Sources of Simple Modules for Weight 2 Blocks of Symmetric Groups

David A. Craven

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Abstract

In this short article we study weight 2 blocks of symmetric groups for a fixed odd prime p, and the relationships between them. We first prove that all sources of simple modules from these blocks come either from S_{2p} or $S_p \wr C_2$; since this latter group is easy to understand, this result gives another indication that the principal block of S_{2p} is the 'most complicated' block of weight 2. We then examine which simple modules for S_{2p} share their source with a simple module from $S_p \wr C_2$ (so that the induced module of these simple modules to the so-called RoCK block is simple).

1 Introduction

The blocks of weight 2 in symmetric groups are well understood, and there is a highly developed theory that relates the various weight 2 blocks of symmetric groups to each other. In this note we will prove three results that describe some connections between these blocks.

Theorem A Let k be a field of characteristic p, and let B be a weight 2 block of a symmetric group over k. If M is a simple B-module then the source of M is isomorphic to the source of a module from the principal p-block of either S_{2p} or $S_p \wr C_2$.

The proof of this uses the abacus and branching rules, and the work of Scopes [5] [6]. It is used in [2] to prove that all simple modules for weight 2 blocks of symmetric groups are algebraic if $p = 3$ or $p = 5$. (An *algebraic* module is one that satisfies a polynomial with integer coefficients in the Green ring.) We must introduce some terminology before we can adequately state the next result. In what follows, k is a field of characteristic p, and if G is a finite group, write $B_0(kG)$ for the principal block of kG.

We construct a directed graph as follows: the vertices of the directed graph are all Puig equivalence classes of block of a given weight w. (There are only finitely many vertices by [5] and [4].) We draw an arrow from one class $\mathcal{B}_1$ to another $\mathcal{B}_2$ if there exist $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that B_1 and B_2 form a $[w : k]$-pair for $k < w$ in the sense of [5] (with B_1 a block of S_n and B_2 a block of S_{n+k} for some n). This directed graph is called the *Scopes digraph* of weight w. It has a unique vertex with no incoming arrows, with representative $B_0(kS_{wp})$, and a unique vertex with no outgoing arrows, with representative the so-called *RoCK block* B_{RoCK}: this is a block whose p-core has $(i-1)(w-1)$ beads on the ith runner of the abacus. We call these the *minimal* and *maximal* vertices respectively. The following result is the restriction to weight 2 of a general result for all blocks (Proposition 3.2).

Proposition B Any path between the minimal and maximal vertices on the Scopes digraph of weight 2 has length $p(p-1)/2$. In other words, if

$$B_0(kS_{2p}) = B_0 \to B_1 \to B_2 \to \cdots \to B_r = B_{\mathrm{RoCK}}$$

is a collection of weight 2 blocks of symmetric groups, where B_i and B_{i+1} form a $[2:k_i]$-pair (and if $k_i = 1$ then B_i is a block of S_n and B_{i+1} is a block of S_{n+1} for some n), then exactly $p(p-1)/2$ of the k_i are equal to 1.

In the proposition above, we allow the $[2:k_i]$-pairs for $k_i \geqslant 2$ to come from either induction or restriction.

The results of [6] prove that in each $B_i \to B_{i+1}$ that forms a $[2:1]$-pair, exactly one simple B_i-module does not branch to a simple B_{i+1}-module, i.e., the sources of all but one B_i-module are isomorphic to the sources of all but one B_{i+1}-module. Hence, of the sources of the $(p+2)(p-1)/2$ simple $B_0(kS_{2p})$-modules, $p(p-1)/2$ of them will be altered, which means that $p-1$ of them will be isomorphic to those of B_{RoCK}. This has the following corollary: define the restriction or induction of a module to a block B is the maximal summand of the restriction or induction of the module to the subgroup or overgroup that belongs to B.

Corollary C Let B be a block of weight 2, and let M be a simple B-module. Let B' denote a block of S_n for some n, that is Puig equivalent to B_{RoCK} and such that there is a series of Scopes and Morita moves connecting B and B' that only involve moving beads to the right. Either the restriction of M to $B_0(kS_{2p})$ or the induction of M to B' is simple. In particular, if the p-core of B is obtained from that of B_{RoCK} by making left bead moves, and M is a simple B-module, then either the restriction of M to $B_0(kS_{2p})$ or the induction of M to B_{RoCK} is simple.

