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RELATING SIMPLE MODULES IN WEIGHT 2 BLOCKS OF SYMMETRIC GROUPS

by DAVID A. CRAVEN†

(Mathematical Institute, 24–29 St Giles', Oxford OX1 3LB, UK)

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Abstract

In this article, we study weight 2 blocks of symmetric groups for a fixed odd prime p, and the relationships between them. We first prove that all sources of simple modules from these blocks come either from S_{2p} or $S_p \wr C_2$; since this latter group is easy to understand, this result gives another indication that the principal block of S_{2p} is the 'most complicated' block of weight 2. We then examine which simple modules for S_{2p} share their source with a simple module from $S_p \wr C_2$.

1. Introduction

The blocks of weight 2 in symmetric groups are well understood, and there is a highly developed theory that relates the various weight 2 blocks of symmetric groups to each other. In this note, we will prove some results that describe connections between the simple modules for these blocks.

THEOREM 1.1 *Let K be a field of odd characteristic p, and let B be a weight 2 block of a symmetric group over K. If M is a simple B-module, then the source of M is isomorphic to the source of a module from the principal p-block of either S_{2p} or $S_p \wr C_2$.*

The proof of this uses the abacus and branching rules, and the work of Scopes [**10**, **11**]. This result should be of use when studying the properties of simple modules in weight 2 blocks, since many properties are captured by the source of a module: for example, it is used in [**3**] to prove that all simple modules for weight 2 blocks of symmetric groups are algebraic if $p = 3$ or $p = 5$. (An *algebraic* module is one that satisfies a polynomial with integer coefficients in the Green ring.)

We must introduce some terminology before we can adequately state the next result. In what follows, K is a field of odd characteristic p, and if G is a finite group, write $B_0(KG)$ for the principal block of KG.

We construct a directed graph as follows: the vertices of the directed graph are all Morita equivalence classes of p-block of a given weight w. (There are only finitely many vertices by [**10**], see also Section 3.) We draw an arrow from one class $\mathcal{B}_1$ to another $\mathcal{B}_2$ if there exist $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ such that B_1 and B_2 form a $[w : k]$-pair for $k < w$ in the sense of [**10**], with B_1 a block of S_n and B_2 a block of S_{n+k} for some n. (We review the theory of $[w : k]$-pairs in Section 3.) We call this directed graph the *Scopes digraph* of weight w. It has a unique vertex with no incoming arrows, with representative $B_0(KS_{wp})$, and a unique vertex with no outgoing arrows, with representative the so-called *RoCK block* B_{RoCK}: this is a block whose p-core has $(i-1)(w-1)$ beads on the ith runner of the abacus (see Section 2 for the definition of the abacus). We call these the *minimal* and *maximal*

†E-mail: David.Craven@maths.ox.ac.uk

vertices, respectively. The following result is the restriction to weight 2 of a general result for all blocks (Proposition 4.1).

PROPOSITION 1.2 *Any path between the minimal and maximal vertices on the Scopes digraph of weight* 2 *has length* $p(p-1)/2$. *In other words, if*

$$B_0(KS_{2p}) = B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots \longrightarrow B_r = B_{\mathrm{RoCK}}$$

is a collection of weight 2 *blocks of symmetric groups, where* B_i *and* B_{i+1} *form a* $[2:k_i]$*-pair (and if* $k_i = 1$ *then* B_i *is a block of* S_n *and* B_{i+1} *is a block of* S_{n+1} *for some* n*), then exactly* $p(p-1)/2$ *of the* k_i *are equal to* 1.

In the proposition above, we allow the $[2:k_i]$-pairs for $k_i \geq 2$ to be either (B_i, B_{i+1}) or (B_{i+1}, B_i), so come from either induction or restriction.

Define the restriction $M\downarrow_B$ or induction $M\uparrow^B$ of a module M to a block B to be the maximal summand of the restriction or induction of the module to the subgroup or overgroup that belongs to B. The results of [11] prove that in each (B_i, B_{i+1}) that forms a $[2:1]$-pair, exactly one simple B_i-module does not block induce to a simple B_{i+1}-module; in particular, the sources of all but one B_i-module are isomorphic to the sources of all but one B_{i+1}-module. Since any path from $B_0(KS_{2p})$ to B_{RoCK} has length exactly $p(p-1)/2$, exactly $p(p-1)/2$ of the sources of the $(p+2)(p-1)/2$ simple B-modules are altered as one moves along the path from the minimal to the maximal vertex. However, it could be that the same simple module is altered twice; our next result, which together with Proposition 1.2 proves Theorem 1.1, states that this cannot happen.

THEOREM 1.3 *Let* B *be a* p*-block of weight* 2 *in a symmetric group, where* p *is odd, and let* M *be a simple* B*-module. Let* B' *be a weight* 2 *block of some symmetric group that is Morita equivalent to* B_{RoCK}*, and such that there is a sequence*

$$B_0(KS_{2p}) = B_0, B_1, \ldots, B_r = B, B_{r+1}, \ldots, B_s = B'$$

with (B_i, B_{i+1}) *a* $[2:k_i]$*-pair for some* k_i*. (In other words, the* $[2:k_i]$*-pairs are always block induction.) Either the block induction* $M\uparrow^{B'}$ *or the block restriction* $M\downarrow_{B_0}$ *is a sum of isomorphic simple modules, and modules of cyclic vertex.*

Since the sources of the simple B'-modules are the same as the sources of the simple B_{RoCK}-modules (by [8]), this yields Theorem 1.1, as we stated earlier. There are $p(p-1)/2$ simple B_{RoCK}-modules that do not restrict to semisimple $B_0(KS_{2p})$-modules, so there are $p-1$ that do. The next theorem identifies the $B_0(KS_{2p})$-modules that always remain unaltered along any path in the Scopes digraph.

