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On the unit conjecture for supersoluble group algebras

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ABSTRACT

We introduce structure theorems for the study of the unit conjecture for group algebras of torsion-free supersoluble groups. Motivated by work of P.M. Cohn we introduce the class of (X, Y, N)-group algebras KG, and following D.S. Passman we define an induced length function $L : KG \to \mathbb{N} \cup \{-\infty\}$ using the fact that G has the infinite dihedral group as a homomorphic image. We develop *splitting theorems* for (X, Y, N)-group algebras, and as an application show that if $\sigma \in KG$ is a unit, then $L(\sigma) = L(\sigma^{-1})$. We extend our analysis of splittings to obtain a canonical *reduced split-form* for all units in (X, Y, N)-group algebras. This leads to the study of group algebras of virtually abelian groups and their representations as subalgebras of suitable matrix rings, where we develop a determinant condition for units in such group algebras. We apply our results to the *fours group*

$$\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2} \rangle$$

and show that over any field K, the group algebra $K\Gamma$ has no non-trivial unit of small L-length. Using this, and the fact that L is equivariant under all $K\Gamma$-automorphisms obtained K-linearly from Γ-automorphisms, we prove that no subset of the Promislow set $\mathscr{P} \subset \Gamma$ is the support of a non-trivial unit in $K\Gamma$ for any field K. In particular this settles a long-standing question and shows that the Promislow set is itself not the support of a unit in $K\Gamma$. We then give an introduction to the theory of *consistent chains* toward a preliminary analysis of units of higher L-length in $K\Gamma$. We conclude our work showing that units in torsion-free-supersoluble group algebras are bounded, in that the supports of units and their inverses

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are related through a property (U) and the induced length function L.

1. Introduction

The unit conjecture for group algebras states that if K is a field and G is a torsion-free group, then all units[1] of the group algebra KG are *trivial*; that is, all units are of the form λg for some $\lambda \in K \setminus \{0\}$ and $g \in G$ [11,13,18,19]. The best result to date is entirely group-theoretic, concerning group algebras of unique-product groups [13,14,20]. (A group G is a *unique-product group* if, given any two non-empty finite subsets X and Y of G, there exists an element $g \in G$ having a unique representation of the form $g = xy$ with $x \in X$ and $y \in Y$.) Unique-product groups typify ordered, locally indicable, and right-ordered groups, and for some time it remained an open question whether there exist torsion-free groups that are not unique-product groups. Using small cancellation theory, Rips and Segev [17] gave the first example of a torsion-free group that is not a unique-product group.

For the unit conjecture beyond unique-product groups, it is natural to consider finitely generated, torsion-free, virtually abelian groups; that is, groups with a short exact sequence

$$1 \to H \to G \to G/H \to 1$$

with H abelian and G/H finite. If G/H is cyclic then G is right-orderable, and therefore a unique-product group, so nothing new occurs. The simplest example where G/H is finite non-cyclic is

$$\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2} \rangle,$$

which satisfies the short exact sequence

$$1 \to \mathbb{Z}^3 \to \Gamma \to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \to 1.$$

Called the 'fours group', Γ was shown by Passman [13, p. 606] to be torsion-free and non-right orderable. Promislow [16], using a random search algorithm, exhibited a 14-element subset $\mathscr{P} \subset \Gamma$ such that $\mathscr{P} \cdot \mathscr{P}$ has no unique product.[2] Since then, very little progress on the unit conjecture has been made, and, in particular, it has been a long-standing question whether the Promislow set $\mathscr{P}$ could be the support of a unit over some field K.

In this paper we show that the answer is 'no', and in fact prove something slightly stronger, namely that no subset of the Promislow set $\mathscr{P}$ is the support of a non-trivial unit in $K\Gamma$ over any field K. It is of interest to note that our techniques are not simply group-theoretic. Motivated by work of Cohn [3] we introduce the class of (X, Y, N)-group algebras KG, and following Passman [13, Theorem 13.3.7] we define an induced length function $L : KG \to \mathbb{N} \cup \{-\infty\}$ using the fact that G has the infinite dihedral group as a homomorphic image. We develop *splitting theorems* for (X, Y, N)-group algebras, and as an application show that if $\sigma \in KG$ is a unit then $L(\sigma) = L(\sigma^{-1})$. We then extend our analysis of splittings to obtain a canonical *reduced split-form* for all units in (X, Y, N)-group algebras. This leads to the study of group algebras of virtually abelian groups and their representations as subalgebras of suitable matrix rings. This viewpoint allows us to develop a determinant condition for units in such group algebras. We apply our results to the *fours group*

$$\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2} \rangle$$

[1] *Unit* here means two-sided unit. If K is a field of characteristic 0, then Kaplansky's theorem [8], [13, p. 38] states that every unit in KG is two-sided. The general result for group algebras over fields of characteristic p remains open [2,4,5].

[2] It is an open question as to whether every unique-product group is right orderable.

noting that over any field K, the group algebra $K\Gamma$ is a virtually abelian (X, Y, N)-group algebra. With respect to the induced length function $L : K\Gamma \to \mathbb{N} \cup \{-\infty\}$ it follows via the splitting theorems, that if $\sigma \in K\Gamma$ is a unit then $L(\sigma) = L(\sigma^{-1})$. On the other hand, the group Γ being abelian-by-finite, with $H = \mathbb{Z}^3$ in the notation above, induces a K-algebra embedding $\theta : K\Gamma \hookrightarrow M_4(KH)$, and we find via our determinant condition, that for $\sigma \in K\Gamma$, σ is a unit of $K\Gamma$ if and only if $\det(\theta(\sigma))$ is a non-zero element of the field. Applying these results we then prove that there is no non-trivial unit $\sigma \in K\Gamma$ with $L(\sigma) \leqslant 3$. From this, and the fact that L is equivariant under all $K\Gamma$-automorphisms obtained K-linearly from Γ-automorphisms, we obtain our result on the Promislow set $\mathscr{P}$. We then give an introduction to the theory of *consistent chains* toward a preliminary analysis of units of higher L-length in $K\Gamma$. Our analysis shows that there is a significant jump in complexity from units of L-length 3 to units of L-length $\geqslant 4$ in $K\Gamma$. Nevertheless in our final section we show that units in torsion-free-supersoluble group algebras are bounded, in that the supports of units and their inverses are related through a *bounding* property (U) and the induced length function L.

Finally a word or two on the presentation of this paper and the philosophy underlying it. To date it has never been clear whether the unit conjecture is a ring-theoretic problem. We hope that this paper suggests that it might be. The techniques found within can best be described as those arising from classical ring theory and the theory of group rings. Our methods are varied, where in some places they are technical or computational, while in others they are structural and somewhat subtle. For these reasons we have written the paper in a rather self-contained way, in part to help guide the reader through our development, but also with the hope of possibly drawing more individuals to this wonderful conjecture and long-standing open problem.

2. Preliminaries and notation

Throughout this paper we let KG denote a group algebra over a field K. Conjugation by $g \in G$ shall be denoted by α^g to mean $g\alpha g^{-1}$ for all $\alpha \in KG$. If $\alpha = \sum_{x \in G} a_x x$ is an element of KG then we define the *support of* α, denoted $\operatorname{Supp}\alpha$, to be the set of $x \in G$ such that $a_x \neq 0$. A *unit* of KG is an invertible element of KG with a two-sided inverse and we denote the group of units of KG by UKG. A unit is said to be *trivial* if its support consists of a single element; that is if it has the form λg for some non-zero $\lambda \in K$ and $g \in G$. Otherwise we say that a unit is *non-trivial* if it is not trivial.

We say that G is an (X, Y, N)-group if there exist subgroups $X, Y \leqslant G$, containing a common subgroup N, normal in G, with $X/N = Y/N = \mathbb{Z}/2\mathbb{Z}$, and such that $G/N = X/N * Y/N$, the infinite dihedral group. If in addition KX and KY have no proper divisors of zero and if KN is an Ore domain then we say that KG is an (X, Y, N)-group algebra. Assume that X/N and Y/N are generated by involutions Nx and Ny and write $X = \langle N, x\rangle$ and $Y = \langle N, y\rangle$. Let W be the set of all alternating words in x and y. We call W the *corresponding set of words in x and y*. Since $G/N = (X/N) * (Y/N)$, it follows from [13, Theorem 9.2.9] that W is a (right-left) transversal for N in G so that by [13, Lemma 1.1.3] KG forms a faithfully free (left-right) KN-module with basis W.

Following [13] we now define a length function $L_W = L : KG \to \mathbb{N} \cup \{-\infty\}$. The *$L$-length*, or simply *length*, of a word $w \in W$, denoted by $L(w)$, is the number of factors that occur in it; the empty word, $w = 1$, has length 0, and for example $xyxy$ has length 4. We extend the length function L in two ways: first, if $g \in G$ then there exists a unique $w \in W$ with $g \in Nw$, and we define $L(g) = L(w)$; and secondly, if $\alpha \in KG$ with α non-zero, then we set $L(\alpha)$ to be the maximum of $L(g)$, where $g \in \operatorname{Supp}\alpha$. Finally, set $L(0)$ to be $-\infty$. From $W \subset G \subset KG$, we see that the definition of L is consistent. For $w_1, w_2 \in W$ we say that the product $w_1 w_2$ is *non-overlapping* if no cancellation occurs. In this case,

$$L(w_1 w_2) = L(w_1) + L(w_2).$$

On the other hand, if the product $w_1 w_2$ overlaps, then $L(w_1 w_2)$ is strictly less than $L(w_1) + L(w_2)$. In this case, if w_1 ends in X (and hence w_2 starts in X) then we say that the overlap is in X, and similarly for overlapping in Y. We define the length function $L = L_W$ to be the *length function on KG induced from W* or simply the *induced length function*.

Since KG is a free left-right KN-module with basis W, we may express any $\sigma \in KG$ as $\sigma = \sum \lambda_w w = \sum w \lambda'_w$, for unique $\lambda_w, \lambda'_w \in KN$ with $w \in W$. The context in which we do so will be clear. Since $N \trianglelefteq G$ it follows that $\lambda_w \neq 0$ if and only if $\lambda'_w \neq 0$. We define a word $w \in W$ to be *in σ* whenever $\lambda_w \neq 0$ (equivalently $\lambda'_w \neq 0$), and define $w \in W$ to be a *maximal-length element in σ* if w in σ and $L(w) = L(\sigma)$. In addition, since KN is a domain, it follows that $L(\sigma\nu) = L(\nu\sigma) = L(\sigma)$ for every $\sigma \in KG$ and $\nu \in KN \setminus \{0\}$.

We say that a group G is a *virtually abelian (X, Y, N)-group* if G is an (X, Y, N)-group containing a normal abelian subgroup H of finite index such that $N \subset H$. We define KG to be a *virtually abelian (X, Y, N)-group algebra* if KG is an (X, Y, N)-group algebra of a virtually abelian (X, Y, N)-group G. In this case we refer to H as the *corresponding abelian subgroup*.

Finally we make the following convention. Throughout this paper we develop our theory *on the left*. All one-sided results have obvious right-analogues whose statements and proofs we safely leave to the reader. In those instances where we do require a right-analogue, we shall use the phrase *by symmetry*.

3. Splittings

The results of this section exist in greater generality and will be the subject of a more comprehensive study in a subsequent paper. We present here a special case that is necessary for our work, easier to describe, and is motivated by earlier work of Cohn [3] and Passman in [13, Theorem 13.3.7].

We assume throughout that KG is an (X, Y, N)-group algebra with corresponding set of words W generated by x and y.

We define a *W-linear term in KG*, to be an expression of the form $\Lambda = \alpha + \beta u$ for some $\alpha, \beta \in KN$ and $u \in \{x, y\}$. For convenience we shall simply say *linear term*, keeping in mind the dependence of a linear term on W and KN. We say that Λ is a *non-zero linear term* if as an element of KG it is non-zero. Since W is a basis for KG as a free left KN-module, it follows that $\Lambda = \alpha + \beta u$ is a non-zero linear term if and only if $\alpha \neq 0$ or $\beta \neq 0$. We say that u *belongs to*, or that Λ *contains* u, if $\beta \neq 0$. Two linear terms in KG are said to be *equal* if as elements of KG they are equal.

We define a *W-splitting in KG* to be a non-empty finite sequence

$$\Lambda_1, \dots, \Lambda_s$$

of *non-zero* linear terms $\Lambda_i \in KG$. For convenience we shall simply say *splitting*, keeping in mind the dependence of a splitting on W and KN. We identify a splitting $\Lambda_1, \dots, \Lambda_s$ in KG with *its corresponding product*

$$\Sigma = \Lambda_1 \dots \Lambda_s.$$

Thus the expression

$$\Lambda_1 \dots \Lambda_s = (\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$$

implies that

$$\Lambda_i = (\alpha_i + \beta_i u_i)$$

for all $i = 1, \dots, s$.

If $\Sigma = \Lambda_1 \dots \Lambda_s$ is a splitting, then we call Λ_i a *term* of this splitting and say that the splitting has $s \geqslant 1$ terms. Two terms of a splitting are said to be *adjacent terms* if they are consecutive terms of the sequence. If adjacent terms Λ and Λ' contain u and u' respectively, then we say that Λ and Λ' *overlap in the same group* if $u = u'$. We remark that our definition of splitting allows for the possibility that adjacent terms overlap in the same group.

We define two splittings in KG to be the *same* if as sequences they are the same. Otherwise we say that they are *different*. We say that two splittings are *equal* if their corresponding products are equal. We note that different splittings can define equal splittings in KG. Note also that not every element of KG has a splitting, for example $1 + xy$ does not.

We define the *L-length*, or simply *length*, *of a splitting* $\Lambda_1, \dots, \Lambda_s$ to be the L-length of its corresponding product, that is,

$$L(\Lambda_1 \dots \Lambda_s).$$

Proposition 3.1. *Let $\Lambda_1 \dots \Lambda_s = (\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$ be a splitting. Then*

$$\Lambda_1 \dots \Lambda_s = {\beta_1}^{w_1} \dots {\beta_s}^{w_s} u_1 \dots u_s + \sum_w \lambda_w w$$

for some $\lambda_w \in KN$ and $w_1, \dots, w_s, w \in W$ with $L(w) < s$. In particular,

$$L(\Lambda_1 \dots \Lambda_s) \leqslant s.$$

Proof. We proceed by induction on $s \geqslant 1$. For $s = 1$ our splitting is $\Lambda_1 = \alpha_1 + \beta_1 u_1 = \beta_1 u_1 + \alpha_1$. Assume $s > 1$ and that the result holds for all splittings with fewer than s terms. Then $\Lambda_2 \dots \Lambda_s$ is a splitting with $s - 1$ terms and therefore the inductive hypothesis implies

$$\Lambda_2 \dots \Lambda_s = {\beta_2}^{w_2} \dots {\beta_s}^{w_s} u_2 \dots u_s + \sum_w \lambda_w w$$

for some $\lambda_w \in KN$ and $w_2, \dots, w_s$, $w \in W$ with $L(w) < s - 1$. Thus

$$\Lambda_1[\Lambda_2 \dots \Lambda_s] = (\alpha_1 + \beta_1 u_1)\left[{\beta_2}^{w_2} \dots {\beta_s}^{w_s} u_2 \dots u_s + \sum_w \lambda_w w\right]$$

for some $\lambda_w \in KN$ and $w_2, \dots, w_s$, $w \in W$ with $L(w) < s - 1$. The result follows immediately by expanding the right side of the foregoing equation. $\square$

Thus the L-length of a splitting is bounded above by the number of its terms. The L-length of any splitting is also bounded below by zero (in other words every splitting is non-zero in KG), but this fact requires additional work and is shown in Proposition 3.4. To this end we give two important results, the first of which is an immediate consequence of Proposition 3.1, and whose proof we safely leave to the reader.