Via Proposition B, it is seen that there are $p(p-1)/2$ simple B_{RoCK}-modules that do not remain simple upon restriction to $B_0(kS_{2p})$, so that there are $p-1$ that do. The next theorem identifies which modules these are.

Theorem D Let λ be either the partition $(2p)$ or $(i^2, 2^{p-i})$ for $3 \leqslant i \leqslant p$. The module D^λ of $B_0(kS_{2p})$, under any series of Scopes and Morita moves, remains simple. In particular, there is a simple module M of B_{RoCK} such that $M\downarrow_{S_{2p}}$ is the sum of D^λ and modules outside of the principal block.

The simple modules in this theorem appear at the top of the quiver of $B_0(kS_{2p})$, as described in [3]; this theorem states that their sources are sources of simple modules for $S_p \wr C_2$, and hence are 'small'. The non-trivial such simple modules are in some sense 'double hook' modules: it is natural to ask to what extent this is a small-weight phenomenon, and whether the 'w-tuple hook' modules for $B_0(S_{wp})$ are an interesting class of modules to consider.

It should be noted that, by using more advanced combinatorics, one may arrive at the results in this article more quickly. In Proposition 3.2 we use Richards's pyramids to easily deduce the version for all blocks of Proposition B. An analogue of Theorem A for blocks of all weight may be proved in a more general form using crystals, which we do not perform here. The author feels however that it is useful to the reader to have fairly short proofs using the basic combinatorial tool of the abacus, in addition to indicating how the general case might be proved, such as the trivial proof of Proposition 3.2.

The structure of this article is as follows: In the next section we recall the notation and results we need from the two papers [5] and [6] of Scopes, and develop some new terminology that we will need to adequately discuss the proofs of the above three theorems. In Section 3 we prove Proposition B, in Section 4 we prove Theorem A, and in the final section we prove Theorem D.

2 Notation and Prior Results

We recall briefly the work of Scopes [5] [6], and assume that the reader is fairly familiar with, in particular, the work in [6]. Recall the Scopes digraph, defined in the introduction.

Our version of the abacus will be (slightly) different from that of [5], in that ours goes upwards. It has p columns, called *runners*, with the first runner on the left. The bottom-left corner of the abacus is the entry 0, and the first-column hook numbers of a partition are placed on the abacus left-to-right, so that the partition $(3, 2, 1, 1)$, with first-column hook numbers $(1, 2, 4, 6)$, are displayed for $p = 5$ as the following. (Here a $\times$ indicates a bead, and a $-$ indicates no beads on that runner.)

	$\times$			
$-$	$\times$	$\times$	$-$	$\times$

We will freely identify a partition with its representation on the abacus. As usual, a p-core is a partition where no bead can fall downwards on the abacus. We will identify a p-block of a symmetric group with its p-core.

A path from the minimal vertex to the maximal vertex in the Scopes digraph is called a *Scopes series*; indeed, there is a sequence $B_0(kS_{wp}) = B_0 \to B_1 \to B_2 \to \cdots \to B_n = B_{\mathrm{RoCK}}$, where each arrow $B_i \to B_{i+1}$ is either a Puig equivalence (so B_i and B_{i+1} represent the same vertex in the Scopes series) or involves shifting k beads ($k < w$) from one runner of the abacus to the runner directly to its right. We call the former a *Morita move* (since it is a Morita equivalence) and the latter a *Scopes move*.

From now on we specialize to the case $w = 2$. In this case, the partitions belonging to a block B with a particular p-core μ are labelled by *symbols* $\langle i \rangle$, $\langle i, i \rangle$ and $\langle i, j \rangle = \langle j, i \rangle$ $(1 \leqslant i < j \leqslant p)$, where $\langle i \rangle$ is the symbol for the partition obtained from μ by raising the highest bead on the ith runner up two places, $\langle i, i \rangle$ is the symbol for the partition obtained from μ by raising the highest two beads on the ith runner up one place, and $\langle i, j \rangle$ is the symbol for the partition obtained from μ by raising the highest bead on each of the ith and jth runners down up place. (In the event of there being insufficient beads on a given runner, we may imagine an extra two rows of beads underneath the abacus and raise those: in this case, the first space (counting from the bottom left as before) should label 0, and all beads above or to the right of this should represent the number relative to this new position 0, so that the symbol $\langle 1, 4 \rangle$ applied to the above 5-core yields the abacus

