite group, and has a p-soluble fusion system.

This is heavily related to work of Kessar and Linckelmann [4] and Onofrei and Stancu [6]; in [4], Theorem A is given for p odd (using the K^∞ and K_∞ characteristic p-functors), but for $p = 2$ this cannot be done using the original method, since K^∞ and K_∞ no longer have the right properties.

To state the theorem from which Theorem A follows, we need the concept of characteristic p-functors. A *characteristic p-functor* is a map W from all finite p-groups to itself, with several properties:

- if $P \neq 1$ then $W(P) \neq 1$;
- $W(P)\,\mathrm{char}\,P$; and
- if $\phi : P \to Q$ is an isomorphism, then $W(P)\phi = W(Q)$.

Theorem B Let H be a finite group. Let W be a positive characteristic p-functor such that, for all H-free finite groups G, $\mathcal{F}_P(G) = \mathcal{F}_P\left(\mathrm{N}_G(W(P))\right)$, where P is a Sylow p-subgroup of G. Then every H-free fusion system arises from a finite group, and consequently $W(P) \leqslant \mathrm{O}_p(\mathcal{F})$ for any H-free saturated fusion system $\mathcal{F}$ on a finite p-group P.

The map $P \mapsto Z(J(P))$ satisfies the conditions on W with $H = Qd(p)$ for p odd (Glauberman's ZJ-Theorem), and Stellmacher proved that there is some functor W with this property for $p = 2$ and $H = S_4 = Qd(2)$, whence Theorem A follows from Theorem B.

In this note, we will use the (mostly standard) notations and conventions from [2], nearly all of which are found for example in [5]. Any definitions not found in [5] will be repeated here.

2 Proof of Theorems A and B

Theorem B will follow quite readily from a few lemmas proved in [4], together with Theorem E from [2]. We will recall the previously known results now. The first is stated only for $H = Qd(p)$ in [4], although the generalization to arbitrary H has exactly the same proof.

Proposition 2.1 ([4, Propositions 6.3 and 6.4]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P that is H-free, and let Q be a subgroup of P.

(i) If Q is fully normalized then $\mathrm{N}_{\mathcal{F}}(Q)$ is H-free.

(ii) If Q is strongly $\mathcal{F}$-closed then $\mathcal{F}/Q$ is H-free.

The next result we need is to be able to pull up control of fusion from normalizers to the whole fusion system.

Proposition 2.2 ([4, Proposition 5.3]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let W be a characteristic p-functor. Suppose that, for any fully normalized subgroup Q of P, we have (writing $\mathcal{E} = \mathrm{N}_{\mathcal{F}}(Q)$)

$$\mathcal{E} = \mathrm{N}_{\mathcal{E}}\left(W(\mathrm{N}_P(Q))\right).$$

Then $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(W(P))$.

The subgroup $\mathrm{O}_p(\mathcal{F})$ is the largest subgroup for which the normalizer subsystem is all of $\mathcal{F}$; define $\mathrm{O}_p^{(i)}(\mathcal{F})$ inductively by

$$\mathrm{O}_p^{(i)}(\mathcal{F})/\mathrm{O}_p^{(i-1)}(\mathcal{F}) = \mathrm{O}_p\left(\mathcal{F}/\mathrm{O}_p^{(i-1)}(\mathcal{F})\right).$$

Recall from [2] that a fusion system is called *p-soluble* if there exists an n such that $\mathrm{O}_p^{(n)}(\mathcal{F}) = P$, where the fusion system $\mathcal{F}$ is based on P.

Theorem 2.3 ([2, Theorem E]) Every p-soluble fusion system arises from a finite group.

Given these preliminaries, we prove that if $\mathcal{F}$ is H-free and W is a characteristic p-functor for which

$$\mathcal{F}_P(G) = \mathrm{N}_{\mathcal{F}_P(G)}(W(P))$$

for all H-free groups G, then $\mathcal{F}$ is a p-soluble fusion system. Suppose that $\mathcal{F}$ is a minimal counterexample; since $\mathcal{F}$ is chosen to be minimal, we must have that $\mathrm{O}_p(\mathcal{F}) = 1$ by Proposition 2.1, and so in particular $\mathrm{N}_\mathcal{F}(Q) \neq \mathcal{F}$ for any non-trivial subgroup Q. However, since $\mathrm{N}_\mathcal{F}(Q) \neq \mathcal{F}$ for any fully normalized Q, we have by choice of minimal counterexample that (writing $\mathcal{E} = \mathrm{N}_\mathcal{F}(Q)$),

$$\mathcal{E} = \mathrm{N}_\mathcal{E}(W(\mathrm{N}_P(Q)),$$

and so $\mathcal{F} = \mathrm{N}_\mathcal{F}(W(P))$ by Proposition 2.2, contradicting the statement that $\mathrm{O}_p(\mathcal{F}) = 1$. Hence $\mathcal{F}$ is p-soluble, and is modelled by a finite group by Theorem 2.3, proving Theorem B.

To derive Theorem A from Theorem B, it suffices to assert the existence of a characteristic p-functor with the desired properties for $H = Qd(p)$. The functor $P \mapsto Z(J(P))$ performs this for p odd (see [3]) and Stellmacher proved the existence of such a functor for $p = 2$ in [7], completing the proof of Theorem A.

References

[1] Carles Broto, Natàlia Castellana, Jesper Grodal, Ran Levi, and Bob Oliver, *Subgroup families controlling p-local finite groups*, Proc. London Math. Soc. **91** (2005), 325–354.

[2] David A. Craven, *Control of fusion and solubility in fusion systems*, Submitted.

[3] George Glauberman, *A characteristic subgroup of a p-stable group*, Cand. J. Math. **20** (1968), 1101–1135.

[4] Radha Kessar and Markus Linckelmann, *ZJ theorems for fusion systems*, Trans. Amer. Math. Soc. **360** (2008), 3093–3106.

[5] Markus Linckelmann, *Introduction to fusion systems*, Group Representation Theory, EPFL Press, Lausanne, 2007, pp. 79–113.

[6] Silvia Onofrei and Radu Stancu, *A characteristic subgroup for fusion systems*, Submitted.

[7] Bernd Stellmacher, *A characteristic subgroup of Σ_4-free groups*, J. Israel Math. Soc. **94** (1996), 367–379.