Proposition 3.2. *Let $\Lambda_1 \dots \Lambda_s = (\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$ be a splitting. The following are equivalent*:

(i) $L(\Lambda_1 \dots \Lambda_s) = s$;
(ii) $\beta_1 \dots \beta_s \neq 0$ *and* $L(u_1 \dots u_s) = s$;
(iii) *This splitting contains a unique maximal-length element of L-length s.*

We observe that the definition of a splitting

$$(\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$$

allows for the possibility that some $\beta_i = 0$ and as we noted earlier that consecutive u_j, u_{j+1} may be equal. This leads to the following. We define a splitting to be *L-reduced*, or simply *reduced*, if its L-length is either 0 or s. In other words, a splitting is reduced if either its corresponding product of terms collapses to define an element of $KN \setminus \{0\}$ or if the L-length of the splitting agrees with the number of terms of the splitting. We say that a splitting can be *brought into reduced form* if it equals a splitting in reduced form.

Proposition 3.3. *Every splitting can be brought into reduced form.*

Proof. We proceed by induction on $s \geqslant 1$ to show that if

$$\Lambda_1 \dots \Lambda_s = (\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$$

is a splitting, then this splitting can be brought into reduced form. For $s = 1$ this splitting is $\Lambda_1 = \alpha_1 + \beta_1 u_1$. If $\beta_1 = 0$ then $\Lambda_1 = \alpha_1 \neq 0$, so that $L(\Lambda_1) = 0$. If $\beta_1 \neq 0$ then $L(\Lambda_1) = 1$. In either instance Λ_1 is reduced and this establishes the case $s = 1$. Assume $s > 1$ and that the result holds for all splittings with fewer than s terms. If some $\beta_j = 0$ then the term $\Lambda_j = \alpha_j \neq 0$ can be absorbed into an adjacent term to produce a linear term Λ. Since KG is a free left KN-module with basis W, and KN has no proper divisors of zero, it follows that $\Lambda \neq 0$, and therefore we obtain an equal splitting with one less term. Hence the inductive hypothesis implies that the original splitting can be brought into reduced form. We may therefore assume that all $\beta_i \neq 0$. If any two consecutive u_j, u_{j+1} are equal, then both Λ_j, Λ_{j+1} lie in KX or both lie in KY. By hypothesis, KX and KY have no proper divisors of zero. Thus with $u = u_j$, we have $\Lambda_j \Lambda_{j+1} = \alpha + \beta u \neq 0$, thereby yielding an equal splitting with fewer than s terms, and again the original splitting can be brought into reduced form. Thus we may assume that the original splitting satisfies $\beta_1 \dots \beta_s \neq 0$ and $L(u_1 \dots u_s) = s$. By Proposition 3.2, $L(\Lambda_1 \dots \Lambda_s) = s$, and the result follows. □

Proposition 3.4. *Every splitting defines a non-zero element of KG. In particular, the L-length of a splitting is bounded below by zero.*

Proof. By Proposition 3.3, every splitting can be brought into reduced form, so by definition, has L-length at least 0, hence lies in $KG \setminus \{0\}$, and the result follows. □

Proposition 3.5. *Let Σ, Σ' be reduced splittings with $L(\Sigma) \geqslant 1$. If $\Sigma\Sigma' \in KN$, then $L(\Sigma) = L(\Sigma')$.*

Proof. We begin with some preliminary remarks. Let $\Sigma = \Lambda_1 \dots \Lambda_s$ with $\Lambda_i = (\alpha_i + \beta_i u_i)$, and let $\Sigma' = \Lambda'_1 \dots \Lambda'_t$, with $\Lambda'_j = (\alpha'_j + \beta'_j u'_j)$. Then $L(\Sigma) = s$ and $L(\Sigma') = t$. We recall that $\alpha_i, \beta_i, \alpha'_j, \beta'_j \in KN$ and that KN has no proper divisors of zero. By Proposition 3.2 we have $\beta_1 \dots \beta_s \neq 0$, $L(u_1 \dots u_s) = s$ and if $t \geqslant 1$ then $\beta'_1 \dots \beta'_t \neq 0$, $L(u'_1 \dots u'_t) = t$. Furthermore $L(\Lambda_1 \dots \Lambda_s) \geqslant 1$ implies $L(\Lambda'_1 \dots \Lambda'_t) \geqslant 1$. Indeed if not, then by Proposition 3.4 the splitting $\Lambda'_1 \dots \Lambda'_t$ would define a non-zero element $\beta \in KN$ which can be absorbed into the first splitting to yield a splitting

$$(\alpha_1 + \beta_1 u_1) \dots \big(\alpha_s \beta + \beta_s \beta^{u_s} u_s\big)$$

with $\beta_1 \dots \beta_s \beta^{u_s} \neq 0$ and $L(u_1 \dots u_s) = s$. Hence by Proposition 3.2,

$$L\big((\alpha_1 + \beta_1 u_1) \dots \big(\alpha_s \beta + \beta_s \beta^{u_s} u_s\big)\big) = s \geqslant 1,$$

an impossibility.

We now consider three possible cases. If $u_s \neq u'_1$, then $L(u_s u'_1 \dots u'_t) = t + 1 > 1$. Moreover $\beta_s \beta'_1 \dots \beta'_t \neq 0$ and thus by Proposition 3.2 $L(\Lambda_s \Lambda'_1 \dots \Lambda'_t) = t + 1 > 1$. If $u_s = u'_1$ and the product $\Lambda_s \Lambda'_1$ does not lie in KN then as above $L(\Lambda_s \Lambda'_1 \dots \Lambda'_t) = t \geqslant 1$. If $u_s = u'_1$ and $\Lambda_s \Lambda'_1 \in KN \setminus \{0\}$, then $L(\Lambda_s \Lambda'_1 \dots \Lambda'_t) = t - 1$.

Assume $s = 1$. Then $\Lambda_1 \Lambda'_1 \dots \Lambda'_t = \Sigma\Sigma' \in KN$ and by the three cases above the only possibility is $0 = L(\Sigma\Sigma') = t - 1$, so that $t = 1$.

Assume by way of contradiction that $\Sigma\Sigma' \in KN$ but that $s \neq t$. Assume Σ, Σ' are chosen so that s is minimal. Then $s > 1$. Apply the map $* : KG \to KG$ defined by

$$\left(\sum a_x x\right)^* = \sum a_x x^{-1}.$$

This defines a K-linear anti-automorphism of KG whose restriction to G is an anti-automorphism of G, so that $L(\alpha^*) = L(\alpha)$ for all $\alpha \in KG$, sending linear terms to linear terms and reduced splittings to reduced splittings. Hence $\Sigma\Sigma' \in KN$ implies $\Sigma'^*\Sigma^* \in KN$ with Σ'^*, Σ^* reduced and such that

$$L\big(\Sigma'^*\big) = t \neq s = L\big(\Sigma^*\big).$$

The minimality of s therefore implies that $1 < s \leqslant t$. Consider $\Lambda_s \Lambda'_1 \ldots \Lambda'_t$ and let Σ'' be a reduced form for this splitting. Then by the three cases above we have $L(\Sigma'') = t+1, t,$ or $t-1$. But $\Lambda_1 \ldots \Lambda_{s-1}$ is reduced and

$$(\Lambda_1 \ldots \Lambda_{s-1})\Sigma'' \in KN.$$

The minimality of s implies $s - 1 = L(\Sigma'') = t+1, t,$ or $t-1$. The first two instances imply $t < s$, an impossibility, and therefore $s-1 = t-1$, so that $s = t$, a contradiction, and the result follows. $\square$

4. Splitting theorems

We assume throughout that *linear term* means *W-linear term* and that *splitting* means *W-splitting*.

We begin with the following result, motivated from the work of Cohn [3] and by Passman [13, Theorem 13.3.7] on the zero-divisor problem. The proof we give is due to Passman for [13, Theorem 13.3.7], and for completeness we include those key ingredients of his proof, modified to suit our needs.

Theorem 4.1. *(See Cohn [3], Passman [13].) Let KG be an (X, Y, N)-group algebra with corresponding set of words W. Assume that $\sigma, \tau \in KG \setminus \{0\}$ such that $\sigma\tau = \zeta \in KN$. If $L(\tau) \geqslant 1$ then there exists a linear term $\Lambda \in KG \setminus \{0\}$ such that*

$$\rho(\Lambda\tau) = \eta\zeta$$

for some $\rho \in KG \setminus \{0\}$ and $\eta \in KN \setminus \{0\}$. Moreover $0 \leqslant L(\Lambda\tau) < L(\tau)$.

Proof. Assume that W is a set of words in x and y. We begin with some preliminary remarks. We recall that since KG is a free left-right KN-module with basis W, we may express any $\alpha \in KG$ as $\alpha = \sum \lambda_w w = \sum w\lambda'_w$, for unique $\lambda_w, \lambda'_w \in KN$ with $w \in W$. The context in which we do so will be clear. Since $N \trianglelefteq G$ it follows that $\lambda_w \neq 0$ if and only if $\lambda'_w \neq 0$. Thus we recall that *maximal-length elements in $\alpha = \sum \lambda_w w$* are those $w \in W$ such that $L(w) = L(\alpha)$ with $\lambda_w \neq 0$ (equivalently $\lambda'_w \neq 0$).

Assume now that σ and τ are non-zero elements of KG with $\sigma\tau = \zeta \in KN$. Assume further that $L(\sigma) = m$ and $L(\tau) = n \geqslant 1$. We begin with two preliminary steps as given by Steps 1 and 2 in the proof of [13, Theorem 13.3.7], adapted to our situation. We stay with the same exposition where possible.

Step 1. *The products of maximal-length elements overlap in the same group.*

Let $\sigma = \sum_{v \in W} v\sigma_v$, $\tau = \sum_{w \in W} \tau_w w$ with $\sigma_v, \tau_w \in KN \setminus \{0\}$, $v \in W$ distinct, and $w \in W$ distinct. Then

$$\sigma\tau = \sum_{v,w} v\sigma_v\tau_w w$$

such that $\operatorname{Supp} v\sigma_v\tau_w w \subset Nvw$. Moreover $v\sigma_v\tau_w w \neq 0$ since KN has no proper divisors of zero. We now consider those terms corresponding to maximal-length elements $v, w \in W$ in σ, τ respectively. Assume $L(v_0) = L(\sigma) = m$ and $L(w_0) = L(\tau) = n$. If the product v_0w_0 is non-overlapping, then $L(v_0w_0) = m+n$, and it is easy to see, then, that $Nvw = Nv_0w_0$ if and only if $v = v_0$ and $w = w_0$. This implies that the non-zero summand $v_0\sigma_{v_0}\tau_{w_0}w_0$ cannot be cancelled by any other terms and this contradicts the fact that $\sigma\tau \in KN$. It therefore follows that all such pairs overlap in the same group, and without loss of generality we may assume they overlap in X. Thus because any maximal-length element v in σ ends in x and any maximal-length element w in τ begins with x, we conclude that v_0 and w_0 are unique maximal-length elements in σ, τ respectively.

Write $\sigma = \sigma' + \sigma''$, where $\operatorname{Supp}\sigma'$ is given by all those elements $g \in \operatorname{Supp}\sigma$ with either $L(g) = L(\sigma) = m$ or with $L(g) = m-1$ and with g ending in Y. All elements of length m in $\operatorname{Supp}\sigma$ end in X so that $\sigma' = v'\varepsilon_1$, with $v' \in W$ of length $m-1$, ending in y, and where $\varepsilon_1 = \alpha + \beta x$, $\alpha, \beta \in KN$, $\beta \neq 0$. Similarly, write $\tau = \tau' + \tau''$, where $\operatorname{Supp}\tau'$ consists of all those elements $g \in \operatorname{Supp}\tau$ with either $L(g) = L(\tau) = n$, or with $L(g) = n-1$ and with g starting in Y. It follows that $\tau' = \delta_1 w'$, with $w' \in W$ of length $n-1$, starting with y, and where $\delta_1 = \alpha' + \beta' x$, $\alpha', \beta' \in KN$, $\beta' \neq 0$.

Step 2. *The product $\varepsilon_1\delta_1$ belongs to $KN \setminus \{0\}$.*

We first claim that the products $\sigma'\tau''$, $\sigma''\tau'$, and $\sigma''\tau''$ all have length at most $m+n-2$. This is immediate for $\sigma''\tau''$ since $L(\sigma'') \leqslant m-1$ and $L(\tau'') \leqslant n-1$. We now consider $\sigma'\tau''$. Since $L(\sigma') = m$ and $L(\tau'') \leqslant n-1$ it follows that $L(\sigma'\tau'') \leqslant n+m-1$. Moreover the only way equality can hold is if there exist elements $g \in \operatorname{Supp}\sigma'$, $h \in \operatorname{Supp}\tau''$ with $L(g) = m$, $L(h) = n-1$, and with gh non-overlapping. But g ends in X and h starts in X, because those elements of $\operatorname{Supp}\tau$ of length $n-1$ starting in Y are contained in $\operatorname{Supp}\tau'$. Thus gh overlaps in X and such elements do not exist. Therefore $L(\sigma'\tau'') \leqslant m+n-2$ and similarly $L(\sigma''\tau') \leqslant m+n-2$. Finally $\sigma\tau = \zeta \in KN$ implies $\sigma'\tau' = \zeta - (\sigma'\tau'' + \sigma''\tau' + \sigma''\tau')$, so that $L(\sigma'\tau') \leqslant m+n-2$. This establishes the claim.

Now $\sigma'\tau' = v'\varepsilon_1\delta_1 w'$. Since KX has no proper divisors of zero, we have $\varepsilon_1\delta_1 \neq 0$. Suppose that $\varepsilon_1\delta_1$ did not lie in KN. Then it would have a summand of the form λx, for some $\lambda \in KN \setminus \{0\}$. Now the product $v'xw'$ is non-overlapping and yields a word in W whose L-length is $(m-1)+1+(n-1) = m+n-1$. Moreover the term $v'\lambda xw'$ is the unique summand of $\sigma'\tau'$ with the support in $Nv'xw'$, so that this term cannot possibly be cancelled by any other summands in $\sigma'\tau'$. But this contradicts the fact that $L(\sigma'\tau') \leqslant m+n-2$. Thus $\varepsilon_1\delta_1 \in KN \setminus \{0\}$.