$$\begin{array}{|c|c|c|c|c|}
 & & & & \\
 & \times & & & \\
\times & \times & \times & \times & \times \\
\hline
- & \times & \times & - & \times \\
\times & \times & \times & \times & \times \\
\hline
\end{array},$$

and the hook numbers $(1, 2, 4, 5, 6, 7, 8, 9, 11)$ and partition $(3, 2^6, 1, 1)$.) Since the symbols are easier to manipulate than the partitions we will concentrate on them, and only relate them to the partitions (and modules D^λ if the partition λ is p-regular) when we consider the main theorems of this article.

The moves $B_m \to B_{m+1}$ above are obtained by swapping adjacent runners of the abacus that contain different numbers of beads (write k for the difference in bead numbers). This induces a bijection between the symbols for B_m and the symbols to B_{m+1}, which we describe now: suppose that runners a and $a + 1$ are swapped.

If $k > 1$ (i.e., it is a Morita move), it is easy to describe the bijection: the partitions $\langle i \rangle$, $\langle i, i \rangle$ and $\langle i, j \rangle$ for block B_m are sent to the same labels for block B_{m+1}, acted on by the transposition $(a, a + 1)$ (so that there is no change unless i or j is one of a and $a + 1$).

If $k = 1$ (i.e., it is a Scopes move, and runner a must have one fewer bead than runner $a + 1$) then the bijection is more complicated: the images of $\langle i \rangle$, $\langle i, i \rangle$ and $\langle i, j \rangle$ are as in the previous case except for $\bar{\alpha} = \langle a+1 \rangle$, $\bar{\beta} = \langle a, a+1 \rangle$ and $\bar{\gamma} = \langle a, a \rangle$ (labelling partitions of B_m), which have images $\alpha = \langle a + 1, a + 1 \rangle$, $\gamma = \langle a \rangle$ and $\beta = \langle a, a + 1 \rangle$ respectively (labelling partitions of B_{m+1}).

If $1 < a < p$ then this description is valid, and we refer to these as *standard moves*. If $a = 1$ (and $k \geqslant 2$ so that this is a Morita move, as Scopes moves to the left are not permitted), the effect of swapping runners $1 = a$ and $2 = a + 1$ is to yield a p-core that has at least one bead in the first column. To get a valid p-core, we must then move the first column to the pth column and remove one bead. Another way to describe this move then is to say that the ith runner moves to the $(i - 1)$th runner for $3 \leqslant i \leqslant p$, the second runner is moved to the pth runner and has a bead removed, and the first runner stays where it is. The effect of this on the symbols is to apply the permutation $(p, p-1, \dots, 2)$ to the entires in the symbols $\langle i \rangle$, $\langle i, i \rangle$ and $\langle i, j \rangle$.

We must also allow the case "$a = p$", which is simply the reverse of the move above. In the reverse however, the case $k = 1$ is also possible, when there are no beads on the pth runner of B_m. This move fixes the first runner, moves the ith to the $(i+1)$th for $2 \leqslant i < p$, and moves the pth runner to the second, adding a bead. The effect on the symbols $\langle i \rangle$, $\langle i, i \rangle$ and $\langle i, j \rangle$ of B_m is to apply the permutation $(2, 3, \dots, p)$ to the entries, as in the previous paragraph, except for the three symbols $\bar{\alpha} = \langle 1 \rangle$, $\bar{\beta} = \langle 1, p \rangle$ and $\bar{\gamma} = \langle p, p \rangle$, which have images $\alpha = \langle 2, 2 \rangle$, $\gamma = \langle 1 \rangle$ and $\beta = \langle 1, 2 \rangle$ respectively, as partitions belonging to B_{m+1}. The Morita and Scopes moves of this type are called *wrap moves*, since they wrap the first runner round to the pth runner or vice versa.