THEOREM 1.4 *Let* λ *be either the partition* $(2p)$ *or* $(i^2, 2^{p-i})$ *for* $3 \leq i \leq p$*. If* B *is any weight* 2 *block of a symmetric group, then* $D^\lambda\uparrow^B$ *is a sum of copies of some simple* B*-module* D^μ*, and modules of cyclic vertex, and similarly* $D^\mu\downarrow_{B_0}$ *is a sum of copies of* D^λ *and modules of cyclic vertex.*

The simple modules in this theorem appear at the top of the quiver of $B_0(KS_{2p})$, as described in [4], and are the completely splittable modules in the sense of [6]; this theorem states that their sources are sources of simple modules for $S_p \wr C_2$ (since the sources of the simple B_{RoCK}-modules are the same as those of $S_p \wr C_2$ by [2]), and hence are 'small'. The non-trivial such simple modules are in some sense 'double hook' modules.

The structure of this article is as follows: Section 2 introduces the basic representation theory of the symmetric groups that we need, and Section 3 summarizes the relevant results of Scopes from [10, 11]. In Section 4, we prove Proposition 1.2 using Richards's pyramids, in Section 5 we prove Theorem 1.3 (and hence Theorem 1.1), and in the final section we prove Theorem 1.4.

2. Abacus combinatorics

For the basics of the representation theory of the symmetric groups, we refer to (for example) [5]. The irreducible ordinary characters of S_n are labelled by partitions λ of n; write S^λ for the *Specht module* corresponding to λ, which is an irreducible representation of S_n in characteristic 0.

If $\lambda = (\lambda_1, \dots, \lambda_r)$, then the first-column hook lengths of λ are

$$\lambda_1 + (r-1), \lambda_2 + (r-2), \dots, \lambda_{r-1} + 1, \lambda_r.$$

(These are the hook lengths of the boxes in the first column of the tableau representation of λ; see for example [5, Section 2.3] for details on hook lengths.) Since λ is a partition, the first-column hook lengths are distinct, so form a subset of $\mathbb{N}_+$. More generally, a *β-set* is a finite subset of $\mathbb{N}_{\geq 0}$. We introduce an equivalence relation $\sim$ on the set of all β-sets, by saying that if X is a β-set then $X \sim X'$, where

$$X' = \{0\} \cup \{x + 1 \,:\, x \in X\},$$

and extending this to an equivalence relation. Each equivalence class contains a unique (possibly empty) representative X not containing 0, and this is the set of first-column hook lengths for a partition λ. We will identify β-sets and partitions; when we define an operation on one of these, we can transport this operation to the other.

We also introduce the p-abacus: it has p columns, called *runners*, with the first runner on the left. The top-left corner of the abacus is the entry 0, and a β-set of a partition are placed on the abacus left-to-right then down the rows (these are called *beads*), so that the partition (3, 2, 1, 1), with first-column hook numbers (1, 2, 4, 6), are displayed for $p = 5$ as the following. (Here a $\times$ indicates a bead, and a $-$ indicates no beads on that runner.)

$-$	$\times$	$\times$	$-$	$\times$
	$\times$			

We number the runners 1 to p, with 1 on the left-hand side.

If X is a β-set, displayed on a p-abacus, then one may add or remove a *p-hook*, which involves sliding a bead down or up one space on its runner, respectively, to produce another β-set. The *p-core* of a β-set X (or partition) is the β-set (or partition) produced by removing all p-hooks from X. Removing a p-hook from a partition λ produces a partition μ with $|\lambda| - |\mu| = p$; the *p-weight* (or just *weight*) of a partition is the number of p-hooks removed from λ to produce the p-core μ, or equivalently $(|\lambda| - |\mu|)/p$.

We now move on to p-blocks. The Nakayama conjecture (which is a theorem—see [5, 6.1.21]) states that two ordinary characters S^λ and S^μ lie in the same p-block of S_n if and only if λ and μ have the same p-core; the p-blocks of S_n are therefore parametrized by the p-cores of the partitions of n.

The irreducible modules D^λ in characteristic p are parametrized by *p-regular* partitions, i.e., partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ such that no p of the λ_i are equal. (The module D^λ is the head of the Specht module S^λ, defined over the field $\mathbb{F}_p$.) Again, two simple modules D^λ and D^μ lie in the same p-block of S_n if and only if λ and μ have the same p-core. The *weight* of a p-block B is the weight of any partition λ such that S^λ belongs to B.

3. Scopes moves

In this section we briefly summarize the theory from [10, 11] that we need. Suppose that $\bar{B}$ and B are p-blocks of symmetric groups, with p-cores $\bar{\kappa}$ and κ, and both of weight w. We say that $(\bar{B}, B)$ forms a $[w : k]$-*pair* if there are β-sets $\bar{X}$ and X for $\bar{\kappa}$ and κ, respectively, and some $1 \le a \le p-1$ such that, in the abacus representation of $\bar{X}$, there are k more beads on runner a than runner $(a+1)$, and by swapping runners a and $(a+1)$ in the abacus for $\bar{X}$, we obtain the abacus for X. In this case, if B is a block of S_n (so that $n = wp + |\kappa|$), we have that $\bar{B}$ is a block of S_{n-k}.