We are now ready for the remainder of the proof, which follows along similar lines as the proof of Step 3 in [13, Theorem 13.3.7]. Since $N \trianglelefteq G$, [13, Lemma 13.3.5(ii)] implies that the set $T = KN \setminus \{0\}$ of regular elements of KN is a left divisor set of regular elements of KG. Now $\varepsilon_1\delta_1 \in T$ and $\sigma\delta_1 \in KG$, so there exist elements $\eta \in T$ and $\rho \in KG$ with

$$\rho(\varepsilon_1\delta_1) = \eta(\sigma\delta_1).$$

Thus, because δ_1 and η are regular elements of KG and σ is non-zero, we conclude that $\rho \neq 0$ and $\rho\varepsilon_1 = \eta\sigma$. This yields

$$\begin{aligned}(\rho\varepsilon_1)\tau &= (\eta\sigma)\tau \\ &= \eta(\sigma\tau) \\ &= \eta\zeta \in KN.\end{aligned}$$

We now compute the length of $\varepsilon_1\tau$. We observe that $\varepsilon_1\tau \neq 0$ since $\tau \neq 0$, and $\varepsilon_1 \neq 0$ lies in KX implies that ε_1 is not a proper divisor of zero in KG. Thus $L(\varepsilon_1\tau) \geqslant 0$. Moreover

$$\varepsilon_1\tau' = \varepsilon_1\delta_1 w',$$

and since $L(w') = n - 1$ and $\varepsilon_1\delta_1 \in KN \setminus \{0\}$, by Step 2, we conclude that $L(\varepsilon_1\tau') \leqslant n - 1$. Since $L(\tau'') \leqslant n - 1$ and $\varepsilon_1 \in KX$, we have

$$L\big(\varepsilon_1\tau''\big) \leqslant L\big(\tau''\big) + L(\varepsilon_1) \leqslant (n-1) + 1 = n.$$

If equality occurs then there exist elements $g \in \operatorname{Supp}\tau''$, $h \in \operatorname{Supp}\tau'$ with $L(g) = n - 1$, $L(h) = 1$, and with hg non-overlapping. However, $L(g) = n - 1$, and $g \in \operatorname{Supp}\tau''$ implies that g starts in X and h ends in X. Therefore, the product does overlap, and this case cannot occur. Hence $L(\varepsilon_1\tau'') \leqslant n - 1$, and from $\varepsilon_1\tau = \varepsilon_1\tau' + \varepsilon_1\tau''$, it follows that

$$0 \leqslant L(\varepsilon_1\tau) \leqslant n - 1 < L(\tau).$$

The result now follows with $\Lambda = \varepsilon_1$. □

Theorem 4.2. *Let KG be an (X, Y, N)-group algebra with corresponding set of words W, and with induced length function L. Assume that σ, τ are non-zero elements of KG with $\sigma\tau = \zeta \in KN$. Then $L(\sigma) = 0$ if and only if $L(\tau) = 0$. Equivalently, $\sigma \in KN \setminus \{0\}$ if and only if $\tau \in KN \setminus \{0\}$.*

Proof. We recall that KG is a free left-right KN-module with basis W, so we may express any $\gamma \in KG$ as $\gamma = \sum \lambda_w w = \sum w\lambda'_w$, for unique $\lambda_w, \lambda'_w \in KN$ with $w \in W$. Since $N \trianglelefteqslant G$ it follows that $\lambda_w \neq 0$ if and only if $\lambda'_w \neq 0$. Furthermore, KN is a domain, and therefore $L(\gamma\nu) = L(\nu\gamma) = L(\gamma)$ for every $\gamma \in KG$ and $\nu \in KN \setminus \{0\}$. If $L(\sigma) = 0$ then $\sigma \in KN \setminus \{0\}$. Thus if $\tau \in KG \setminus \{0\}$ and $\sigma\tau = \zeta \in KN$, then $\zeta \neq 0$ so that $L(\sigma\tau) = L(\zeta) = 0$, and hence

$$L(\tau) = L(\sigma\tau) = 0$$

as desired. The result now follows, reversing the roles of σ and τ. □

Theorem 4.3 *(Left-splitting: weak form). Let KG be an (X, Y, N)-group algebra with corresponding set of words W. Assume that σ, τ are non-zero elements of KG with $\sigma\tau = \zeta \in KN$. There exists a splitting $\Lambda_1 \ldots \Lambda_s$ such that*

$$(\Lambda_1 \ldots \Lambda_s)\tau = \eta\zeta$$

for some $\eta \in KN \setminus \{0\}$.

Proof. Assume that W is a set of words in x and y. Since σ, τ are non-zero elements of KG with $\sigma\tau \in KN$, it follows that $L(\sigma) \geqslant 0$ and $L(\tau) \geqslant 0$. We now proceed by induction on $L(\tau)$. If $L(\tau) = 0$ then by Theorem 4.2 $\sigma \in KN \setminus \{0\}$, so that

$$(\sigma + 0 \cdot x)\tau = (\Lambda_1)\tau = 1 \cdot \zeta$$

yields the desired result with $\eta = 1 \in KN \setminus \{0\}$. Therefore, we may assume that $L(\tau) > 0$, and that the result holds for all such σ' and τ' with $L(\tau') < L(\tau)$. By Theorem 4.1 there exists a linear term $\Lambda \in KG \setminus \{0\}$ and $\rho \in KG \setminus \{0\}$ such that $\rho(\Lambda\tau) = \eta_1\zeta$ for some $\eta_1 \in KN \setminus \{0\}$ and such that $0 \leqslant L(\Lambda\tau) < L(\tau)$.

By induction there exists a splitting $\Lambda_1 \ldots \Lambda_{s-1}$ such that

$$(\Lambda_1 \ldots \Lambda_{s-1})(\Lambda\tau) = \eta_2\eta_1\zeta$$

for some $\eta_2 \in KN \setminus \{0\}$. The result now follows noting that $\eta = \eta_2\eta_1 \in KN \setminus \{0\}$ and that

$$(\Lambda_1 \dots \Lambda_{s-1})\Lambda$$

is a splitting. □

We can now establish an alternate proof of the zero-divisor conjecture for such group algebras first established by Cohn [3] and Lewin [10]:

Theorem 4.4. *(See Cohn [3], Lewin [10].) Let KG be an (X, Y, N)-group algebra with corresponding set of words W. Then KG contains no proper divisors of zero.*

Proof. Assume σ, τ are non-zero elements of KG such that $\sigma\tau = 0$. By Theorem 4.3 there exists a splitting $\Lambda_1 \dots \Lambda_s$ such that

$$(\Lambda_1 \dots \Lambda_s)\tau = \eta \cdot 0.$$

By symmetry there exists a splitting $\Lambda'_1 \dots \Lambda'_t$ such that

$$(\Lambda_1 \dots \Lambda_s)\left(\Lambda'_1 \dots \Lambda'_t\right) = 0 \cdot \eta' = 0.$$

This contradicts Proposition 3.4. □

Corollary 4.5. *Let KG be an (X, Y, N)-group algebra. Then G is torsion-free.*

Proof. If G is not torsion-free then KG has proper divisors of zero by [13, Lemma 13.1.1]. □

Corollary 4.6. *Let KG be an (X, Y, N)-group algebra. Then KG is von Neumann finite; that is for $\sigma, \tau \in KG$, $\sigma\tau = 1$ implies $\tau\sigma = 1$.*

Proof. If $\sigma, \tau \in KG$ with $\sigma\tau = 1$ then $\tau\sigma$ is a non-zero trivial idempotent by Theorem 4.4. □

Theorem 4.7 *(Left-splitting: strong form). Let KG be an (X, Y, N)-group algebra with corresponding set of words W. Assume that σ, τ are non-zero elements of KG with $\sigma\tau = \zeta \in KN$. Then there exists a splitting $\Lambda_1 \dots \Lambda_s$ such that*

$$(\Lambda_1 \dots \Lambda_s)\tau = \eta\zeta$$

for some $\eta \in KN \setminus \{0\}$. For any such splitting we have

$$L(\sigma) = L(\Lambda_1 \dots \Lambda_s).$$

If $L(\sigma) \geqslant 1$ then we may choose the above splitting to satisfy

$$L(\sigma) = s = L(\Lambda_1 \dots \Lambda_s).$$

Proof. By Theorem 4.3 there exists a splitting $\Lambda_1 \dots \Lambda_s$ such that

$$(\Lambda_1 \dots \Lambda_s)\tau = \eta\zeta$$

for some $\eta \in KN \setminus \{0\}$. As $(\eta\sigma)\tau = \eta\zeta$, and KG has no proper divisors of zero by Theorem 4.4, it follows that

$$\eta\sigma = \Lambda_1 \dots \Lambda_s.$$

Furthermore since KG is a free left KN-module with basis W, we have

$$L(\sigma) = L(\eta\sigma) = L(\Lambda_1 \dots \Lambda_s).$$

Since $L(\sigma) \geqslant 1$, Proposition 3.3 implies that we can bring any splitting into reduced form and therefore for such a reduced splitting $\Lambda_1 \dots \Lambda_s$ we have

$$L(\sigma) = L(\Lambda_1 \dots \Lambda_s) = s$$

and the result follows. □

The next result is a special case of Theorem 4.7 and is of independent interest. Within a virtually abelian (X, Y, N)-group algebra KG, the term $\eta \in KN \setminus \{0\}$ can be chosen to be central in KG.

Theorem 4.8 *(Left-splitting: virtually abelian). Let KG be a virtually abelian (X, Y, N)-group algebra with corresponding set of words W, and with corresponding abelian subgroup H containing N. Suppose for some $\sigma, \tau \in KG \setminus \{0\}$ we have $\sigma\tau = \zeta \in KN$. Then there exists a splitting $\Lambda_1 \dots \Lambda_s$ such that*

$$(\Lambda_1 \dots \Lambda_s)\tau = \eta\zeta$$

for some $\eta \in KN \setminus \{0\}$, central in KG, and $u_1, \dots, u_s \in \{x, y\}$. For any such splitting we have

$$L(\sigma) = L(\Lambda_1 \dots \Lambda_s).$$

If $L(\tau) \geqslant 1$ then we may choose the above splitting to satisfy

$$L(\sigma) = s = L(\Lambda_1 \dots \Lambda_s).$$

Proof. Assume that σ and τ are non-zero elements of KG with $\sigma\tau = \zeta \in KN$. If $L(\tau) = 0$ then by Theorem 4.2 $L(\sigma) = 0$ and

$$(\sigma + 0 \cdot x)\tau = (\Lambda_1)\tau = 1 \cdot \zeta,$$

so the result holds with $\eta = 1$. Suppose that $L(\tau) \geqslant 1$, and by Theorem 4.7 there exists a splitting $(\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$ such that

$$(\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)\tau = \nu\zeta$$

for some $\nu \in KN \setminus \{0\}$. Fix a transversal $\{1, a_2, \dots, a_k\}$ for H in G, and let $\nu' = \prod_{i=2}^{k} \nu^{a_i}$. Then

$$\Lambda_1 \dots \Lambda_s = \left(\nu'\alpha_1 + \nu'\beta_1 u_1\right) \dots (\alpha_s + \beta_s u_s)$$

is a splitting such that

$$(\Lambda_1 \dots \Lambda_s)\tau = \nu'\nu\zeta$$

and with $\eta = \nu'\nu$ central in KG as desired. □

The splitting theorems are powerful tools for analysing units in (X, Y, N)-group algebras. The following results give information on the inverse of a unit σ.

Theorem 4.9. *Let KG be an (X, Y, N)-group algebra with corresponding set of words W. Let $\sigma, \tau \in KG \setminus \{0\}$. If $\sigma\tau \in KN$, then $L(\sigma) = L(\tau)$.*

Proof. Assume that σ and τ are non-zero elements of KG with $\sigma\tau = \zeta \in KN$. Then $L(\sigma) \geqslant 0$ and $L(\tau) \geqslant 0$. If $L(\sigma) = 0$ then by Theorem 4.2 $L(\tau) = 0$. Therefore we may assume that $L(\sigma) \geqslant 1$. By symmetry we have $L(\tau) \geqslant 1$. By Theorem 4.7 there exists a reduced splitting $\Lambda_1 \ldots \Lambda_s$ such that

$$(\Lambda_1 \ldots \Lambda_s)\tau \in KN$$

and satisfying

$$L(\sigma) = s = L(\Lambda_1 \ldots \Lambda_s).$$

By symmetry there exists a reduced splitting $\Lambda'_1 \ldots \Lambda'_t$ such that

$$(\Lambda_1 \ldots \Lambda_s)\bigl(\Lambda'_1 \ldots \Lambda'_t\bigr) \in KN$$

and satisfying

$$L(\tau) = t = L\bigl(\Lambda'_1 \ldots \Lambda'_t\bigr).$$

Since each splitting is reduced and $L(\sigma) \geqslant 1$, Proposition 3.5 applies to yield $s = t$ as desired. □

Theorem 4.10. *Let KG be an (X, Y, N)-group algebra with corresponding set of words W. Let $\sigma, \tau \in KG$. If $\sigma\tau = 1$, then $L(\sigma) = L(\tau)$.*

5. Localisations and split-forms for units

The results of this section form the key ingredients to the paper.

If KG is an (X, Y, N)-group algebra with corresponding set of words W, then in particular $N \trianglelefteq G$ and KN is an Ore domain. Therefore by [13, Lemma 13.3.5(ii)] the set of non-zero elements of KN forms both a left and right *divisor* (or *denominator*) set $T = KN \setminus \{0\}$ of KG. Thus we may localise and by [15, Lemma 25.4] conclude that

$$KG \subset T^{-1}KG = KGT^{-1}.$$

It is convenient therefore to note that if $\sigma\tau = 1$ then by Theorem 4.7 there exists a reduced splitting $\Lambda_1 \ldots \Lambda_s$ such that

$$(\Lambda_1 \ldots \Lambda_s)\tau = \eta \cdot 1$$

for some non-zero element η of KN. Observing $(\eta\sigma)\tau = \eta$ and using that KG has no proper divisors of zero, by Theorem 4.4, we have

$$\eta\sigma = \Lambda_1 \ldots \Lambda_s.$$

Thus up to a factor in KN, any unit of KG is a reduced splitting.

We define a *left split-form for* σ, or simply *split-form*, to be an ordered pair $(\eta, \Lambda_1 \dots \Lambda_s)$ such that

$$\eta\sigma = \Lambda_1 \dots \Lambda_s$$

for some $\eta \in KN \setminus \{0\}$ and splitting $\Lambda_1 \dots \Lambda_s$ in KG. To stress the dependence of the splitting on the unit σ we shall write the split-form as $(\eta, \bar{\sigma})$.

Of course the foregoing gives an alternate perspective on the structure of any unit in KG. By definition, any split-form $(\eta, \bar{\sigma})$ satisfies $\eta\sigma = \bar{\sigma}$, and therefore in the localisation $T^{-1}KG$ we have $\sigma = \eta^{-1}\bar{\sigma}$, which, in some sense, expresses a *uniqueness* result derived from all split-forms for σ.

It is convenient at this point to introduce the following concept. If KG is an (X, Y, N)-group algebra with corresponding set of words W, then we may express any $\sigma \in KG$ as

$$\sigma = \sum_{w \in W} \mu_w w$$

for unique $\mu_w \in KN$. We define the W*-support* of σ to be

$$\operatorname{Supp}_W \sigma = \{w \in W \mid \mu_w \neq 0\}.$$

Proposition 5.1. *Let KG be an (X, Y, N)-group algebra with corresponding set of words W. Let $\sigma \in KG$ and $\eta \in KN \setminus \{0\}$. Then $\operatorname{Supp}_W \sigma = \operatorname{Supp}_W \eta\sigma$.*

Proof. Since KG is a free left KN-module with basis W, we can write

$$\sigma = \sum_{w \in W} \mu_w w$$

for uniquely determined $\mu_w \in KN$. Hence

$$\eta\sigma = \sum_{w \in W} (\eta\mu_w)w.$$

Since KN has no proper divisors of zero we have $\eta\mu_w \neq 0$ if and only if $\mu_w \neq 0$, and the result follows. □

This leads to the next result, which is also a consequence of Step 1 of the proof of Theorem 4.1.