If $B_m \to B_{m+1}$ is a Scopes move then the only simple B_{m+1}-module whose restriction to B_m (i.e., take the restriction between the relevant symmetric group and then keep only the summand that belongs to the block B_m) is not simple is the one whose (p-regular) partition corresponds to the symbol $\langle i, i \rangle$, i.e., to α. Thus the sources of the simple B_{i+1}-modules and the simple B_i-modules are isomorphic, and in correspondence via the bijection of symbols, except for the modules corresponding to the symbols $\bar{\alpha}$ and α.

3 The Scopes Series

This section contains a proof of Proposition B using the abacus, and then notes the general case in Proposition 3.2. To each p-core λ we associate a number $f(\lambda)$, which is the number of pairs (i, j) with $1 \leqslant i < j \leqslant p$, and such that either there are more beads on runner j than runner i, or there are at least two more beads on runner i than runner j (call such a pair *acceptable* for λ). Notice that if λ is the empty partition (i.e., the core of $B_0(kS_{2p})$) then $f(\lambda) = 0$, and if λ is the p-core of the RoCK block (so that its abacus representation has $i - 1$ beads on runner i) then $f(\lambda) = p(p-1)/2$. We prove the following proposition, which immediately implies Proposition B.

Proposition 3.1 Let λ and μ be p-cores, and suppose that they label weight 2-blocks B_λ and B_μ that form a $[2 : k]$-pair (with B_λ a block of S_n and B_μ a block of S_{n+k}).

(i) If $k \geqslant 2$ then $f(\mu) = f(\lambda)$.

(ii) If $k = 1$ then $f(\mu) = f(\lambda) + 1$.

Proof: Since B_λ and B_μ form a $[2 : k]$-pair, μ is obtained from λ by swapping runners i and $i+1$ on the abacus, and there are more beads on runner i than runner $i+1$ in λ.

Firstly suppose that this is a standard move (so that $i > 1$). Notice that runners i and $i+1$ maintain their positions relative to all other runners, and so we are only concerned with whether $(i, i+1)$ is an acceptable pair for λ and μ. If $k \geqslant 2$ then it is acceptable for both λ and μ, and so $f(\lambda) = f(\mu)$. If $k = 1$ then it is *not* acceptable for λ but is for μ, so that $f(\mu) = f(\lambda) + 1$.

Finally, suppose that we have a wrap move. We might as well assume that this is the case "$a = p$" in the description in Section 2, as we will show that Morita moves do not affect $f(\lambda)$, so will not when performed in the opposite direction. This move from λ to μ fixes the first runner, moves the ath runner to the $(a+1)$th runner for $2 \leqslant a < p$, and moves the pth runner to the second, adding a bead. If $a = 1$ and $2 \leqslant b \leqslant p-1$, or $2 \leqslant a < b \leqslant p-1$, then the number of beads on runners a and b, and their relative position, do not change in the move from λ to μ, so $(1, b)$ and (a, b) (for $a \geqslant 2$) are acceptable for λ if and only if $(1, b+1)$ and $(a+1, b+1)$ are acceptable for μ. The pair $(1, p)$ is acceptable for λ if and only if there are beads on the pth runner, and this is true if and only if $k \geqslant 2$ (i.e., the move is a Morita move rather than a Scopes move). Note that $(1, 2)$ is always acceptable for μ, since there must be at least one bead on the 2nd runner of μ (as a wrap move has taken place).

It remains to deal with the case where $2 \leqslant j \leqslant p-1$ and $k = p$: there are more beads on runner p than runner j in λ if and only if there are at least two more beads on runner 2 than runner $j+1$ in μ; and there are at least two more beads on runner j than runner p in λ if and only if there are more beads on runner $j+1$ then runner 2 on μ. This implies that (j, p) is acceptable for λ if and only if $(2, j+1)$ is acceptable for μ. Hence $f(\mu) = f(\lambda)$ if it is a Morita move, and $f(\mu) = f(\lambda)+1$ if it is a Scopes move. This completes the proof in this case as well. $\square$

This immediately shows that there are exactly $p(p-1)/2$ Scopes moves in any sequence of moves (either Scopes moves, or Morita moves in either direction) between $B_0(kS_{2p})$ and B_{RoCK}, as claimed in Proposition B.