In [10], Scopes proved that, if $(\bar{B}, B)$ forms a $[w : k]$-pair for $k \ge w$, then $\bar{B}$ and B are Morita equivalent. This proves that there are only finitely many Morita equivalence classes of p-blocks of symmetric groups of a given weight w, and these are the vertices of the Scopes digraph of weight w (as defined in the introduction). If $(\bar{B}, B)$ forms a $[w : k]$-pair, then we draw an arrow from $\bar{\mathcal{B}}$ to $\mathcal{B}$, where $\bar{B} \in \bar{\mathcal{B}}$ and $B \in \mathcal{B}$. (We could label this arrow with 'k' to keep track of the k's involved, but since we will specialize to weight 2, we do not need to do this here.)

A path from the minimal vertex to the maximal vertex in the Scopes digraph is called a *Scopes series*; this is a sequence $B_0(KS_{wp}) = B_0 \to B_1 \to B_2 \to \cdots \to B_n = B_{\mathrm{RoCK}}$, where each arrow $B_i \to B_{i+1}$ is either a Morita equivalence (so, B_i and B_{i+1} represent the same vertex in the Scopes series) or (B_i, B_{i+1}) forms a $[w : k]$-pair for $k < w$. We call the former a *Morita move* (since it is a Morita equivalence) and the latter a *Scopes move*.

From now on we specialize to the case $w = 2$. In this case, the partitions belonging to a block B with a particular p-core κ, and a fixed β-set X, are labelled by *symbols* $\langle i\rangle$, $\langle i, i\rangle$ and $\langle i, j\rangle = \langle j, i\rangle$ $(1 \le i < j \le p)$, where $\langle i\rangle$ is the symbol for the partition with β-set that obtained from X by lowering the lowest bead on the ith runner two places, $\langle i, i\rangle$ is the symbol for the partition with β-set that obtained from X by lowering the lowest two beads on the ith runner down one place, and $\langle i, j\rangle$ is the symbol for the partition with β-set that obtained from X by lowering the lowest bead on each of the ith and jth runners down one place. (In the event of there being insufficient beads on a given runner, we may imagine extra rows of beads above the abacus and lower those, or equivalently replace the β-set by one with a multiple of p more elements in it.)

An important remark is that this labelling is *not* canonical, and depends on the choice of β-set X; as suggested in the previous paragraph, if X' is another β-set, then the symbols correspond to the same partition if and only if $|X| - |X'|$ is a multiple of p.

In the example 5-core above, the symbol $\langle 1, 4\rangle$ (with five extra elements in the β-set) has abacus

—	×	×	—	×
×	×	×	×	×
	×			

yielding hook numbers $(1, 2, 4, 5, 6, 7, 8, 9, 11)$ and partition $(3, 2^6, 1, 1)$. Since the symbols are easier to manipulate than the partitions, we will concentrate on them, and only relate them to the partitions (and modules D^λ if the partition λ is p-regular) when we consider the main theorems of this article.

We now describe a bijection between the symbols for $\bar{B}$ and the symbols for B, when $(\bar{B}, B)$ forms a $[w : k]$-pair, achieved by swapping runners a and $a + 1$. If $k > 1$ (i.e., it is a Morita move), it is easy to describe the bijection: the partitions $\langle i\rangle$, $\langle i, i\rangle$ and $\langle i, j\rangle$ for the block $\bar{B}$ are sent to the same labels for block B, acted on by the transposition $(a, a + 1)$ (so that, for example, if $a = 2$, then $\langle 2, 4\rangle$ goes to $\langle 3, 4\rangle$ and $\langle 1, 4\rangle$ is left invariant).

If $k = 1$ (i.e., it is a Scopes move), then the bijection is more complicated: the images of $\langle i\rangle$, $\langle i, i\rangle$ and $\langle i, j\rangle$ are as in the previous case, except for $\bar{\alpha} = \langle a + 1\rangle$, $\bar{\beta} = \langle a, a + 1\rangle$ and $\bar{\gamma} = \langle a, a\rangle$ (labelling partitions of $\bar{B}$), which have images $\alpha = \langle a + 1, a + 1\rangle$, $\gamma = \langle a\rangle$ and $\beta = \langle a, a + 1\rangle$, respectively (labelling partitions of B).

The above yields a bijection $\Phi_{\bar{B},B}$ from symbols for B to symbols for $\bar{B}$ (or vice versa). If $k > 1$, λ is p-regular, and if D^λ is a simple B-module, then $D^\lambda\downarrow_{\bar{B}}$ is a sum of $k!$ copies of $D^{\bar{\lambda}}$ for $\bar{\lambda} = \Phi_{\bar{B},B}(\lambda)$ (see [10]). If $k = 1$, then for $\lambda \neq \alpha, \beta, \gamma$, we get $D^\lambda\downarrow_{\bar{B}} = D^{\bar{\lambda}}$, as before (see [11, Corollary 3.7]). If β is p-regular, then $D^\beta\downarrow_{\bar{B}} = D^{\bar{\gamma}}$, and if γ is p-regular, then $D^\gamma\downarrow_{\bar{B}} = D^{\bar{\beta}}$ (see [11, Section 4]). The module D^α does *not* have a simple restriction to $\bar{B}$ (it has composition length at least three, and has both head and socle isomorphic to $D^{\bar{\alpha}}$).