Theorem 5.2. *Let KG be an (X, Y, N)-group algebra with corresponding set of words W. If σ is a unit of KG, then σ contains a unique word of maximal L-length.*

Proof. If $L(\sigma) = 0$ then $\sigma \in KN$, and $\sigma = \sigma \cdot 1$ implies that the identity $w = 1$ is the unique word of maximal L-length in σ. Assume $L(\sigma) = s \geqslant 1$ and let $(\eta, \bar{\sigma})$ be a split-form for σ such that $\bar{\sigma}$ is reduced. Then $\bar{\sigma}$ has s terms, and by Proposition 3.2 has a unique word of maximal L-length $s = L(\sigma)$. By Proposition 5.1 we have $\operatorname{Supp}_W \sigma = \operatorname{Supp}_W \bar{\sigma}$, and the result follows. □

In the following, we let $\gcd(\alpha, \beta)$ denote the *greatest common divisor* of α and β.

Proposition 5.3. *If KG is an (X, Y, N)-group algebra with N finitely generated abelian, then every splitting in KG is expressible as*

$$\nu(\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$$

for some $\nu \in KN \setminus \{0\}$, $(\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$ reduced and such that $\gcd(\alpha_i, \beta_i) = 1$ for all $i = 1, \dots, s$.

Proof. Let W be a corresponding set of words in x and y. Since N is finitely generated abelian, KN is a Laurent polynomial ring, and so is a unique-factorisation domain. Thus the gcd of any two elements of KN, with at least one of them non-zero, is well-defined. By Proposition 3.3 a splitting can be brought into reduced form and by Proposition 3.4 it defines a non-zero element of KG. If a reduced splitting defines a non-zero element ν of KN then this reduced splitting equals $\nu(1+0\cdot x)$, an expression satisfying the conclusion of the theorem. If a reduced splitting has length $s \geqslant 1$, then by Proposition 3.2 it is expressible as

$$\left(\alpha_1' + \beta_1' u_1\right)\dots\left(\alpha_s' + \beta_s' u_s\right)$$

with $\beta_1' \dots \beta_s' \neq 0$ and $L(u_1 \dots u_s) = s \geqslant 1$. If $s = 1$ then we can pull out $\nu_s = \gcd(\alpha_s', \beta_s')$ to the left to obtain an expression

$$\nu(\alpha_s + \beta_s u_s)$$

with $\nu = \nu_s \in KN \setminus \{0\}$, $\alpha_s = \alpha_s'/\nu_s$, $\beta_s = \beta_s'/\nu_s \neq 0$ in KN, and with $\gcd(\alpha_s, \beta_s) = 1$. By Proposition 3.2, the single-term splitting $(\alpha_s + \beta_s u_s)$ is reduced, and therefore the expression $\nu(\alpha_s + \beta_s u_s)$ satisfies the conclusion of the theorem. If $s \geqslant 2$, then we can pull out $\nu_s = \gcd(\alpha_s', \beta_s')$ to the left and absorb it into the previous term to obtain a splitting

$$\left(\alpha_1' + \beta_1' u_1\right)\dots\left(\alpha_{s-2}' + \beta_{s-2}' u_{s-2}\right)\left(\alpha_{s-1}'' + \beta_{s-1}'' u_{s-1}\right)(\alpha_s + \beta_s u_s)$$

with $\alpha_{s-1}'' = (\alpha_{s-1}')(\nu_s)$, $\beta_{s-1}'' = (\beta_{s-1}')(\nu_s^{u_{s-1}})$, $\alpha_s = \alpha_s'/\nu_s$, and $\beta_s = \beta_s'/\nu_s$ in KN. Since KN is a domain and $\nu_s^{u_{s-1}} \neq 0$, it follows that $\beta_1' \dots \beta_{s-2}' \beta_{s-1}'' \beta_s \neq 0$. Moreover $L(u_1 \dots u_s) = s$. Thus by Proposition 3.2 this splitting is reduced, with $\gcd(\alpha_s, \beta_s) = 1$. Proceeding inductively, we can pull out gcd's to the left as outlined above to arrive at the desired expression. $\square$

We now come to the main result of this section.

Theorem 5.4. *Assume KG is an (X, Y, N)-group algebra with N finitely generated abelian. Assume σ is a unit of L-length $s \geqslant 1$ in KG. Then*

$$\varepsilon\sigma = (\alpha_1 + \beta_1 u_1)\dots(\alpha_s + \beta_s u_s)$$

for some $\varepsilon \in KN \setminus \{0\}$, $(\alpha_1 + \beta_1 u_1)\dots(\alpha_s + \beta_s u_s)$ reduced and such that $\gcd(\alpha_i, \beta_i) = 1$ for all $i = 1, \dots, s$.

Proof. By Theorem 4.7 we can write

$$(\Lambda_1 \dots \Lambda_s)\tau = \eta \cdot 1$$

for some $\eta \in KN \setminus \{0\}$ and reduced splitting $\Lambda_1 \dots \Lambda_s$. By the previous Proposition 5.3 we can write our reduced splitting as

$$\nu(\alpha_1 + \beta_1 u_1)\dots(\alpha_s + \beta_s u_s)$$

for some $\nu \in KN \setminus \{0\}$, $(\alpha_1 + \beta_1 u_1)\dots(\alpha_s + \beta_s u_s)$ reduced and such that $\gcd(\alpha_i, \beta_i) = 1$ for all $i = 1, \dots, s$. By our remarks above we conclude

$$\eta\sigma = \nu(\alpha_1 + \beta_1 u_1)\dots(\alpha_s + \beta_s u_s).$$

Now let W be the corresponding set of words for this algebra and view KG as a free left KN-module with basis W. Then

$$\nu(\alpha_1+\beta_1 u_1)\dots(\alpha_s+\beta_s u_s)=\sum_{w\in W}(\nu\lambda_w)w$$

for some uniquely determined $\lambda_w \in KN$. It follows that η divides $\nu\lambda_w$ in KN for each $w\in W$. We claim that this forces ν to divide η in KN. Suppose by way of contradiction that ν does not divide η. Then there exists a prime divisor p of ν that does not divide η so that η divides $(\nu/p)\lambda_w$ for every $w\in W$. It then follows that

$$\sigma = p\cdot\sum_{w\in W}\eta^{-1}(\nu/p)\lambda_w w$$

with

$$\sum_{w\in W}\eta^{-1}(\nu/p)\lambda_w w\in KG.$$

But this implies that the prime $p\in KN$ is a unit of KG hence a unit of KN by [13, Lemma 1.1.4], so that KN contains a non-trivial unit, an impossibility by [11, Theorem 8.5.3] as N is torsion-free abelian. Hence $\varepsilon=\nu^{-1}\eta\in KN$ and it follows that

$$\varepsilon\sigma=(\alpha_1+\beta_1 u_1)\dots(\alpha_s+\beta_s u_s)$$

with $\varepsilon\in KN\setminus\{0\}$, $(\alpha_1+\beta_1 u_1)\dots(\alpha_s+\beta_s u_s)$ reduced and such that $\gcd(\alpha_i,\beta_i)=1$ for all $i=1,\dots,s$ as desired. $\square$

If σ is a unit of L-length $\geqslant 1$, then we will say that $(\varepsilon,\bar{\sigma})$ is a *left-reduced split-form for* σ or simply a *reduced split-form*, if ε and $\bar{\sigma}$ satisfy the conclusion of the previous Theorem 5.4.

6. Representation theorems

In this section *group* means arbitrary group and not necessarily an (X,Y,N)-group.

For the convenience of the reader we begin with a brief review necessary for our work and refer the reader to [13, Chapters 1.1 and 5.1] for a more thorough account. We follow closely the exposition found there.

Let G be a group and H a subgroup. Then KG defines a free left-right KH-module with basis any right-left transversal for H in G. The map $\pi_H: KG\to KH$ given by

$$\sum_{x\in G}a_x x\mapsto\sum_{x\in H}a_x x$$

is a KH-bimodule map with $\pi_H(1)=1$. Moreover for any $g\in G$, $\sigma\in KG$ we have

$$\pi_H\big(g\sigma g^{-1}\big)=g\pi_H(\sigma)g^{-1}.$$

Let $X=\{x_j\}$ be a right transversal for H in G, and let $V=KG$ be the free left KH-module with basis X. Then for any $\sigma\in KG$ there exist $\sigma_j\in KG$ with $\operatorname{Supp}\sigma_j\subset Hx_j$ such that

$$\sigma=\sum\sigma_j$$

with all but finitely many $\sigma_j = 0$. Observing that

$$\operatorname{Supp} \sigma_j x_j^{-1} \subset H,$$

allows us to rewrite the foregoing expression for σ as

$$\sigma = \sum \left(\sigma_j x_j^{-1}\right) x_j.$$

Using properties of the map π_H, we have $\pi_H(\sigma x_j^{-1}) = \sigma_j x_j^{-1}$ so that

$$\sigma = \sum \pi_H\left(\sigma x_j^{-1}\right) x_j,$$

thereby giving us a convenient expression for σ as a (left) KH-linear combination of the right transversal $X = \{x_j\}$, noting that the $\pi_H(\sigma x_j^{-1})$ are uniquely determined.

Assume H has finite index n in G and write $X = \{x_1, \ldots, x_n\}$. The free left KH-module, $V = KG$, with basis X is also a right KG-module that is faithful. Right and left multiplications commute as operators on V and therefore $KG \subset M_n(KH)$, the ring of all $n \times n$ matrices over KH; that is, KG embeds as KH-linear maps on the n-dimensional free KH-module V. This embedding is obtained by computing the right action of KG on any basis. With respect to X we have

$$x_i \sigma = \sum_j \pi_H\left((x_i \sigma) x_j^{-1}\right) x_j$$

so that the desired embedding with respect to this basis X is

$$\rho_X : KG \to M_n(KH)$$

with

$$\sigma \mapsto \left[\pi_H\left(x_i \sigma x_j^{-1}\right)\right].$$

We shall refer to the map ρ_X as the (*right*) *regular embedding* of KG in $M_n(KH)$ (with respect to the right transversal X). By a *regular embedding* $\rho : KG \to M_n(KH)$ we shall mean an embedding such that $\rho = \rho_X$ for some right transversal X. We summarise all of this in the following theorem as given in [13, Lemma 5.1.1].

Theorem 6.1. *Let H be a subgroup of finite index n in a group G and let $X = \{x_1, \ldots, x_n\}$ be a right transversal for H in G. Then there exists a regular embedding*

$$\rho_X : KG \to M_n(KH)$$

given by

$$\sigma \mapsto \left[\pi_H\left(x_i \sigma x_j^{-1}\right)\right].$$

We are now ready for the results of this section. We begin with three straightforward, but necessary, propositions.

Proposition 6.2. *Let H be a normal subgroup of finite index n in G. Then G acts by conjugation on $M_n(KH)$, and this action does not depend on any particular embedding of G in $M_n(KH)$.*

Proof. Since $H \trianglelefteq G$, G acts by conjugation on KH and hence G acts on $M_n(KH)$. More precisely if $[\alpha_{ij}] \in M_n(KH)$ and $g \in G$ then we define

$$[\alpha_{ij}]^g = \left[\alpha_{ij}^g\right],$$

and this yields an action of G by conjugation on $M_n(KH)$ that does not depend on any specific embedding of G in $M_n(KH)$. $\square$

Proposition 6.3. *Let H be a subgroup of finite index n in G and let ρ_X and ρ_Y be regular embeddings of KG in $M_n(KH)$. Then for any $\sigma \in KG$ the matrices $\rho_X(\sigma)$ and $\rho_Y(\sigma)$ are conjugate by an invertible matrix in $M_n(KH)$.*

Proof. The regular embeddings ρ_X and ρ_Y are defined in terms of right transversals X and Y respectively. Replacing X by Y is a change of basis for the module $V = KG$ and therefore for any $\sigma \in KG$ the matrices ρ_X and ρ_Y are conjugate by an invertible matrix in $M_n(KH)$. $\square$

Proposition 6.4. *Let H be a normal subgroup of finite index n in G, and assume that $\rho : KG \to M_n(KH)$ is a regular embedding. Then for any $\sigma \in KG$ and $g \in G$ the matrices $\rho(\sigma)$ and $\rho(\sigma)^g$ are conjugate by an invertible matrix in $M_n(KH)$.*

Proof. Let $X = \{x_1, \dots, x_n\}$ be a right transversal for H in G such that $\rho = \rho_X$. For any $\sigma \in KG$ and $g \in G$ we have:

$$\begin{aligned}\rho_X(\sigma)^g &= \left[g\pi_H\left(x_i\sigma x_j^{-1}\right)g^{-1}\right]\\ &= \left[\pi_H\left(gx_i\sigma x_j^{-1}g^{-1}\right)\right]\\ &= \left[\pi_H\left((gx_i)\sigma(gx_j)^{-1}\right)\right]\\ &= \rho_{gX}(\sigma).\end{aligned}$$

Since gX is another right transversal for H in G, Proposition 6.3 applies and therefore the matrices $\rho_X(\sigma)$ and $\rho_{gX}(\sigma) = \rho_X(\sigma)^g$ are conjugate by an invertible matrix in $M_n(KH)$. $\square$

We now give the main results of this section. We thank the referee for suggesting them to us as generalisations to our original theorems.

Theorem 6.5. *Let H be a normal abelian subgroup of finite index n in G and assume $KG \subset M_n(KH)$ via a regular embedding. Let ZKG denote the centre of KG. Then the determinant map sends KG into $KH \cap ZKG$ and is independent of the choice of regular embedding.*

Proof. Since H is abelian, KH is a commutative ring so the determinant map

$$\det : M_n(KH) \to KH$$

certainly exists. Let $\sigma \in KG$, $g \in G$ and assume

$$\rho : KG \to M_n(KH)$$

is a regular embedding. By Proposition 6.4 the matrices $\rho(\sigma)$ and $\rho(\sigma)^g$ are conjugate by an invertible matrix in $M_n(KH)$ and therefore

$$\det\rho(\sigma) = \det\bigl(\rho(\sigma)^g\bigr).$$

Moreover

$$\det\bigl(\rho(\sigma)^g\bigr) = \bigl(\det\rho(\sigma)\bigr)^g.$$

Thus

$$\det\rho(\sigma) = \bigl(\det\rho(\sigma)\bigr)^g$$

for all $g \in G$ and therefore

$$\det\rho(\sigma) \in KH \cap ZKG.$$

If X and Y are right transversals for H in G then by Proposition 6.3 the matrices $\rho_X(\sigma)$ and $\rho_Y(\sigma)$ are conjugate by an invertible matrix in $M_n(KH)$. Hence

$$\det\rho_X(\sigma) = \det\rho_Y(\sigma)$$

and this shows that the determinant map sends KG into $KH \cap ZKG$ and is independent of the choice of regular embedding. □

Theorem 6.6. *Let H be a normal abelian subgroup of finite index n in G and let $\rho : KG \to M_n(KH)$ be a regular embedding. If $\sigma \in KG$ is invertible then*

$$\det\rho(\sigma) \in UKH \cap ZKG$$

where UKH denotes the units of KH and is independent of the regular embedding.