More generally, using Richards's pyramids, it is easy to see the following more general result.

Proposition 3.2 Any path between the minimal and maximal vertices on the Scopes digraph of weight w has length $wp(p-1)/2$. Moreover, if

$$B_0(kS_{wp}) = B_0 \to B_1 \to B_2 \to \cdots \to B_r = B_{\mathrm{RoCK}}$$

is a collection of weight w blocks of symmetric groups, where B_i and B_{i+1} form a $[w : k_i]$-pair (and if $k_i < w$ then B_i is a block of S_n and B_{i+1} is a block of S_{n+1} for some n), then exactly $p(p-1)/2$ of the k_i are equal to j for each $1 \leqslant j \leqslant w-1$.

Proof: The block B_{RoCK} has pyramid consisting solely of $p(p-1)/2$ zeros, and $B_0(kS_{wp})$ has a pyramid consisting solely of $p(p-1)/2$ copies of $w-1$. Each non-Morita $B_i \to B_{i+1}$ move decrements exactly one of the entries by 1, and if the entry changes from j to $j-1$ then B_i and B_{i+1} form a $[w : w-j]$ pair. This completes the proof. $\square$

4 Sources of Weight 2 Blocks

Having dealt with cores, we now move on to the partitions associated to cores. As each Scopes and Morita move induces a bijection between the symbols $\langle i\rangle$, $\langle i, i\rangle$ and $\langle i, j\rangle$ of the two blocks, a composition of moves yields a composition of bijections of symbols. In addition, since each move induces a permutation of the runners, a composition of moves yields a permutation of the runners, which is related to the bijection of the symbols above.

Our first claim is that, if

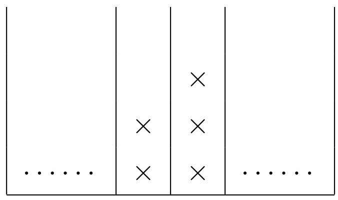
$$B_0(kS_{2p}) = B_0 \to B_1 \to B_2 \to \cdots \to B_m = B_{\mathrm{RoCK}}$$

is a sequence of moves, that λ is a symbol for B_0, and that it (or rather its image under the bijection between symbols) plays the role of $\bar{\gamma}$ in some Scopes move $B_r \to B_{r+1}$, then it never plays the role of $\bar{\beta}$ in any Scopes move $B_s \to B_{s+1}$ for $r < s$. More formally, we have the following proposition.

Proposition 4.1 Let λ be the symbol of a p-regular partition associated to $B_0(kS_{2p}) = B_0$, and let $B_0 \to B_1 \to B_2 \to \cdots \to B_m = B_{\mathrm{RoCK}}$ be a sequence of Morita and Scopes moves. Let ϕ_t denote the bijection of partitions induced by the first t maps (from those belonging to B_0 to those belonging to B_t). If $\phi_r(\lambda)$ is the symbol $\bar{\gamma}$ for some Scopes move $B_r \to B_{r+1}$ then none of the $\phi_s(\lambda)$, for $s > r$, is the symbol $\bar{\beta}$ for some Scopes move $B_s \to B_{s+1}$.

Proof: Suppose that $\phi_r(\lambda)$ is the symbol $\bar{\gamma}$ for a Scopes move $B_r \to B_{r+1}$. If this Scopes move is a wrap move, then $\phi_r(\lambda) = \langle 1, 1\rangle$, which is easily seen not to be the symbol of a p-regular partition. Since the ϕ_t send p-regular partitions to p-regular partitions [6, Lemma 3.5(2)], we arrive at a contradiction. Hence this Scopes move is a standard move.

Label the runners involved in the swap a and $a+1$, with b and $b+1$ beads respectively in B_{r+1}.