In later applications we wish to chain together sequences of Scopes and Morita moves, and for this purpose we need to fix a β-set from the beginning, as we have seen that the symbols depend on the β-set chosen. The obvious β-set we choose is the set of first-column hook lengths of a p-core, so the first runner should not have beads on it after a sequence of runner swaps; hence, after each move, we will always readjust the β-set so that the abacus display has an empty first runner. Swapping runners a and $a + 1$ for $1 < a < p$ presents no problem, and we refer to these cases as *standard moves*. However, we also need to be able to swap runners 1 and 2 (which results in there being beads on the first runner), and swap runners 1 and p (because these runners can be adjacent if one uses a different β-set).

We deal with the case $a = 1$ first. Since there cannot be more beads on runner 1 than runner 2, swapping runners 1 and 2 must be the 'reverse' of a move, and we only need to define these in the Morita case. The effect of swapping runners $1 = a$ and $2 = a + 1$ is to yield a p-core that has at least one bead in the first column. To get the first-column hook lengths for a p-core, we must then move the first column to the pth column and remove one bead. Another way to describe this move then is to say that the ith runner moves to the $(i - 1)$th runner for $3 \leq i \leq p$, the second runner is moved to the pth runner and has a bead removed, and the first runner stays where it is. The effect of this on the symbols is to apply the permutation $(p, p - 1, \dots, 2)$ to the entries in the symbols $\langle i\rangle$, $\langle i, i\rangle$ and $\langle i, j\rangle$. (Since this is a Morita move, we do not need to worry about the symbols α, β and γ.)

For $a = p$, this is essentially the reverse of the previous case. Here, however, the case $k = 1$ is also possible, which is when there are no beads on the pth runner of $\bar{B}$. This move fixes the first runner, moves the ith to the $(i + 1)$th for $2 \leq i < p$ and moves the pth runner to the second, adding a bead. The effect on the symbols $\langle i\rangle$, $\langle i, i\rangle$ and $\langle i, j\rangle$ of $\bar{B}$ is to apply the permutation $(2, 3, \dots, p)$ to the entries, as in the previous paragraph, except for the three symbols $\bar{\alpha} = \langle 1\rangle$, $\bar{\beta} = \langle 1, p\rangle$ and $\bar{\gamma} = \langle p, p\rangle$, which have images $\alpha = \langle 2, 2\rangle$, $\gamma = \langle 1\rangle$ and $\beta = \langle 1, 2\rangle$, respectively, as partitions belonging to B. The Morita and Scopes moves of this type are called *wrap moves*, since they wrap the first runner round to the pth runner or vice versa.

The previous paragraphs give, for any $[2 : k]$-pair $(\bar{B}, B)$, a bijection from the symbols for $\bar{B}$ to the symbols for B, and this bijection induces a bijection between the simple $\bar{B}$-modules and the simple B-modules; this bijection is compatible with block induction for all modules except $D^{\bar{\alpha}}$ in the case of a [2 : 1]-pair, and this induced module is not a direct sum of copies of a single simple B-module.

4. The scopes series

This section contains a proof of Proposition 1.2 using Richards's pyramids, which appear in [9] (although we slightly modify the definition, in keeping with modern convention). We describe them briefly now: let B be a p-block of a symmetric group, with p-core κ and weight w. Let $X = \{x_1, \dots, x_s\}$ with $x_i > x_{i+1}$ be a β-set for κ; consider the subset Y of all $x \in X$ such that $x + p$ is not in X (i.e., the beads that lie at the end of their runner), and note that if $|X|$ is sufficiently large then $|Y| = p$. Inheriting the ordering from X, we have $y_1 > y_2 > \cdots > y_p$, yielding $p(p-1)/2$ numbers $a_{i,j}$, for $1 \le i < j \le p$, defined by

$$a_{i,j} = \begin{cases} 0 & 0 < y_i - y_j < p \\ 1 & p < y_i - y_j < 2p \\ & \vdots \\ w-2 & (w-2)p < y_i - y_j < (w-1)p \\ w-1 & (w-1)p < y_i - y_j \end{cases}.$$

(Notice that, since y_i and y_j lie on different runners of the abacus, $y_i - y_j$ is not divisible by p, and also that they do not depend on the β-set X, only on κ.) We may arrange these numbers, which lie in the range 0,…,$(w-1)$, into a pyramid, with $a_{1,p}$ at the top, then $a_{1,p-1}$ and $a_{2,p}$ in the second row, down to $a_{i,i+1}$ for $1 \le i \le p-1$ at the bottom. By [9, Lemma 3.1], if B and $\bar{B}$ are related by a sequence of Morita moves, then the p-cores labelling B and $\bar{B}$ have the same pyramid.