Proof. Immediate by properties of determinants and the previous theorem. □

The converse is the following. Some of our work is implicit in [13, Lemma 5.1.15] and follows along similar lines.

Theorem 6.7. *Let H be a normal abelian subgroup of finite index n in G and $\rho : KG \to M_n(KH)$ be a regular embedding. Let $\sigma \in KG$. If*

$$\det\rho(\sigma) \in UKH \cap ZKG$$

then $\sigma \in UKG$.

Proof. Let $p(T) = T^n + \alpha_{n-1}T^{n-1} + \cdots + \alpha_0$ be the characteristic polynomial of $\rho(\sigma)$, with $\alpha_i \in KH$. For any $g \in G$ the characteristic polynomial of $\rho(\sigma)^g$ is

$$p^g(T) = T^n + \alpha_{n-1}^g T^{n-1} + \cdots + \alpha_0^g.$$

By Proposition 6.4, the matrices $\rho(\sigma)$ and $\rho(\sigma)^g$ are conjugate by an invertible matrix in $M_n(KH)$ and therefore have the same characteristic polynomial. Thus

$$p(T) = p^g(T).$$

Therefore for all i, g we have $\alpha_i^g = \alpha_i$ and hence $\alpha_i \in KH \cap ZKG$. By the Cayley–Hamilton theorem the matrix $\rho(\sigma)$ satisfies its characteristic polynomial, where the coefficients α_i are viewed as scalar matrices $\alpha_i I$, and where I is the identity matrix. Thus

$$\rho(\sigma)^n + (\alpha_{n-1}I)\rho(\sigma)^{n-1} + \cdots + (\alpha_0 I) = 0$$

and therefore

$$\det \rho(\sigma) = (-1)^n \alpha_0 = \alpha \in UKH \cap ZKG.$$

Thus

$$\alpha^{-1} \in UKH \cap ZKG.$$

We can therefore write

$$\rho(\sigma)\big(\rho(\sigma)^{n-1} + (\alpha_{n-1}I)\rho(\sigma)^{n-2} + \cdots + (\alpha_1 I)\big)\big(-\alpha^{-1}I\big) = I.$$

Moreover for any $\delta \in KH \cap ZKG$ we have $\rho(\delta) = \mathrm{diag}[\delta, \ldots, \delta]$; that is, the matrix with δ in each position along the main diagonal and 0 off the main diagonal. Hence

$$\big(\rho(\sigma)^{n-1} + (\alpha_{n-1}I)\rho(\sigma)^{n-2} + \cdots + (\alpha_1 I)\big)\big(-\alpha^{-1}I\big)$$

lies in the image of ρ and is the inverse of $\rho(\sigma)$. Thus $\sigma \in UKG$. □

As a consequence we have the following special case that is important for our work in the next sections.

Theorem 6.8. *Let H be a torsion-free normal abelian subgroup of finite index in G, and assume $Z(G) = \{1\}$. Let $\rho : KG \to M_n(KH)$ be any regular embedding. Then σ is a unit in KG if and only if $\det \rho(\sigma) \in K \setminus \{0\}$.*

Proof. Let σ be a unit in KG and let $\rho : KG \to M_n(KH)$ be any regular embedding. Then by Theorem 6.5 we have

$$\det \rho(\sigma) \in UKH \cap Z(KG).$$

Since H is torsion-free abelian it follows from [11, Theorem 8.5.3] that $\det \rho(\sigma)$ is a trivial unit of KH and therefore $\det \rho(\sigma) = \lambda h$ for some $\lambda \in K \setminus \{0\}$ and $h \in H$. Thus $\det \rho(\sigma) = \lambda h$ is central in KG, and it follows that $h = 1$, as $Z(G) = \{1\}$, so that $\det \rho(\sigma) \in K \setminus \{0\}$. The converse follows immediately from Theorem 6.7. □

We conclude this section with the following useful result on localisations and their representations. The proof we give extends our comments on localisations given at the beginning of the previous section.

Theorem 6.9. *Let KG be an (X, Y, N)-group algebra with subgroup H of finite index n in G such that $N \subset H$. Let $T = KN \setminus \{0\}$. Then any regular embedding*

$$\rho : KG \to M_n(KH)$$

extends to an embedding

$$\rho : T^{-1}KG \to M_n\big(T^{-1}KH\big)$$

such that

$$\rho\big(T^{-1}\big) = \big[\rho(T)\big]^{-1}.$$

Proof. Let KG be an (X, Y, N)-group algebra with subgroup H of finite index n in G such that $N \subset H$. Let $T = KN \setminus \{0\}$. Then by Theorem 6.1 we have a regular embedding

$$\rho : KG \to M_n(KH).$$

Since $N \trianglelefteq G$ and KN is an Ore domain, it follows from [13, Lemma 13.3.5(ii)] that $T = KN \setminus \{0\}$ is both a left and right divisor (or denominator) set in KG. Hence we may localise at T. The same argument also shows that we may localise the subalgebra KH at T to get $T^{-1}KH \subset T^{-1}KG$. Thus we may view our map ρ as an embedding

$$\rho : KG \to M_n\big(T^{-1}KH\big).$$

Inside $M_n(T^{-1}KH)$ all elements of $\rho(T)$ are invertible. Thus by the universal property of T-inverting morphisms [9, Corollary 10.11] and by [15, Lemma 25.2] it follows that our map ρ extends to an embedding

$$\rho : T^{-1}KG \to M_n\big(T^{-1}KH\big)$$

such that

$$\rho\big(T^{-1}\big) = \big[\rho(T)\big]^{-1}.$$

7. The fours group

In this section we study properties of $\Gamma = \langle x, y \mid xy^2x^{-1} = y^{-2},\ yx^2y^{-1} = x^{-2}\rangle$ that we will use in subsequent sections. We fix the following notation. Let $z = xy$, $a = x^2$, $b = y^2$ and $c = xyxy = z^2$. We recall our convention that for any group G conjugation by $g \in G$ shall be denoted by α^g to mean $g\alpha g^{-1}$ for all $\alpha \in KG$.

We have the following straightforward, but important, result. The statements are presented in the logical order in which they would be proved.

Proposition 7.1. *For any $g \in \Gamma$, $\alpha \in K\Gamma$ we have $\alpha^{xy} = \alpha^{yx}$ and $\alpha^{g^2} = \alpha$. Moreover we have*

(i) $a^x = a$, $a^y = a^{-1}$, $a^z = a^{-1}$;
(ii) $b^x = b^{-1}$, $b^y = b$, $b^z = b^{-1}$;
(iii) $c^{-1} = yxyx$;
(iv) $c^x = c^{-1}$, $c^y = c^{-1}$, $c^z = c$.

Proof. We will establish (iii). To this end we see that $c(yxyx) = xyxyyxyx = xyxy^2xyx = y^2xyxxyx = y^2x^{-2}xyyx = y^2x^{-2}y^{-2}x^2 = 1$ by (iii). $\square$

With significantly more work one can then show the following.

Theorem 7.2. *Let $X = \langle x, y^2\rangle$ and $Y = \langle x^2, y\rangle$ be subgroups of Γ. Then*

(i) *$N = \langle a, b\rangle$ is a normal free abelian subgroup of Γ of rank 2;*
(ii) *$\Gamma/N = X/N * Y/N$ is infinite dihedral;*
(iii) *A right transversal for N in G is given by W, the set of all words in x and y;*
(iv) *There exists a length function $L : \Gamma \to \mathbb{N}$ induced from W;*
(v) *$H = \langle a, b, c\rangle$ is a normal free abelian subgroup of Γ of rank 3 with Γ/H a fours group;*
(vi) *A right transversal for H in Γ is given by $\{1, x, y, xy\}$.*

Proof. Most of the details can be found in [13, Lemma 13.3.3 and Theorem 13.3.7], with the remainder safely left to the reader. $\square$

Theorem 7.3. *The group Γ is supersoluble.*

Proof. The normal series

$$\langle 1\rangle \subset \langle x^2\rangle \subset \langle x^2, y^2\rangle \subset \langle x^2, y^2, xy\rangle \subset \Gamma$$

shows that Γ is supersoluble. $\square$

There are other length functions on Γ. To better see this, we give the following result.

Proposition 7.4. *There are exactly three normal subgroups, $N_1 = N$, $N_2 = \langle a, c\rangle$, and $N_3 = \langle b, c\rangle$, such that if $\phi : \Gamma \to D_\infty$ is a surjective homomorphism then $\ker\phi = N_i$ for some i. Furthermore, there is an automorphism ψ of Γ such that $N_i^\psi = N_{i+1}$ (where the indices are taken modulo 3).*

Proof. We use the calculus from Proposition 7.1. Notice that $(z^2)^x = c^x = c^{-1} = z^{-2}$. Similarly $(z^2)^y = z^{-2}$, $(x^2)^z = x^{-2}$, and $(y^2)^z = y^{-2}$. Therefore any ordered pair from $\{x, y, z\}$ satisfies the relations of the group, and so there are (outer) automorphisms interchanging (x, y) with (u, v), where $u, v \in \{x, y, z\}$. In particular, all of the N_i are $\mathrm{Aut}(\Gamma)$-conjugate.

We now prove that if $M \trianglelefteq \Gamma$ and $\Gamma/M \cong D_\infty$, then $M = N_i$. Firstly, let $G \cong D_\infty$ be generated by elements g and h. Since every element of G is either of order 2 or lies inside the cyclic subgroup of index 2, it cannot be that both g and h have infinite order. Also, if one has infinite order, then their product (either gh or hg) has order 2 as well. This will be important in what follows.

Let M be a normal subgroup of Γ such that Γ/M is infinite dihedral. Then $\Gamma/M = \langle Mx, My\rangle$, and so by the previous paragraph exactly two of Mx, My, and Mxy, must have order 2 in the quotient. Hence M contains one of the N_i, say N_1. (Since they are all $\mathrm{Aut}(\Gamma)$-conjugate, we may assume that $N_1 \leqslant M$.) Since any proper quotient of D_∞ is finite, and we know that Γ/N_1 is infinite dihedral, we see that $M = N_1$, as claimed. $\square$

We can see that $\bigcap N_i = 1$, and so for a group element $g \in G$, its images modulo each of the quotients Γ/N_i is enough to determine it uniquely. Also, since each of the three normal subgroups N_i are $\mathrm{Aut}(\Gamma)$-conjugate, any result proved using one of the length functions is automatically applicable for the other two length functions obtained in this way.

Thus we see how to form other length functions on the group, simply by taking two other generators for Γ that satisfy the group relations: for example, consider the pair (x, xyx), which together generate Γ. Then $\langle x^2, (xyx)^2\rangle = \langle x^2, y^{-2}\rangle = N$, but here the elements x and xyx are considered to have length 1, and the element $y = x(xyx)x$ has length 3.

A useful method for determining the length of an element under an automorphism of KG, using a new length function, is given by the next result.

Proposition 7.5. *Let KG be an (X, Y, N)-group algebra with W the corresponding set of words in $\{x, y\}$ and $L : KG \to \mathbb{N} \cup \{-\infty\}$ the induced length function. Let $\phi : G \to G$ be an isomorphism extending K-linearly to an isomorphism $\phi : KG \to KG$. Then KG is a $(\phi(X), \phi(Y), \phi(N))$-group algebra with $\phi(W)$ the corresponding set of words in $\{\phi(x), \phi(y)\}$ and with $\phi L : KG \to \mathbb{N} \cup \{-\infty\}$ the induced length function. Moreover for any $\alpha \in KG$ we have $(\phi L)(\alpha) = L(\phi(\alpha))$.*

This leads us to the following result which we will use in analysing the Promislow set.

Proposition 7.6. *The map sending $x \mapsto xy$ and $y \mapsto y$ extends to an automorphism*

$$\phi : \Gamma \to \Gamma$$

and yields a length function

$$\phi L : \Gamma \to \mathbb{N} \cup \{-\infty\},$$

induced from $\phi(W)$, the set of all words in $\phi(x) = xy$ and $\phi(y) = y$.

Proof. This follows immediately from the fact that $\Gamma = \langle xy, y\rangle$, with xy, y satisfying the defining relations, and noting that $N = \langle x^2, y^2\rangle = \langle (xy)^2, y^2\rangle$. $\square$

8. Structure theorems in $K\Gamma$

In this section we summarise results of the previous sections applied to $K\Gamma$.

Theorem 8.1. *Let $N = \langle a, b\rangle$ and $H = \langle a, b, c\rangle$. Then $K\Gamma$ is a virtually abelian (X, Y, N)-group algebra, with corresponding abelian subgroup H containing N and with corresponding set of words W in x and y.*

Proof. By Theorem 7.2, Γ is an (X, Y, N)-group with $X = \langle x, y^2\rangle$ and $Y = \langle x^2, y\rangle$. The groups X and Y are poly-infinite cyclic, and therefore right-orderable by [13, Lemma 13.1.6]. Thus KX and KY are domains by [13, Lemma 13.1.11], and it remains to show that KN is an Ore domain. To this end we note that KN is a domain since $N \subset X \cap Y$, and that KN is an Ore ring by Theorem 7.3 and [13, Lemma 13.3.6(iii)]. $\square$

Theorem 8.2. *The group algebra $K\Gamma$ is a domain (and hence all invertible elements have two-sided inverses).*

Proof. This follows from Theorem 4.4. $\square$

Theorem 8.3 *(Left-splitting: strong form). Let $N = \langle a, b\rangle$, and let σ, τ be non-zero elements of $K\Gamma$ with $\sigma\tau = \zeta \in KN$. Then there exists a splitting $\Lambda_1 \dots \Lambda_s$ such that*

$$(\Lambda_1 \dots \Lambda_s)\tau = \eta\zeta$$

for some $\eta \in KN \setminus \{0\}$. For any such splitting we have

$$L(\sigma) = L(\Lambda_1 \dots \Lambda_s).$$

If $L(\sigma) \geqslant 1$ then we may choose the above splitting to satisfy

$$L(\sigma) = s = L(\Lambda_1 \dots \Lambda_s).$$

Proof. This follows immediately from Theorem 4.7. □

Theorem 8.4 *(Left-reduced split-form). Assume σ is a unit of L-length $s \geqslant 1$ in $K\Gamma$. Then σ has a left-reduced split-form*

$$\varepsilon\sigma = (\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$$

with $\varepsilon \in KN \setminus \{0\}$, $(\alpha_1 + \beta_1 u_1) \dots (\alpha_s + \beta_s u_s)$ reduced and such that $\gcd(\alpha_i, \beta_i) = 1$ for all $i = 1, \dots, s$.

Proof. The subgroup N is finitely generated abelian. The result now follows by Theorem 5.4. □

Theorem 8.5 *(Determinant condition). Let $H = \langle a, b, c\rangle$ be the rank-3 free abelian subgroup of index 4 in Γ. Let X be any right transversal for H in Γ and let*

$$\rho_X : K\Gamma \to M_4(KH)$$

be the induced regular embedding. Let $\sigma \in K\Gamma$. Then σ is a unit of $K\Gamma$ if and only if $\det \rho_X(\sigma) \in K \setminus \{0\}$.