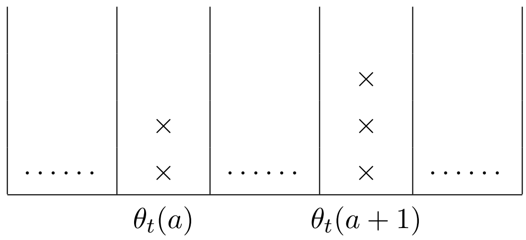


a

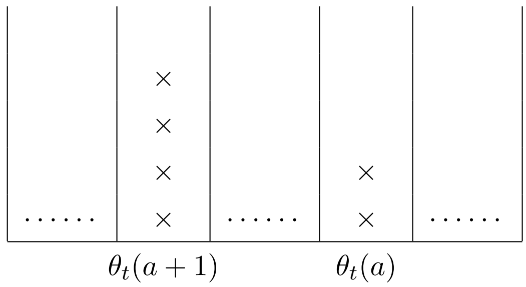
We suppose that there is an $s > r$ such that $\phi_s(\lambda)$ is the symbol $\bar{\gamma}$ for a Scopes move $B_s \to B_{s+1}$ (and choose the smallest such s), and for $r+1 \leqslant t \leqslant s$ write θ_t for the permutation of the runners induced by the Morita and Scopes moves from B_{r+1} to B_t.

Notice that $\phi_{r+1}(\lambda)$ has label $\langle a, a+1\rangle$, and for $r+1 \leqslant t \leqslant s$, $\phi_t(\lambda) = \langle \theta_t(a), \theta_t(a+1)\rangle$, until $\phi_t(\lambda)$ becomes the partition $\bar{\beta}$ for some Scopes move, i.e., $t = s$. However, in order for $\langle \theta_t(a), \theta_t(a+1)\rangle$ to be $\bar{\beta}$ in some Scopes move, the runners $\theta_t(a)$ and $\theta_t(a+1)$ must be adjacent, and there must be one fewer bead on the runner to the left than the one to the right.

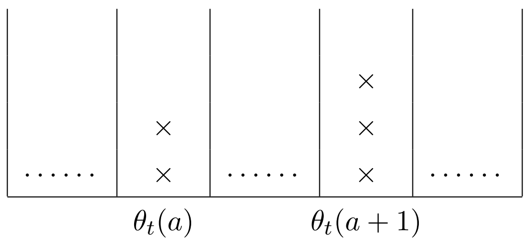
After any sequence of standard moves, the runner $\theta_t(a)$ still has b beads on it, and lies to the left of $\theta_t(a+1)$, which has $b+1$ beads.



Hence, unless wrap moves are performed, $\langle \theta_t(a), \theta_t(a+1) \rangle$ cannot be $\bar{\beta}$. In this case, a wrap move involving $\theta_t(a)$ (which must wrap from the left-hand side to the right-hand side of the abacus) will lose one bead, so the difference in beads will be 2, similarly for $\theta_t(a+1)$ wrapping from the right-hand side to the left.



Wrap moves in these directions can only *increase* the difference in beads between the two runners, and so the last wrap move made before we reach the Scopes move $B_s \to B_{s+1}$ (in which the two runners differ by one bead) must be in the other direction. This still leaves $\theta_t(a+1)$ with one more bead than $\theta_t(a)$ (as in order to wrap from the other side the two runners must swap at some point, and they can only do this if the difference in beads is at least 2, so that it is a Morita move), and must be on the right of $\theta_t(a)$, as in the following situation (again).



This means that a second Scopes move involving $\theta_t(a)$ and $\theta_t(a+1)$ cannot be performed, proving the proposition. □

Corollary 4.2 Let λ be the symbol of a p-regular partition associated to $B_0(kS_{2p}) = B_0$, and let $B_0 \to B_1 \to B_2 \to \cdots \to B_m = B_{\mathrm{RoCK}}$ be a sequence of Morita and Scopes moves. Let ϕ_t denote the bijection of partitions induced by the first t maps (from those belonging to B_0 to those belonging to B_t). If $\phi_r(\lambda)$ is the symbol $\bar{\alpha}$ for some Scopes move $B_r \to B_{r+1}$ then none of the $\phi_s(\lambda)$, for $s > r$, is the symbol $\bar{\alpha}$ again, for some Scopes move $B_s \to B_{s+1}$.