Suppose that $(\bar{B}, B)$ form a $[w : k]$-pair for some $k < w$: we let $\bar{X}$ be a β-set for the p-core of $\bar{B}$, define $\bar{Y}$ as above, and suppose that runners a and $a+1$ are swapped by the Scopes move. This move replaces some y_α with $y_\alpha + 1$ and some y_β with $y_\beta - 1$, with $\alpha < \beta$. Furthermore, since there are k more beads on runner a than $a+1$, we see that $y_\alpha - y_\beta = kp - 1$, and $(y_\alpha + 1) - (y_\beta - 1) = kp + 1$. Hence, the pyramid number $a_{\alpha,\beta}$ goes from $k-1$ to k, so increments a single element of the pyramid by 1. (Clearly, all other numbers in the pyramid are unaffected.)

This yields the following proposition.

PROPOSITION 4.1 *Any path between the minimal and maximal vertices on the Scopes digraph of weight w has length $wp(p-1)/2$. Moreover, if*

$$B_0(KS_{wp}) = B_0, B_1, B_2, \dots, B_r = B_{\mathrm{RoCK}}$$

is a collection of weight w blocks of symmetric groups, where either (B_i, B_{i+1}) forms a $[w : k_i]$-pair, or (B_{i+1}, B_i) forms a $[w : k]$-pair for $k \ge w$ (i.e., $B_i \to B_{i+1}$ is a Morita move), then exactly $p(p-1)/2$ of the k_i are equal to j for each $1 \le j \le w-1$.

Proof. It is easy to see that the block B_{RoCK} has pyramid consisting solely of $p(p-1)/2$ copies of $w-1$, and $B_0(KS_{wp})$ has a pyramid consisting solely of $p(p-1)/2$ copies of 0. Each Scopes move $B_i \to B_{i+1}$ increments exactly one of the entries by 1, and if the entry changes from k to $k+1$ then (B_i, B_{i+1}) forms a $[w : k]$-pair. This completes the proof.

In the case of weight 2 blocks, pyramids have only entry 0 or 1, so for a p-core κ one may associate $f(\kappa)$, which is the number of pairs (i, j) with $1 \le i < j \le p$ such that either there are more beads on runner j than runner i, or there are at least two more beads on runner i than runner j. This statistic is the sum of the $a_{i,j}$: it is not difficult to see that a Morita move does not alter $f(\kappa)$, and a Scopes move increases $f(\kappa)$ by 1, which is the specialization to $w = 2$ of Proposition 4.1.

5. Sources of weight 2 blocks

Having dealt with cores, we now move on to the partitions associated with cores, and prove Theorem 1.3. As each Scopes and Morita move induces a bijection between the symbols $\langle i \rangle$, $\langle i, i \rangle$ and $\langle i, j \rangle$ of the two blocks, a composition of moves yields a composition of bijections of symbols. In addition, since each move induces a permutation of the runners, a composition of moves yields a permutation of the runners, which is related to the bijection of the symbols above.

Our first claim is that, if

$$B_0(KS_{2p}) = B_0 \longrightarrow B_1 \longrightarrow B_2 \longrightarrow \cdots \longrightarrow B_m = B_{\mathrm{RoCK}}$$

is a sequence of moves, that λ is a symbol for B_0, and that it (or rather its image under the bijection between symbols) plays the role of $\bar{\gamma}$ in some Scopes move $B_r \to B_{r+1}$, then it never plays the role of $\bar{\beta}$ in any Scopes move $B_s \to B_{s+1}$ for $r < s$. More formally, we have the following proposition.

PROPOSITION 5.1 *Let λ be the symbol of a p-regular partition associated to $B_0(KS_{2p}) = B_0$, and let $B_0 \to B_1 \to B_2 \to \cdots \to B_m = B_{\mathrm{RoCK}}$ be a sequence of Morita and Scopes moves. Let ϕ_t denote the bijection of symbols induced by the first t maps (from those belonging to B_0 to those belonging to B_t). If $\phi_r(\lambda)$ is the symbol $\bar{\gamma}$ for some Scopes move $B_r \to B_{r+1}$ then none of the $\phi_s(\lambda)$, for $s > r$, is the symbol $\bar{\beta}$ for some Scopes move $B_s \to B_{s+1}$.*

Proof. Suppose that $\phi_r(\lambda)$ is the symbol $\bar{\gamma}$ for a Scopes move $B_r \to B_{r+1}$. If this Scopes move is a wrap move, then $\phi_r(\lambda) = \langle 1, 1 \rangle$, which is easily seen not to be the symbol of a p-regular partition. Since the ϕ_t send p-regular partitions to p-regular partitions [11, Lemma 3.5(2)], we arrive at a contradiction. Hence, this Scopes move is a standard move.

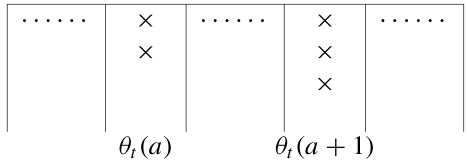
Label the runners involved in the Scopes move $B_r \to B_{r+1}$ by a and $a+1$, with b and $b+1$ beads, respectively, in B_{r+1}.

$\cdots\cdots$	$\times$	$\times$	$\cdots\cdots$
	$\times$	$\times$	
		$\times$	
	a		

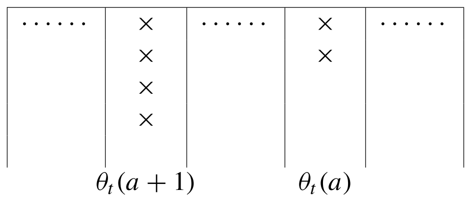
We suppose that there is an $s > r$ such that $\phi_s(\lambda)$ is the symbol $\bar{\gamma}$ for a Scopes move $B_s \to B_{s+1}$ (and choose the smallest such s), and for $r + 1 \le t \le s$ write θ_t for the permutation of the runners induced by the Morita and Scopes moves from B_{r+1} to B_t.