Proof. This follows immediately from Theorem 6.8, noting that the centre $Z(\Gamma) = \{1\}$. □

By Theorem 7.2, $H = \langle a, b, c\rangle$ is a rank-3 free abelian subgroup of Γ of index 4 with $\{1, x, y, xy\}$ a right transversal for H in Γ. We can therefore write any element σ of the group algebra $K\Gamma$ as a sum

$$\sigma = A + Bx + Cy + Dz,$$

where A, B, C and D are elements of the subalgebra KH.

Theorem 8.6. *For any $\sigma \in K\Gamma$, the map given by*

$$\sigma \mapsto \begin{pmatrix} A & B & C & D \\ B^x a & A^x & D^x a & C^x \\ C^y b & D^y a^{-1} c^{-1} & A^y & B^y a^{-1} b c^{-1} \\ D^z c & C^z b^{-1} & B^z b^{-1} c & A^z \end{pmatrix}$$

is the regular embedding

$$\theta : K\Gamma \to M_4(KH)$$

in the basis $X = \{1, x, y, xy\}$.

Proof. We fix an ordering of the basis as $x_1 = 1$, $x_2 = x$, $x_3 = y$, $x_4 = xy$. By Theorem 6.1,

$$\theta(\sigma) = \left[\pi_H(x_i \sigma x_j)\right],$$

and in particular for any $\alpha \in KN$

$$\theta(\alpha) = \begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha^x & 0 & 0 \\ 0 & 0 & \alpha^y & 0 \\ 0 & 0 & 0 & \alpha^{xy} \end{pmatrix}.$$

Since

$$\theta(A + Bx + Cy + Dxy) = \theta(A + Bx) + \theta(C + Dx)\theta(y),$$

it is enough to find $\theta(x)$ and $\theta(y)$.

Since $\theta(x) = [\pi_H(x_i x x_j^{-1})]$, we have:

(i) $1xx_j^{-1} \in H$ if and only if $x_j = x$, so $\pi_H(1xx^{-1}) = 1$;
(ii) $xxx_j^{-1} \in H$ if and only if $x_j = 1$, so $\pi_H(xx1^{-1}) = a$;
(iii) $yxx_j^{-1} \in H$ if and only if $x_j = xy$, since $yxy^{-1}x^{-1} = yxyy^{-2}xx^{-2} = yxyxx^{-2}y^2 = c^{-1}a^{-1}b$, so $\pi_H(yx(xy)^{-1}) = a^{-1}bc^{-1}$;
(iv) $xyxx_j^{-1} \in H$ if and only if $x_j = y$, since $xyxy^{-1} = xyxyy^{-2} = cb^{-1}$, so $\pi_H(xyxy^{-1}) = b^{-1}c$.

From $\theta(y) = [\pi_H(x_i y x_j^{-1})]$, we have:

(i) $1yx_j^{-1} \in H$ if and only if $x_j = y$, so $\pi_H(1yy^{-1}) = 1$;
(ii) $xyx_j^{-1} \in H$ if and only if $x_j = xy$, so $\pi_H(xy(xy)^{-1}) = 1$;
(iii) $yyx_j^{-1} \in H$ if and only if $x_j = 1$, so $\pi_H(yy1^{-1}) = b$;
(iv) $xyyx_j^{-1} \in H$ if and only if $x_j = x$, since $xyyx^{-1} = b^{-1}$, so $\pi(xyyx^{-1}) = b^{-1}$.

Thus we have

$$\theta(x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & 0 & 0 & a^{-1}bc^{-1} \\ 0 & 0 & b^{-1}c & 0 \end{pmatrix} \quad \text{and} \quad \theta(y) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \end{pmatrix}.$$

Hence

$$\theta(A + Bx) = \theta(A) + \theta(B)\theta(x) = \begin{pmatrix} A & B & 0 & 0 \\ B^x a & A^x & 0 & 0 \\ 0 & 0 & A^y & B^y a^{-1}bc^{-1} \\ 0 & 0 & B^z b^{-1}c & A^z \end{pmatrix},$$

so

$$\begin{aligned} \theta(C + Dx)\theta(y) &= \begin{pmatrix} C & D & 0 & 0 \\ D^x a & C^x & 0 & 0 \\ 0 & 0 & C^y & D^y a^{-1}bc^{-1} \\ 0 & 0 & D^z b^{-1}c & C^z \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ b & 0 & 0 & 0 \\ 0 & b^{-1} & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & C & D \\ 0 & 0 & D^x a & C^x \\ C^y b & D^y a^{-1}c^{-1} & 0 & 0 \\ D^z c & C^z b^{-1} & 0 & 0 \end{pmatrix}, \end{aligned}$$

and the result follows. $\square$

This embedding extends naturally to $T^{-1}K\Gamma$.

Theorem 8.7. *Let $N=\langle a,b\rangle$, and let $T=KN\setminus\{0\}$. Let $H=\langle a,b,c\rangle$ be the rank-3 free abelian subgroup of index 4 in Γ, and let $X=\{1,x,y,xy\}$ be the right transversal for H in Γ. Then for any $\eta\in KN\setminus\{0\}$ and $\sigma=A+Bx+Cy+Dxy$, the map*

$$\theta\left(\eta^{-1}\sigma\right)=\begin{pmatrix}\eta^{-1}&0&0&0\\0&(\eta^x)^{-1}&0&0\\0&0&(\eta^y)^{-1}&0\\0&0&0&(\eta^z)^{-1}\end{pmatrix}\begin{pmatrix}A&B&C&D\\B^xa&A^x&D^xa&C^x\\C^yb&D^ya^{-1}c^{-1}&A^y&B^ya^{-1}bc^{-1}\\D^zc&C^zb^{-1}&B^zb^{-1}c&A^z\end{pmatrix}$$

is the extension of the regular embedding

$$\theta:K\Gamma\to M_4(KH)$$

to an embedding

$$\theta:T^{-1}K\Gamma\to M_n\left(T^{-1}KH\right)$$

such that

$$\theta\left(T^{-1}\right)=\left(\theta(T)\right)^{-1}.$$

Proof. By Theorem 6.9 it is enough to evaluate $\theta(\eta)$. By Theorem 8.6 we have

$$\begin{pmatrix}\eta&0&0&0\\0&\eta^x&0&0\\0&0&\eta^y&0\\0&0&0&\eta^z\end{pmatrix},$$

and the result follows. □

9. Special terms

We stay with the notation and conventions of Section 7. In particular we recall that $N=\langle a,b\rangle$ is a normal, free rank-2 abelian subgroup of Γ and that KN is a Laurent polynomial ring in the commuting variables a and b. We begin with two results whose proofs are straightforward and safely left to the reader.

Proposition 9.1. *Let $\alpha+\beta x\in K\Gamma$ be a linear term, and let $D=\alpha\alpha^x-\beta\beta^x a\in KN$. Then*

(i) $D^x=D$;
(ii) $D^y=D^z=\alpha^y\alpha^z-\beta^y\beta^z a^{-1}$;
(iii) DD^y *is central in* $K\Gamma$.

Proposition 9.2. *Let $\alpha+\beta y\in K\Gamma$ be a linear term, and let $D=\alpha\alpha^y-\beta\beta^y b\in KN$. Then*

(i) $D=D^y$;
(ii) $D^x=D^z=\alpha^x\alpha^z-\beta^x\beta^z b^{-1}$;
(iii) DD^x *is central in* $K\Gamma$.

Let $D_1 = \alpha\alpha^x - \beta\beta^x a$ and $D_2 = \alpha\alpha^y - \beta\beta^y b$. These terms arise in an essential way.

Proposition 9.3. *Let $\theta : K\Gamma \to M_4(KH)$ be the regular embedding. Then*

(i) $\det\theta(\alpha + \beta x) = (\alpha\alpha^x - \beta\beta^x a)(\alpha^y\alpha^z - \beta^y\beta^z a^{-1}) = D_1 D_1^y$;
(ii) $\det\theta(\alpha + \beta y) = (\alpha\alpha^y - \beta\beta^y b)(\alpha^x\alpha^z - \beta^x\beta^z b^{-1}) = D_2 D_2^x$.

Proof. Applying Theorem 8.6 we have

$$\det\theta(\alpha + \beta x) = \det\begin{pmatrix} \alpha & \beta & 0 & 0 \\ \beta^x a & \alpha^x & 0 & 0 \\ 0 & 0 & \alpha^y & \beta^y a^{-1} b c^{-1} \\ 0 & 0 & \beta^z b^{-1} c & \alpha^z \end{pmatrix} = D_1 D_1^y$$

and

$$\det\theta(\alpha + \beta y) = \det\begin{pmatrix} \alpha & 0 & \beta & 0 \\ 0 & \alpha^x & 0 & \beta^x \\ \beta^y b & 0 & \alpha^y & 0 \\ 0 & \beta^z b^{-1} & 0 & \alpha^z \end{pmatrix} = D_2 D_2^x.$$

Of course the second determinant is simply $\det(\psi(\psi^{-1}(\alpha) + \psi^{-1}(\beta)x))$, where ψ is the automorphism on $K\Gamma$ induced K-linearly by interchanging x with y. $\square$

The following is of independent interest, and provides a description for the inverse of a unit σ of $K\Gamma$ in terms of any reduced split-form $(\varepsilon, \bar{\sigma})$ for σ.

Theorem 9.4. *Let $T = KN \setminus \{0\} \subset K\Gamma$. Assume σ is a unit of L-length $s \geqslant 1$ in $K\Gamma$ and that*

$$\eta\sigma = (\alpha_1 + \beta_1 u_1)\dots(\alpha_s + \beta_s u_s)$$

is a split-form for σ. Then in $T^{-1}K\Gamma$ we have

$$\sigma^{-1} = \left(\alpha_s^{u_s} - \beta_s u_s\right) D_s^{-1} \dots \left(\alpha_1^{u_1} - \beta_1 u_1\right) D_1^{-1} \eta$$

where

$$D_i = \left(\alpha_i \alpha_i^{u_i} - \beta_i \beta_i^{u_i} u_i^2\right)$$

and $u_i \in \{x, y\}$ for all $i = 1, \dots, s$.

Proof. In $T^{-1}K\Gamma = K\Gamma T^{-1}$ we write

$$\sigma = \eta^{-1}(\alpha_1 + \beta_1 u_1)\dots(\alpha_s + \beta_s u_s).$$

For any $u_i \in \{x, y\}$ we have

$$(\alpha_i + \beta_i u_i)\left(\alpha_i^{u_i} - \beta_i u_i\right) = D_i,$$

and the result follows by induction. $\square$

The foregoing also shows directly that invertible elements in $K\Gamma$ have two-sided inverses.

We conclude this section with an important result. Before doing so, however, we provide the following elementary result concerning Laurent polynomials.

Proposition 9.5. *Let K be an infinite field, and let $f(X,Y) \in K[X, X^{-1}, Y, Y^{-1}]$ be a Laurent polynomial in two variables. If $f(k,Y) = 0$ for infinitely many $k \in K$, then $f(X,Y) = 0$.*

Proof. Write

$$f(X,Y) = \frac{p(X,Y)}{X^i Y^j}$$

for some polynomial $p(X,Y) \in K[X,Y]$. Then

$$p(X,Y) = p_n(X)Y^n + \cdots + p_1(X)Y^1 + p_0(X)$$

for some polynomials $p_i(X) \in K[X]$. If $p(k,Y) = 0$, then

$$p_n(k)Y^n + \cdots + p_1(k)Y^1 + p_0(k) = 0,$$

and therefore $p_i(k) = 0$ for $i = 1, \ldots, s$. Since no non-zero K-polynomial can have infinitely many roots in K, it follows that each $p_i(X) = 0$, and therefore $f(X,Y) = 0$. $\square$

Theorem 9.6. *Let α and β be elements of KN, and suppose that $\alpha\alpha^y - \beta\beta^y b$ is a unit. Then either $\alpha = 0$ or $\beta = 0$.*

Proof. We think of $\alpha\alpha^y - \beta\beta^y b$ as a Laurent polynomial in the commuting variables a and b. By extending K if necessary, we assume that K is infinite. If $\alpha\alpha^y - \beta\beta^y b$ is a unit in KN, then we may specialise a to be any element of $K \setminus \{0\}$ and the specialisation of $\alpha\alpha^y - \beta\beta^y b$ remains a unit. Hence specialising $a = k \in K \setminus \{0\}$ yields a non-zero Laurent polynomial $(\bar\alpha)^2 - (\bar\beta)^2 b = \lambda b^i$ for some $\lambda \in K \setminus \{0\}$. Suppose by way of contradiction that both $\bar\alpha$ and $\bar\beta$ are non-zero. Then $\bar\alpha^2$ and $\bar\beta^2 b$ are non-zero. Notice that the highest and lowest powers of b in $(\bar\alpha)^2$ are of even degree, and the highest and lowest powers of b in $(\bar\beta)^2 b$ are of odd degree. Assume

$$\max\deg_b \bar\alpha^2 = 2m < 2n+1 = \max\deg_b \bar\beta^2 b.$$

Then

$$\bar\alpha^2 - \bar\beta^2 b = \lambda b^{2n+1}$$

and therefore all powers of b appearing in $\bar\alpha^2$ cancel off with powers of b appearing in $\bar\beta^2 b$. Therefore $\bar\beta^2 b$ has at least two terms in powers of b so that

$$\min\deg_b \bar\beta^2 b < \max\deg_b \bar\beta^2 b.$$

Thus

$$\min\deg_b \bar\beta^2 b \leqslant \min\deg_b \bar\alpha^2.$$

But $\min\deg_b \bar\beta^2 b$ is odd and $\min\deg_b \bar\alpha^2$ is even so that

$$\min \deg_b \bar{\beta}^2 b < \min \deg_b \bar{\alpha}^2.$$

But this implies that $(\bar{\alpha})^2 - (\bar{\beta})^2 b$ contains at least two terms in powers of b, contradicting the fact that $(\bar{\alpha})^2 - (\bar{\beta})^2 b$ is a unit. Thus our assumption $2m < 2n+1$ is incorrect and therefore

$$\max \deg_b \bar{\beta}^2 b = 2n+1 < 2m = \max \deg_b \bar{\alpha}^2.$$

A symmetric argument as above, reversing the roles of $\bar{\beta}^2 b$ and $\bar{\alpha}^2$ under the assumption $2n+1 < 2m$, yields a similar contradiction. Thus for each specialisation $a = k \in K \setminus \{0\}$, either $\bar{\alpha} = 0$ or $\bar{\beta} = 0$. Since K is infinite, Proposition 9.5 applies to yield either $\alpha = 0$ or $\beta = 0$ as claimed. □

10. Units of L-length 0, 1 and 2 in $K\Gamma$

In this section, and the next, we analyse the structure of units in $K\Gamma$ of small L-length. We fix the notation and conventions of the previous sections. In particular we let $X = \langle x, y^2\rangle$ and $Y = \langle x^2, y\rangle$ be subgroups of Γ, each containing $N = \langle a, b\rangle$. We shall view $K\Gamma$ as an (X, Y, N)-group algebra. We fix W to be the set of all words in x and y, creating a transversal for N in Γ, and we write L for the induced length function on Γ. All splittings are W-splittings with respect to the foregoing conventions. We recall that KN is a Laurent polynomial ring in two variables and is therefore a unique-factorisation domain. For $\alpha, \beta \in KN$ we shall say that α *divides* β if α divides β in KN. In this case we write $\alpha \mid \beta$. If α does not divide β, then we write $\alpha \nmid \beta$. We write gcd to abbreviate *greatest common divisor*, denoting the greatest common divisor of $\alpha, \beta \in KN$, by $\gcd(\alpha, \beta)$ or simply by (α, β) when the context is clear.