Proof: From the definition of a Scopes move, unless the symbols involved are $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$, the move permutes the symbols of the form $\langle i\rangle$, and those of the form $\langle i,i\rangle$, and those of the form $\langle i,j\rangle$ for $i\neq j$. Hence if $\phi_r(\lambda)$ is the symbol $\bar{\alpha}$, in order for $\phi_s(\lambda)$ to also be the symbol $\bar{\alpha}$ there must exist $r<u<v<s$ such that $\phi_u(\lambda)$ is $\bar{\gamma}$ and $\phi_v(\lambda)$ is $\bar{\beta}$, for Scopes moves $B_u\to B_{u+1}$ and $B_v\to B_{v+1}$. This is impossible by the previous proposition. $\square$

From here the proof of the theorem is clear: if $B_i\to B_{i+1}$ is a Scopes move, the only simple B_{i+1}-module that does not share a source with a B_i module is that corresponding to α. If no partition can be $\bar{\alpha}$ twice in a Scopes series, then if N replaces M in some Scopes move $B_i\to B_{i+1}$ (so that the sources of the other simple B_{i+1}-modules are the same as the sources of the other B_i-modules) then M cannot be replaced in any later Scopes move, so it shares a source with a simple module from the RoCK block, and N can never have been replaced in any previous Scopes move, so that N shares a source with a simple $B_0(kS_{2p})$-module. Thus the source of any simple module of any block of weight 2 is isomorphic with that of a simple module for $B_0(kS_{2p})$, or that of a simple module for the RoCK block. The final step is that the RoCK block is Puig equivalent to the principal block of $S_p\wr C_2$, which is proved in [1]; this completes the proof of Theorem A.

5 Restrictions of Partitions

The objective of this section is to prove Theorem D. In order to prove this we will show that the partitions labelled by the symbols $\langle i\rangle$ for $1\leqslant i\leqslant p-1$, $\langle 1,i\rangle$ for $3\leqslant i\leqslant p$ and $\langle i,j\rangle$ for $2\leqslant i<j\leqslant p$ and $j-i>1$, are all $\bar{\alpha}$ for some Scopes move; since this accounts for $p(p-1)/2$ partitions, this must be all partitions that have their sources altered, and so the branching rule proves the result (noting that the remaining symbols – $\langle p\rangle$ and $\langle i,i+1\rangle$ for $2\leqslant i\leqslant p-1$ – yield the partitions $(2p)$ and $(i^2,2^{p-i})$).

We now describe sequences of Scopes moves that prove that each of these can be $\bar{\alpha}$; each of the sequence of Scopes moves described here results in a p-core that has runners with either zero or one bead on them. A symbol or partition λ of $B_0(kS_{2p})$ is called *replaced* if there is a series of Scopes and Morita moves, ending with a Scopes move, for which the image of λ under the previous moves is the symbol or partition $\bar{\alpha}$ in the final Scopes move.

From the empty p-core we may only perform a wrap Scopes move, for which the symbol $\langle 1\rangle$ is $\bar{\alpha}$. At this stage, if one performs the standard Scopes moves with runner swaps the transpositions $(2,3)$, $(3,4)$, $(4,5)$, $\dots$, $(i,i+1)$, then the last move has as $\bar{\alpha}$ the symbol $\langle i\rangle$ of $B_0(kS_{2p})$. This is easy to see, as the wrap move sends $\langle i\rangle$ to $\langle i+1\rangle$, and all swaps leave $\langle i+1\rangle$ fixed until the final one, for which it is $\bar{\alpha}$. This proves that the symbols $\langle i\rangle$ for $1\leqslant i\leqslant p-1$ label replaced partitions.

Alternatively, one may repeatedly perform wrap Scopes moves, and after i wrap Scopes moves ($i\geqslant 2$), the partition $\langle 1,p-i+2\rangle$ has been replaced: to see this, a wrap Scopes move replaces $\langle 1\rangle$, sends $\langle 1,p\rangle$ to $\langle 1\rangle$, and maps $\langle 1,p-i+2\rangle$ to $\langle 1,p-i+3\rangle$, which by induction proves the claim. Hence the symbols $\langle 1,i\rangle$ for $3\leqslant i\leqslant p$ are replaced symbols.

In general, let $\lambda = \langle i, j\rangle$ be a symbol with $2 \leqslant i < j \leqslant p$ and $j - i > 1$. There is a four-step process to produce a series of Scopes moves, whose last one replaces λ.