Notice that $\phi_{r+1}(\lambda)$ has label $\langle a, a+1 \rangle$, and for $r+1 \le t \le s$, $\phi_t(\lambda) = \langle \theta_t(a), \theta_t(a+1) \rangle$, until $\phi_t(\lambda)$ becomes the partition $\bar{\beta}$ for some Scopes move, i.e., $t = s$. However, in order for $\langle \theta_t(a), \theta_t(a+1) \rangle$ to be $\bar{\beta}$ in some Scopes move, the runners $\theta_t(a)$ and $\theta_t(a+1)$ must be adjacent, and there must be one fewer bead on the runner to the left than the one to the right.

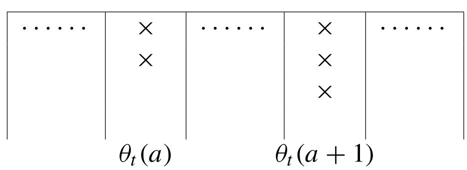
After any sequence of standard moves, the runner $\theta_t(a)$ still has b beads on it, and lies to the left of $\theta_t(a+1)$, which has $b+1$ beads.



Hence, unless wrap moves are performed, $\langle \theta_t(a), \theta_t(a+1)\rangle$ cannot be $\bar{\beta}$. In this case, a wrap move involving $\theta_t(a)$ (which must wrap from the left-hand side to the right-hand side of the abacus) will lose one bead, so the difference in beads will be 2, similarly for $\theta_t(a+1)$ wrapping from the right-hand side to the left.



Wrap moves in these directions can only *increase* the difference in beads between the two runners, and so the last wrap move made before we reach the Scopes move $B_s \to B_{s+1}$ (in which the two runners differ by one bead) must be in the other direction. This still leaves $\theta_t(a+1)$ with one more bead than $\theta_t(a)$ (as in order to wrap from the other side the two runners must swap at some point, and they can only do this if the difference in beads is at least 2, so that it is a Morita move), and must be on the right of $\theta_t(a)$, as in the following situation (again).



This means that a second Scopes move involving $\theta_t(a)$ and $\theta_t(a+1)$ cannot be performed, proving the proposition.

COROLLARY 5.2 *Let λ be the symbol of a p-regular partition associated with $B_0(KS_{2p}) = B_0$, and let $B_0 \to B_1 \to B_2 \to \cdots \to B_m = B_{\mathrm{RoCK}}$ be a sequence of Morita and Scopes moves. Let ϕ_t denote the bijection of partitions induced by the first t maps (from those belonging to B_0 to those belonging to B_t). If $\phi_r(\lambda)$ is the symbol $\bar{\alpha}$ for some Scopes move $B_r \to B_{r+1}$ then none of the $\phi_s(\lambda)$, for $s > r$, is the symbol $\bar{\alpha}$ again, for some Scopes move $B_s \to B_{s+1}$.*

Proof. From the definition of a Scopes move, unless the symbols involved are $\bar{\alpha}$, $\bar{\beta}$ and $\bar{\gamma}$, the move permutes the symbols of the form $\langle i\rangle$, those of the form $\langle i, i\rangle$ and those of the form $\langle i, j\rangle$ for $i \neq j$. Hence, if $\phi_r(\lambda)$ is the symbol $\bar{\alpha}$, in order for $\phi_s(\lambda)$ to also be the symbol $\bar{\alpha}$ there must exist $r < u < v < s$ such that $\phi_u(\lambda)$ is $\bar{\gamma}$ and $\phi_v(\lambda)$ is $\bar{\beta}$, for Scopes moves $B_u \to B_{u+1}$ and $B_v \to B_{v+1}$. This is impossible by the previous proposition.

From here we can prove Theorem 1.3 (using the notation of that theorem), via another proposition. If $B_i \to B_{i+1}$ is a Scopes move, the only simple B_{i+1}-module that does not share a source with a B_i module is that corresponding to α. If no partition can be $\bar{\alpha}$ twice in a Scopes series, then if N replaces M in some Scopes move $B_i \to B_{i+1}$ (so that the sources of the other simple B_{i+1}-modules are the same as the sources of the other B_i-modules) then N cannot be replaced in any later Scopes move, and M can never have been replaced in any previous Scopes move. In particular, this means that $M\downarrow_{B_0}$ has a simple summand S with $M \mid S\uparrow^{B}$, and $N\uparrow^{B'}$ has a simple summand.

The next proposition completes the proof of Theorem 1.3, by specializing to the case $w = 2$.

PROPOSITION 5.3 *Let $m < n$ be integers, and let B_1 be a block of $G_1 = S_m$ and let B_2 be a block of $G_2 = S_n$, both of weight $w < p$, with (the same) defect group P. Suppose that D^{λ_i} is a simple B_i-module, and $D^{\lambda_1}\uparrow^{B_2}$ has D^{λ_2} as a summand. We have that, up to modules with vertex strictly contained in P,*

$$D^{\lambda_1}\uparrow^{B_2} = a \cdot D^{\lambda_2}, \quad D^{\lambda_2}\downarrow_{B_1} = b \cdot D^{\lambda_1}$$

for some $a, b \in \mathbb{N}$.