Proposition 10.1. *Assume*

$$\Sigma = (\alpha_1 + \beta_1 u_1) \ldots (\alpha_s + \beta_s u_s)$$

is a splitting. If Σ is a unit of $K\Gamma$, then Σ is trivial.

Proof. We proceed by induction on the number of terms of Σ. If $s = 1$ then Σ lies in the group algebra of $\langle N, u_1\rangle$, which is poly-$\mathbb{Z}$, so satisfies the unit conjecture by [13, Theorem 13.1.11]. Assume $s > 1$ and that the result holds for all splittings with fewer than s terms. Then $\alpha_1 + \beta_1 u_1$ is a unit and therefore trivial. The result now follows noting that the splitting

$$(\alpha_2 + \beta_2 u_2) \ldots (\alpha_s + \beta_s u_s)$$

satisfies the inductive hypothesis and that the product of trivial units is a trivial unit. □

We shall refer to the following elementary result repeatedly.

Proposition 10.2. *Let $\sigma \in K\Gamma$, $\eta \in KN \setminus \{0\}$, and assume*

$$\eta\sigma = \sum_{w \in W} \lambda_w w$$

for some $\lambda_w \in KN$. Then η divides λ_w in KN for every $w \in W$.

Proof. Since $K\Gamma$ is a free left KN-module with basis W, we can write

$$\sigma = \sum_{w \in W} \mu_w w$$

for some $\mu_w \in KN$. Then

$$\sum_{w\in W} (\eta\mu_w)w = \sum_{w\in W} \lambda_w w,$$

and therefore $\eta\mu_w = \lambda_w$ for all $w \in W$ as desired. □

Proposition 10.3. *Assume σ is a unit of L-length $\geqslant 1$ in $K\Gamma$ with reduced split-form $\varepsilon\sigma = \bar{\sigma}$. Then ε is a unit in KN if and only if σ is trivial.*

Proof. If ε is a unit in KN, then ε is trivial by [11, Theorem 8.5.3] since N is torsion-free abelian. By Proposition 10.1, $\bar{\sigma}$ is a trivial unit, and therefore so is $\sigma = \varepsilon^{-1}\bar{\sigma}$. Conversely assume σ is a trivial unit of L-length $\geqslant 1$ in $K\Gamma$, and let $(\varepsilon, \bar{\sigma})$ be a reduced split-form for σ. Then

$$\varepsilon\sigma = \lambda_w w$$

for uniquely determined $\lambda_w \in KN \setminus \{0\}$. By Proposition 10.2, ε divides λ_w in KN, so that

$$\sigma = (\lambda_w/\varepsilon)w,$$

with λ_w/ε in KN. Since σ is a unit, λ_w/ε is a (trivial) unit of KN, and therefore we have $\gcd(0, \lambda/\varepsilon) = 1$. Hence the expression

$$\varepsilon\sigma = \bar{\sigma} = \varepsilon\bigl(0 + (\lambda_w/\varepsilon)w\bigr)$$

is a reduced split-form for σ, and it follows that ε is a unit in KN. □

We now proceed to analyse units.

Theorem 10.4. *If σ is a unit of L-length 0 in $K\Gamma$, then σ is trivial.*

Proof. Assume that σ is a unit in $K\Gamma$ such that $L(\sigma) = 0$. Then $\sigma \in KN$, and therefore is trivial by [11, Theorem 8.5.3] since N is torsion-free abelian. □

Theorem 10.5. *If σ is a unit of L-length 1 in $K\Gamma$, then σ is trivial.*

Proof. Assume that σ is a unit in $K\Gamma$ such that $L(\sigma) = 1$. By Theorem 5.2 we may assume that σ has unique maximal-length word of the form $w = x$. By Theorem 8.4, σ has a reduced split-form which we may assume looks like

$$\varepsilon\sigma = (\alpha_1 + \beta_1 x) = \bar{\sigma}$$

with $\varepsilon \in KN \setminus \{0\}$, $(\alpha_1 + \beta_1 x)$ reduced and such that $\gcd(\alpha_1, \beta_1) = 1$. By Proposition 10.2, $\varepsilon \mid \alpha_1$ and $\varepsilon \mid \beta_1$. Thus ε is a unit in KN, and the result follows by Proposition 10.3. □

Theorem 10.6. *If σ is a unit of L-length 2 in $K\Gamma$, then σ is trivial.*

Proof. Assume that σ is a unit in $K\Gamma$ such that $L(\sigma)=2$. By Theorem 5.2 we may assume that σ has unique maximal-length word of the form $w=xy$. By Theorem 8.4, σ has a reduced split-form which we may assume looks like

$$\varepsilon\sigma=(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 y)=\bar{\sigma}$$

with $\varepsilon\in KN\setminus\{0\}$, $(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 y)$ reduced and such that $\gcd(\alpha_i,\beta_i)=1$ for all $i=1,2$. Expanding out $(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 y)$ we get

$$\bar{\sigma}=\alpha_1\alpha_2+\alpha_1\beta_2 y+\beta_1\alpha_2^x x+\beta_1\beta_2^x xy.$$

By Proposition 10.2, ε divides

$$\mu=\gcd\bigl(\alpha_1\alpha_2,\alpha_1\beta_2,\beta_1\alpha_2^x,\beta_1\beta_2^x\bigr).$$

Assume p is a prime of KN such that $p\mid\mu$. Then $p\mid\alpha_1\alpha_2$, so either $p\mid\alpha_1$ or $p\mid\alpha_2$. Moreover $p\mid\mu$ implies $p\mid\beta_1\alpha_2^x$ and $p\mid\beta_1\beta_2^x$. As $\gcd(\alpha_2^x,\beta_2^x)=1$, we have $p\mid\beta_1$. Thus $p\nmid\alpha_1$ since $\gcd(\alpha_1,\beta_1)=1$ and therefore $p\mid\alpha_2$. But $p\mid\mu$ implies $p\mid\alpha_1\beta_2 y$, and therefore $p\mid\beta_2$. This contradicts $\gcd(\alpha_2,\beta_2)=1$. Thus $\mu=1$ and therefore ε is a unit in KN. The result now follows by Proposition 10.3. $\square$

We remark that the essential ingredient in the foregoing two proofs was to take the expression

$$(\alpha_1+\beta_1 u_1)\dots(\alpha_s+\beta_s u_s)=\sum\lambda_w w$$

with $\gcd(\alpha_i,\beta_i)=1$ for all $i=1,\dots,s$ and show that this implied $\gcd\{\lambda_w\}_{w\in W}=1$. Unfortunately such expressions involving reduced splittings of higher L-length do not yield similar conclusions as the following example illustrates. Thus a similar strategy in showing triviality of units will not work in general.

Example 10.7. Choose

$$\alpha_1=\alpha_2=\alpha_3=\beta_1=1,\qquad\beta_2=1-a,\qquad\beta_3=-a.$$

We have

$$(1+x)\bigl(1+(1-a)y\bigr)(1-ax)=(a-1)\bigl(a^{-1}xyx+a^{-1}yx-xy-y-x+(1+a)\bigr).$$

11. Units of L-length 3 in $K\Gamma$

We stay with the notation and conventions of the previous section.

Assume σ is a unit in $K\Gamma$ such that $L(\sigma)=3$. By Theorem 5.2 and Theorem 8.4, σ has a reduced split-form which we may assume looks like

$$\varepsilon\sigma=(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 y)(\alpha_3+\beta_3 x)=\bar{\sigma}$$

with $\varepsilon\in KN\setminus\{0\}$, $(\alpha_1+\beta_1 x)(\alpha_2+\beta_2 y)(\alpha_3+\beta_3 x)$ reduced and such that $\gcd(\alpha_i,\beta_i)=1$ for all $i=1,2,3$. Since $K\Gamma$ forms a free left KN-module with basis W we can express $\bar{\sigma}$ as

$$\bar{\sigma}=\sum\lambda_w w$$

for uniquely determined $\lambda_w\in KN$.

The following table records the coefficients λ_w in front of the words w when one expands out the product $\bar{\sigma}$.

Word $= w$	Coefficient $= \lambda_w$
xyx	$\beta_1\beta_2^x\beta_3^{yx}$
yx	$\alpha_1\beta_2\beta_3^y$
xy	$\beta_1\beta_2^x\alpha_3^{yx}$
y	$\alpha_1\beta_2\alpha_3^y$
x	$\alpha_1\alpha_2\beta_3+\beta_1\alpha_2^x\alpha_3^x$
1	$\alpha_1\alpha_2\alpha_3+\beta_1\alpha_2^x\beta_3^x x^2$

Proposition 11.1. *Let p be a prime of KN that divides each of the coefficients λ_w in the expression for $\bar{\sigma}$. We have that $p \mid \beta_2, \beta_2^x$, and $p \nmid \alpha_2, \alpha_2^x, \alpha_1, \beta_1$. In particular, if $p \mid \varepsilon$ then $p \mid \beta_2$ and $p \mid \beta_2^x$.*

Proof. We first recall that $\gcd(\alpha_i, \beta_i) = 1$ for $i = 1, 2, 3$, and now proceed in stages, reducing the problem one step at a time.

Step 1. *Either $p \mid \alpha_1$ or $p \mid \beta_2$, and either $p \mid \beta_1$ or $p \mid \beta_2^x$.* Considering the coefficients of yx and y, we see that p divides both $\alpha_1\beta_2\beta_3^y$ and $\alpha_1\beta_2\alpha_3^y$. As p cannot divide both α_3^y and β_3^y, we must have that either $p \mid \alpha_1$ or $p \mid \beta_2$. Similarly, considering the coefficients of xyx and xy, we see that p divides both $\beta_1\beta_2^x\beta_3^{yx}$ and $\beta_1\beta_2^x\alpha_3^{yx}$, so divides either β_1 or β_2^x, proving the claim.

Notice that since p cannot divide both α_1 and β_1, if $p \mid \alpha_1$ then $p \mid \beta_2^x$, and similarly if $p \mid \beta_1$ then $p \mid \beta_2$.

Step 2. *$p \nmid \alpha_1$, and so $p \mid \beta_2$.* Suppose that $p \mid \alpha_1$. Since this means that $p \mid \beta_2^x$, we must have that $p \nmid \alpha_2^x$. Considering the coefficients of x and 1, we see that p divides the first expression in both cases, and so $p \mid \beta_1\alpha_2^x\alpha_3^x, \beta_1\alpha_2^x\beta_3^x a$. This yields a contradiction, since $p \nmid \beta_1$ and $p \nmid \alpha_2^x$. Hence $p \nmid \alpha_1$, so by Step 1, $p \mid \beta_2$.

Step 3. *$p \nmid \beta_1$, and so $p \mid \beta_2^x$.* Suppose that $p \mid \beta_1$. Since this means that $p \mid \beta_2$, we must have that $p \nmid \alpha_2$. Considering the coefficients of x and 1, we see that p divides the second expression in both cases, and so $p \mid \alpha_1\alpha_2\beta_3, \alpha_1\alpha_2\alpha_3$. This yields a contradiction, since $p \nmid \alpha_1$ and $p \nmid \alpha_2$. Hence $p \nmid \beta_1$, so by Step 1, $p \mid \beta_2^x$. This completes the proof, since $p \nmid \alpha_2, \alpha_2^x$ now. $\square$

Theorem 11.2. *If σ is a unit of L-length 3 in $K\Gamma$, then σ is trivial.*

Proof. Assume that σ is a unit in $K\Gamma$ such that $L(\sigma) = 3$. By Theorem 5.2 we may assume that σ has unique maximal-length word of the form $w = xyx$. By Theorem 8.4, σ has a reduced split-form which we may assume looks like

$$\varepsilon\sigma = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)(\alpha_3 + \beta_3 x) = \bar{\sigma}$$

with $\varepsilon \in KN \setminus \{0\}$, $(\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)(\alpha_3 + \beta_3 x)$ reduced and such that $\gcd(\alpha_i, \beta_i) = 1$ for all $i = 1, 2, 3$. We will show that ε is a unit, and conclude by Proposition 10.3 that σ is trivial.

To this end we apply the regular embedding of Theorem 8.6

$$\theta : K\Gamma \to M_4(KH)$$

to $\varepsilon\sigma = \bar{\sigma}$ and compute determinants. Since $\varepsilon \in KN$ we have

$$\theta(\varepsilon)=\begin{pmatrix}\varepsilon & 0 & 0 & 0\\ 0 & \varepsilon^x & 0 & 0\\ 0 & 0 & \varepsilon^y & 0\\ 0 & 0 & 0 & \varepsilon^z\end{pmatrix}$$

and therefore

$$\det\theta(\varepsilon)=\varepsilon\varepsilon^x\varepsilon^y\varepsilon^z.$$

Since σ is a unit we have by Theorem 8.5

$$\det\theta(\sigma)=\lambda$$

for some $\lambda\in K\setminus\{0\}$. Finally let $D_1=\alpha_1\alpha_1^x-\beta_1\beta_1^x a$, $D_2=\alpha_2\alpha_2^y-\beta_2\beta_2^y b$ and let $D_3=\alpha_3\alpha_3^x-\beta_3\beta_3^x a$. Then by Proposition 9.3 we have

$$\det\theta(\bar{\sigma})=D_1D_1^yD_2D_2^xD_3D_3^y.$$

We therefore get

$$\varepsilon\varepsilon^x\varepsilon^y\varepsilon^z\lambda=D_1D_1^yD_2D_2^xD_3D_3^y.$$

Claim. $D_2=(\alpha_2,\beta_2^y)(\alpha_2^y,\beta_2)$.

Proof. Suppose $p\in KN$ is a prime divisor of D_2. Since KN is a unique-factorisation domain we have $p\mid\varepsilon$, $p\mid\varepsilon^x$, $p\mid\varepsilon^y$, or $p\mid\varepsilon^z$. We now use Proposition 11.1 in analysing the following four possibilities:

(i) If $p\mid\varepsilon$ then $p\mid\beta_2$ so $p\mid\alpha_2^y$ and therefore $p\mid(\alpha_2^y,\beta_2)$.
(ii) If $p\mid\varepsilon^x$ then $p^x\mid\varepsilon$, so $p^x\mid\beta_2^x$. Thus $p\mid\beta_2$, so $p\mid\alpha_2^y$ and $p\mid(\alpha_2^y,\beta_2)$.
(iii) If $p\mid\varepsilon^y$ then $p^y\mid D_2^y=D_2$ and $p^y\mid\varepsilon$ so $p^y\mid(\alpha_2^y,\beta_2)$ so $p\mid(\alpha_2,\beta_2^y)$.
(iv) If $p\mid\varepsilon^{xy}$ then $p^y\mid D_2^y=D_2$ and $p^y\mid\varepsilon^x$ so $p^y\mid(\alpha_2^y,\beta_2)$ so $p\mid(\alpha_2,\beta_2^y)$.