Step 1: perform $p - j + 1$ wrap Scopes moves. Letting ϕ_1 denote the bijection of the symbols obtained from these moves, we have $\phi_1(\lambda) = \langle 2, x\rangle$, where $x = p+1-(j-i)$. (As $j - i > 1$, $x < p$.)

$-$	$\times$	$\times$	$\cdots\cdots$	$\times$	$-$	$\cdots\cdots$	$-$	$\cdots\cdots$	$-$
	2						x		
	$\underbrace{\qquad\qquad\qquad}_{p-j+1}$								

Step 2: move all but one of the beads to the last but one runner, so that the core looks as follows.

$-$	$\times$	$-$	$\cdots\cdots$	$-$	$\cdots\cdots$	$-$	$\times$	$\cdots\cdots$	$\times$	$-$
	2			$i+1$			$\underbrace{\qquad\qquad}_{p-j}$			

Let ϕ_2 denote the bijection of the symbols obtained from these moves. As we have passed $p - j$ beads through the runner labelled x, we get $\phi_2(\phi_1(\lambda)) = \phi_2(\langle 2, x\rangle) = \langle 2, i+1\rangle$, as each bead reduces x by 1.

Step 3: move the bead on the 2nd runner as far to the right as it may go. Let ϕ_3 denote the bijection of the symbols obtained from these moves. This final bead has the property that as it moves from the 2nd to the ath runner, the symbol $\langle 2, i + 1\rangle$ becomes $\langle a, i + 1\rangle$ until $a = i$, when it becomes $\langle i, i + 1\rangle$. At this point the bead is on the ith runner, and so the next move changes $\langle i, i + 1\rangle$ into $\langle i\rangle$ (as it is $\bar{\beta}$ for this move), and subsequent moves do not affect this symbol. Hence $\phi_3(\langle 2, i + 1\rangle) = \langle i\rangle$.

To see that this is possible, as $j - i > 1$, the $(i + 1)$th runner on the abacus at the end of Step 2 is empty, and so the bead on the 2nd runner may be passed through it. At this stage the core looks as follows.

$-$	$\cdots\cdots$	$-$	$\cdots\cdots$	$-$	$\times$	$\cdots\cdots$	$\times$	$-$
		i			$\underbrace{\qquad\qquad}_{p-j+1}$			

Step 4: perform one wrap move then move the new bead on the 2nd runner to the $(i+1)$th runner. Performing a wrap move alters the symbol $\langle i\rangle$ to $\langle i + 1\rangle$, and makes the core look as follows.

$-$	$\times$	$-$	$\cdots\cdots$	$-$	$\cdots\cdots$	$-$	$\times$	$\cdots\cdots$	$\times$
				i			$\underbrace{\qquad\qquad}_{p-j+1}$		

Moving the bead on the 2nd runner to the ith runner does not alter the symbol $\langle i+1\rangle$, but swapping the ith and $(i + 1)$th runners has the effect of replacing $\langle i + 1\rangle$, as it is the symbol $\bar{\alpha}$ for this move. Hence λ is replaced, as claimed.

Theorem D now follows since we have shown that all other p-regular partitions are replaced, so those remaining – $\langle p\rangle$ and $\langle i, i+1\rangle$ for $2 \leqslant i \leqslant p-1$, which label $(2p)$ and $(i^2, 2^{p-1})$ for $3 \leqslant i \leqslant p$ – form the $p-1$ partitions that are not replaced.

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References

[1] Joseph Chuang, *The derived categories of some blocks of symmetric groups and a conjecture of Broué*, J. Algebra **217** (1999), 114–155.

[2] David A. Craven, *On tensor products of simple modules for simple groups*, In preparation.

[3] Karin Erdmann and Stuart Martin, *Quiver and relations for the principal p-block of* Σ_{2p}, J. London Math. Soc. **49** (1994), 442–462.

[4] Lluís Puig, *On Joanna Scopes' criterion of equivalence for blocks of symmetric groups*, Algebra Colloq. **1** (1994), 25–55.

[5] Joanna Scopes, *Cartan matrices and Morita equivalence for blocks of the symmetric groups*, J. Algebra **142** (1991), 441–455.

[6] ______, *Symmetric group blocks of defect two*, Quart. J. Math. Oxford **46** (1995), 201–234.