Proof. Since $w < p$, the defect groups of B_1 and B_2 are abelian, so that D^λ and D^μ have vertex the whole defect group, by [7]. Let $H = \mathrm{Sym}(1, \ldots, wp)$, and choose P to be a Sylow p-subgroup of H. Let $L_1 = \mathrm{Sym}(wp + 1, \ldots, m)$ and $L_2 = \mathrm{Sym}(wp + 1, \ldots, n)$, and note that $N_{G_i}(P) \leq H \times L_i$. Hence, since P is a defect group of B_i, there is a block $\bar{B}_i$ of $K(H \times L_i)$ with defect group P that is the Brauer correspondent of B_i.

Let S be an indecomposable KH-module such that $D^{\lambda_1} \mid S\uparrow^{G_1}$ (and hence $D^{\lambda_2} \mid S\uparrow^{G_2}$). We have that $S\uparrow^{H\times L_i}$ is the (outer) tensor product of S with the free KL_i-module KL_i, so is the direct sum of modules of the form $S \otimes \mathcal{P}$, where $\mathcal{P}$ is a projective KL_i-module. Notice that the block containing $S \otimes \mathcal{P}$ has defect group P if and only if $\mathcal{P}$ is a simple module; in this case, copies of $S \otimes \mathcal{P}$ are the only summands of $S\uparrow^{H\times L_i}$ that come from this block, and so the block induction of S to this block is the sum of copies of $S \otimes \mathcal{P}$.

We now apply Green correspondence to B_i and $\bar{B}_i$: there is a unique $K(H \times L_i)$-module M_i, which belongs to $\bar{B}_i$ by [1, Corollary 6.3.2], such that $D^{\lambda_i} \mid M_i\uparrow^{B_i}$. However, as $D^{\lambda_i} \mid S\uparrow^{G_i} = (S\uparrow^{H\times L_i})\uparrow^{G_i}$, we see that $M_i = S \otimes \mathcal{P}_i$ for some projective KL_i-module $\mathcal{P}_i$. As M_i belongs to a block with defect group P, by the previous paragraph we see that $\mathcal{P}_i$ is a projective simple KL_i-module, and $S\uparrow^{\bar{B}_i} = a_i \cdot M_i$, for some $a_i \in \mathbb{N}$. By Green correspondence, up to modules with vertex strictly contained in P,

$$S\uparrow^{B_i} = a_i \cdot D^{\lambda_i}.$$

We see from this that $D^{\lambda_1}\uparrow^{B_2}$, which is a summand of $S\uparrow^{B_2}$, has the required form.

We now consider the converse. In this case, $D^{\lambda_i}\downarrow_H$ is a summand of $(S\uparrow^{G_i})\downarrow_H$, which by the Mackey formula is a sum of modules of the form $(S\downarrow_X)\uparrow^H$, for X a subgroup of the form $H \cap H^g$. However, since $H \cong S_{wp}$, $H \cap H^g$ is a symmetric group on at most wp letters, so that either $H = H^g$ or $H \cap H^g$ has a smaller Sylow p-subgroup than H. This proves that $(S\uparrow^{G_i})\downarrow_H$ is a sum of copies of S and modules of smaller vertex, whence so is $D^{\lambda_i}\downarrow_H$. Now suppose that Y is a summand of $D^{\lambda_2}\downarrow_{B_1}$ with vertex P. Since $D^{\lambda_2}\downarrow_H$ is a sum of copies of S and modules of smaller vertex, so is $Y\downarrow_H$; however, since Y has vertex $P \leq H$, $Y \mid S\uparrow^{G_1}$, and actually $Y \mid S\uparrow^{B_1}$. However, the only modules in $S\uparrow^{B_1}$ with vertex P are D^{λ_1}, so $Y \cong D^{\lambda_1}$, as claimed.

This proposition might be of independent interest, as in some sense it allows you to deduce information about a general block induction just going in stages. The restrictions of $w < p$ and G_i being symmetric groups can be replaced by weaker conditions, with the result still holding.

6. Restrictions of partitions

The objective of this section is to prove Theorem 1.4. In order to prove this, we will show that the partitions of $2p$ labelled by the symbols $\langle i\rangle$ for $1 \leq i \leq p-1$, $\langle 1, i\rangle$ for $3 \leq i \leq p$ and $\langle i, j\rangle$ for $2 \leq i < j \leq p$ and $j - i > 1$, are all eventually $\bar{\alpha}$ for some series of Scopes moves; since this accounts for $p(p-1)/2$ partitions, this must be all partitions that have their sources altered, and so the branching rule proves the result (noting that the remaining symbols – $\langle p\rangle$ and $\langle i, i+1\rangle$ for $2 \leq i \leq p-1$ – yield the partitions $(2p)$ and $(i^2, 2^{p-i})$).

We now describe sequences of Scopes moves that prove that each of these can be $\bar{\alpha}$; each of the sequence of Scopes moves described here results in a p-core that has runners with either zero or one bead on them. A symbol or partition λ of $B_0(KS_{2p})$ is called *replaced* if there is a series of Scopes and Morita moves, ending with a Scopes move, for which the image of λ under the previous moves is the symbol or partition $\bar{\alpha}$ in the final Scopes move.

From the empty p-core we may only perform a wrap Scopes move, for which the symbol $\langle 1\rangle$ is $\bar{\alpha}$. At this stage, if one performs the standard Scopes moves with runner swaps the transpositions $(2, 3)$, $(3, 4)$, $(4, 5)$, …, $(i, i+1)$, then the last move has as $\bar{\alpha}$ the symbol $\langle i\rangle$ of $B_0(KS_{2p})$. This is easy to see, as the wrap move sends $\langle i\rangle$ to $\langle i+1\rangle$, and all swaps leave $\langle i+1\rangle$ fixed until the final one, for which it is $\bar{\alpha}$. This proves that the symbols $\langle i\rangle$ for $1 \leq i \leq p-1$ label replaced partitions.

Alternatively, one may repeatedly perform wrap Scopes moves, and after i wrap Scopes moves ($i \geq 2$), the partition $\langle 1, p-i+2\rangle$ has been replaced: to see this, a wrap Scopes move replaces $\langle 1\rangle$, sends $\langle 1, p\rangle$ to $\langle 1\rangle$ and maps $\langle 1, p-i+2\rangle$ to $\langle 1, p-i+3\rangle$, which by induction proves the claim. Hence, the symbols $\langle 1, i\rangle$ for $3 \leq i \leq p$ are replaced symbols.

In general, let $\lambda = \langle i, j\rangle$ be a symbol with $2 \leq i < j \leq p$ and $j - i > 1$. There is a four-step process to produce a series of Scopes moves, whose last one replaces λ.

Step 1: perform $p - j + 1$ wrap Scopes moves. Letting ϕ_1 denote the bijection of the symbols obtained from these moves, we have $\phi_1(\lambda) = \langle 2, x\rangle$, where $x = p + 1 - (j - i)$. (As $j - i > 1$, $x < p$.)

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|}
- & \times & \times & \cdots\cdots & \times & - & \cdots\cdots & - & \cdots\cdots & - \\
& 2 & & & & & & x & & \\
\multicolumn{1}{c}{} & \multicolumn{4}{c}{\underbrace{\hspace{4cm}}_{p-j+1}} & \multicolumn{5}{c}{}
\end{array}$$

Step 2: move all but one of the beads to the last but one runner, so that the core looks as follows.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}
- & \times & - & \cdots\cdots & - & \cdots\cdots & - & \times & \cdots\cdots & \times & - \\
& 2 & & & i+1 & & & & & & \\
\multicolumn{7}{c}{} & \multicolumn{3}{c}{\underbrace{\hspace{3.5cm}}_{p-j}} & \multicolumn{1}{c}{}
\end{array}$$

Let ϕ_2 denote the bijection of the symbols obtained from these moves. As we have passed $p - j$ beads through the runner labelled x, we get $\phi_2(\phi_1(\lambda)) = \phi_2(\langle 2, x\rangle) = \langle 2, i+1\rangle$, as each bead reduces x by 1.

Step 3: move the bead on the 2nd runner as far to the right as it may go. Let ϕ_3 denote the bijection of the symbols obtained from these moves. This final bead has the property that as it moves from the 2nd to the ath runner, the symbol $\langle 2, i+1\rangle$ becomes $\langle a, i+1\rangle$ until $a = i$, when it becomes $\langle i, i+1\rangle$. At this point, the bead is on the ith runner, and so the next move changes $\langle i, i+1\rangle$ into $\langle i\rangle$ (as it is $\bar{\beta}$ for this move), and subsequent moves do not affect this symbol. Hence, $\phi_3(\langle 2, i+1\rangle) = \langle i\rangle$.

To see that this is possible, as $j - i > 1$, the $(i+1)$th runner on the abacus at the end of Step 2 is empty, and so the bead on the 2nd runner may be passed through it. At this stage, the core looks as follows.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|} - & \cdots\cdots & - & \cdots\cdots & - & \times & \cdots\cdots & \times & - \\ & & i & & & \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad}_{p-j+1}} & \end{array}$$

Step 4: perform one wrap move then move the new bead on the 2nd runner to the $(i+1)$th runner. Performing a wrap move alters the symbol $\langle i\rangle$ to $\langle i+1\rangle$, and makes the core look as follows.

$$\begin{array}{|c|c|c|c|c|c|c|c|c|c|} - & \times & - & \cdots\cdots & - & \cdots\cdots & - & \times & \cdots\cdots & \times \\ & & & & i & & & \multicolumn{3}{c}{\underbrace{\qquad\qquad\qquad}_{p-j+1}} \end{array}$$

Moving the bead on the 2nd runner to the ith runner does not alter the symbol $\langle i+1\rangle$, but swapping the ith and $(i+1)$th runners has the effect of replacing $\langle i+1\rangle$, as it is the symbol $\bar{\alpha}$ for this move. Hence, λ is replaced, as claimed.

We have shown that, of the p-regular partitions of $2p$, all but $p-1$ of them are replaced, so those remaining ($\langle p\rangle$ and $\langle i, i+1\rangle$ for $2 \le i \le p-1$, which label $(2p)$ and $(i^2, 2^{p-1})$ for $3 \le i \le p$) form the $p-1$ partitions that are not replaced. This proves that, along any series of Morita or Scopes moves, repeated block induction yields a semisimple module; Proposition 5.3, again specialized to the case $w = 2$, completes the proof.

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