Conversely if $p\mid(\alpha_2^y,\beta_2)$ then $p\mid D_2$ and if $p\mid(\alpha_2,\beta_2^y)$ then $p\mid D_2$. Finally we observe if $p\mid(\alpha_2^y,\beta_2)$ then $p\mid\beta_2$ so $p\nmid\alpha_2$ as $(\alpha_2,\beta_2)=1$, and therefore $p\nmid(\alpha_2,\beta_2^y)$ so that $\gcd((\alpha_2^y,\beta_2),(\alpha_2,\beta_2^y))=1$. Thus

$$D_2=\bigl(\alpha_2,\beta_2^y\bigr)\bigl(\alpha_2^y,\beta_2\bigr)$$

as claimed. □

Now certainly

$$\begin{aligned}1&=\frac{\alpha_2\alpha_2^y}{D_2}-\frac{\beta_2\beta_2^y b}{D_2}\\&=\left(\frac{\alpha_2}{(\alpha_2,\beta_2^y)}\right)\left(\frac{\alpha_2^y}{(\alpha_2^y,\beta_2)}\right)-\left(\frac{\beta_2}{(\alpha_2^y,\beta_2)}\right)\left(\frac{\beta_2^y}{(\alpha_2,\beta_2^y)}\right)b.\end{aligned}$$

Since $(\alpha_2,\beta_2^y)^y=(\alpha_2^y,\beta_2)$, we can write the foregoing as

$$1=\left(\frac{\alpha_2}{(\alpha_2,\beta_2^y)}\right)\left(\frac{\alpha_2}{(\alpha_2,\beta_2^y)}\right)^y-\left(\frac{\beta_2}{(\alpha_2^y,\beta_2)}\right)\left(\frac{\beta_2}{(\alpha_2^y,\beta_2)}\right)^y b$$

with each parenthetic term defining an element in KN. Thus by Theorem 9.6 we have either

$$\frac{\alpha_2}{(\alpha_2,\beta_2^y)}=0$$

or

$$\frac{\beta_2}{(\alpha_2^y,\beta_2)}=0.$$

In the latter case, $\beta_2=0$ implies $L(\sigma)<3$ an impossibility. Thus $\alpha_2=0$. But

$$1=(\alpha_2,\beta_2)=(0,\beta_2)$$

implies β_2 is a unit of KN.

But any prime divisor of ε divides β_2 and therefore it follows that ε is a unit of KN, and the result follows by Proposition 10.3. □

We remark that the foregoing proof does not require the full use of Theorem 8.5, namely that $\det\theta(\sigma)=\lambda\in K\setminus\{0\}$. Rather, $\det\theta(\sigma)\in UKH$ is only needed in order to conclude that any prime divisor of D_2 divides $\varepsilon\varepsilon^x\varepsilon^y\varepsilon^z$. The specific use of $\det\theta(\sigma)\in K\setminus\{0\}$ is discussed in Section 15.

12. The Promislow set

In [16], Promislow found a fourteen-element subset $\mathscr{P}$ of the fours group Γ such that $\mathscr{P}\cdot\mathscr{P}$ has no unique product. It has been a long-standing question whether this subset can be the support of a non-trivial unit in $K\Gamma$ for some field K. Using the techniques developed in this paper we have the following.

Theorem 12.1. *There is no non-trivial unit of $K\Gamma$ whose support is a subset of $\mathscr{P}$. In particular, there is no unit in $K\Gamma$ whose support is $\mathscr{P}$.*

Proof. We will use freely the results of Proposition 7.1 to make our calculations in Γ. By [14, p. 393] we write the Promislow set explicitly as

$$\mathscr{P}=\mathscr{A}\cup\mathscr{B}x\cup\mathscr{C}y,$$

such that

$$\mathscr{A}=\{c,c^{-1}\},\qquad \mathscr{B}=\{1,a^{-1},a^{-1}b,b,a^{-1}c^{-1},c\},\qquad \mathscr{C}=\{1,a,b^{-1},b^{-1}c,c,ab^{-1}c\}.$$

Rewriting the elements in terms of x and y, expressing these terms using words in W yields

$$\mathscr{P}=\{xyxy,yxyx\}\cup\{x,x^{-2}x,x^{-2}y^2x,y^2x,yxy,xyxyx\}\cup\{y,x^2y,y^{-2}y,xyx,y^2xyx,x^2xyx\}.$$

We observe that

$$\max\{L(g)\}_{g\in\mathscr{P}}=5$$

with $xyxyx\in\mathscr{P}$ the unique element of $\mathscr{P}$ of L-length 5.

We now seek an automorphism $\phi : \Gamma \to \Gamma$ such that $\max\{L(\phi(g))\}_{g\in\mathscr{P}} \leqslant 3$ so that we may appeal to Theorems 10.4, 10.5, 10.6 and 11.2. By Proposition 7.5 L is equivariant under any automorphism $\phi : \Gamma \to \Gamma$; that is $(\phi L)(g) = L(\phi(g))$ for all $g \in \Gamma$. This property, together with an inspection of $\mathscr{P}$, leads us to consider the outer automorphism $\phi : \Gamma \to \Gamma$ of Proposition 7.6, sending $x \mapsto xy$ and $y \mapsto y$.

We now determine the image of $\mathscr{P}$ under the map ϕ, and do so via a direct calculation in order to more clearly indicate the effect of the automorphism $\phi : \Gamma \to \Gamma$ upon lengths of words from $\mathscr{P}$.

We begin by observing that $\phi(x^2) = xyxy$, $\phi(x^{-2}) = yxyx$, $\phi(y^2) = y^2$, and $\phi(y^{-2}) = y^{-2}$. We then have:

$$
\begin{aligned}
xyxy &\mapsto (xy)y(xy)y = xy^2xy^2 = x^2,\\
yxyx &\mapsto y(xy)y(xy) = yxy^2xy = x^{-2},\\
x &\mapsto xy,\\
x^{-2}x &\mapsto (yxyx)(xy) = yxyx^2y = x^2y^{-2}yx,\\
x^{-2}y^2x &\mapsto (yxyx)y^2(xy) = yxyxy^2xy = x^2yx,\\
y^2x &\mapsto y^2(xy) = y^2xy,\\
yxy &\mapsto y(xy)y = yxy^2 = y^{-2}yx,\\
xyxyx &\mapsto (xy)y(xy)y(xy) = xy^2xy^2xy = x^2xy,\\
y &\mapsto y,\\
x^2y &\mapsto (xyxy)y = xyxy^2 = y^2xyx,\\
y^{-2}y &\mapsto y^{-2}y,\\
xyx &\mapsto (xy)y(xy) = xy^2xy = x^2y^{-2}y,\\
y^2xyx &\mapsto y^2(xy)y(xy) = y^2xy^2xy = x^2y,\\
x^2xyx &\mapsto (xyxy)(xy)y(xy) = xyxyxy^2xy = x^2xyx.
\end{aligned}
$$

Thus the image of $\mathscr{P}$ under the automorphism ϕ becomes

$$\phi(\mathscr{P}) = \{x^2, x^{-2}, xy, x^2y^{-2}yx, x^2yx, y^2xy, y^{-2}yx, x^2xy, y, y^2xyx, y^{-2}y, x^2y^{-2}y, x^2y, x^2xyx\},$$

and by inspection we see that

$$\max\{L(\phi(g))\}_{g\in\mathscr{P}} = 3.$$

Extending $\phi : \Gamma \to \Gamma$ K-linearly to an automorphism $\phi : K\Gamma \to K\Gamma$, we see that any element of $K\Gamma$ with support a subset of $\phi(\mathscr{P})$ has L-length $\leqslant 3$, so by Theorems 10.4, 10.5, 10.6 and 11.2 is not a non-trivial unit of $K\Gamma$. The result now follows noting that $\phi^{-1} : K\Gamma \to K\Gamma$ is an automorphism of $K\Gamma$, such that $\phi^{-1}(\operatorname{Supp}\alpha) = \operatorname{Supp}\phi^{-1}(\alpha)$ for every $\alpha \in K\Gamma$. $\square$

13. The higher-length case: consistent chains

We stay with the notation and conventions of the previous sections. Unless otherwise specified, we assume that all splittings are W-splittings in x and y.

Let σ be a non-trivial unit in $K\Gamma$, with reduced split-form

$$\varepsilon\sigma = \bar{\sigma} = \sum_{w \in W} \lambda_w w.$$

By Proposition 10.2, ε must divide the coefficients λ_w. If $L(\sigma) = 3$, then by Proposition 11.1 all KN-primes dividing ε must divide β_2 and β_2^x. If, however, the L-length of σ is greater than 3, then there is no unique collection of conjugates of the α_i, β_j that a KN-prime dividing ε needs to divide. This leads to the notion of *consistent chains*.

Let $n \geqslant 1$ be an integer, and let $\Sigma_{n,x}$ denote a reduced splitting of L-length n starting with x. Then

$$\Sigma_{n,x} = (\alpha_1 + \beta_1 u_1)(\alpha_2 + \beta_2 u_2) \ldots (\alpha_n + \beta_n u_n),$$

with $u_1 = x$, $u_i \in \{x, y\}$, $L(u_1 \ldots u_n) = n$, and $\beta_1 \ldots \beta_n \neq 0$. Since $K\Gamma$ is a free left KN-module with basis W, we can write

$$\Sigma_{n,x} = \sum_{w \in W} \lambda_w w,$$

the sum taken over distinct $w \in W$, with $\lambda_w \neq 0$ uniquely determined. Let

$$\mathrm{Supp}_W(\Sigma_{n,x}) = \{w \in W \mid \lambda_w \neq 0\}$$

and

$$V_{n,x} = \bigl\{(w, \lambda_w) \bigm| w \in \mathrm{Supp}_W(\Sigma)\bigr\}.$$

We identify (w, λ_w) with λ_w so that

$$V_{n,x} = \bigl\{\lambda_w \bigm| w \in \mathrm{Supp}_W(\Sigma)\bigr\}$$

and observe that each λ_w is a k-fold sum, $1 \leqslant k \leqslant n$, of monomials with each monomial being the product of n conjugates α_i^u, β_j^v, for $u, v \in \{1, x, y, xy\}$, of various α_i, β_j arising from the expansion of the splitting $\Sigma_{n,x}$ above. We define a *term of* λ_w to be any such monomial, and we say that a term *contains* a conjugate of α_i or β_j if that conjugate appears as one of the n factors of the monomial. We let $S_{n,x}$ be the set of all conjugates of the α_i, β_j appearing in terms of the λ_w. Similar definitions for $\Sigma_{n,y}$, $V_{n,y}$, and $S_{n,y}$ exist by interchanging x with y in the above definitions.

Example 13.1. With $n = 3$, let

$$\Sigma_{3,x} = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)(\alpha_3 + \beta_3 x) = \sum \lambda_w w.$$

We then have the following chart:

Word $= w$	Coefficient $= \lambda_w$
xyx	$\beta_1 \beta_2^x \beta_3^{xy}$
yx	$\alpha_1 \beta_2 \beta_3^y$
xy	$\beta_1 \beta_2^x \alpha_3^{xy}$
y	$\alpha_1 \beta_2 \alpha_3^y$
x	$\alpha_1 \alpha_2 \beta_3 + \beta_1 \alpha_2^x \alpha_3^x$
1	$\alpha_1 \alpha_2 \alpha_3 + \beta_1 \alpha_2^x \beta_3^x x^2$

so that

$$S_{3,x} = \{\alpha_1, \alpha_2, \alpha_2^x, \alpha_3, \alpha_3^y, \alpha_3^{xy}, \beta_1, \beta_2, \beta_2^x, \beta_3^x, \beta_3^y, \beta_3^{xy}\}$$

and

$$V_{3,x} = \{\lambda_1, \lambda_x, \lambda_y, \lambda_{xy}, \lambda_{yx}, \lambda_{xyx}\}.$$

If $w = xyx$ then λ_{xyx} has only one term, namely the monomial $\beta_1\beta_2^x\beta_3^{xy}$, and this term contains β_1, β_2^x, and β_3^{xy}. If $w = x$ then λ_x has two terms, $\alpha_1\alpha_2\beta_3$ and $\alpha_2^x\alpha_3^x\beta_1$, and these terms contain α_1, α_2, β_3 and α_2^x, α_3^x, β_1 respectively. We observe that each monomial is the product of $n = 3$ conjugates of various α_i, β_j for $i, j = 1, 2, 3$ and that $S_{3,x}$ does not contain all conjugates α_i^u, β_j^v for all $u, v \in \{1, x, y, xy\}$.

We now have the following.

A *consistent chain for* $\Sigma_{n,x}$, or simply a *consistent chain*, is defined to be a set C of conjugates of α_i, β_j satisfying the following four conditions:

C1. $C \subset S_{n,x}$.
C2. If $v \in V_{n,x}$ has only one term then v contains precisely one element of C.
C3. If $v \in V_{n,x}$ has precisely $s \geqslant 2$ terms $t_1, \dots, t_s$ and t_i contains an element $c_i \in C$, for $i = 1, \dots, s-1$, then t_s contains precisely one element $c_s \in C$.
C4. For $u \in \{1, x, y, xy\}$, whenever α_i^u lies in C, β_i^u does not, and whenever β_i^u lies in C, α_i^u does not.

A similar definition of *consistent chain for* $\Sigma_{n,y}$ follows by interchanging x with y in the preceding.

To illustrate, we look at consistent chains for $\Sigma_{n,x}$, $n = 1, 2, 3$.

Proposition 13.2. *There are no consistent chains for* $\Sigma_{1,x}$.

Proof. By definition

$$\Sigma_{1,x} = (\alpha_1 + \beta_1 x)$$

so that

$$S_{1,x} = \{\alpha_1, \beta_1\}$$

and

$$V_{1,x} = \{\lambda_1, \lambda_x\} = \{\alpha_1, \beta_1\}.$$

If C is a consistent chain for $\Sigma_{1,x}$, then C must contain both α_1 and β_1 by **C2**, and this contradicts **C4**. $\square$

Proposition 13.3. *There are no consistent chains for* $\Sigma_{2,x}$.

Proof. By definition

$$\Sigma_{2,x} = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)$$

so that

$$\Sigma_{2,x} = \alpha_1\alpha_2 + \alpha_2^x\beta_1 x + \alpha_1\beta_2 y + \beta_1\beta_2^x xy.$$

Thus

$$S_{2,x} = \{\alpha_1, \alpha_2, \alpha_2^x, \beta_1, \beta_2, \beta_2^x\}$$

and

$$V_{2,x} = \{\lambda_1, \lambda_x, \lambda_y\lambda_{xy}\} = \{\alpha_1\alpha_2, \alpha_2^x\beta_1, \alpha_1\beta_2, \beta_1\beta_2^x\}.$$

Let C be a consistent chain for $\Sigma_{2,x}$. By **C2**, $\alpha_1\alpha_2 \in V_{2,x}$ implies $\alpha_1 \in C$ or $\alpha_2 \in C$. Assume $\alpha_1 \in C$. By **C2**, $\alpha_2^x\beta_1 \in V_{2,x}$ yields $\alpha_2^x \in C$ or $\beta_1 \in C$. By **C4** it then follows that β_1 is not in C and so $\alpha_2^x \in C$. But by **C2**, $\beta_1\beta_2^x \in V_{2,x}$ implies $\beta_1 \in C$ or $\beta_2^x \in C$. Either case is impossible by **C4**, and therefore α_1 does not lie in C, so that $\alpha_2 \in C$. By **C2**, $\alpha_1\beta_2 \in V_{2,x}$ implies $\beta_2 \in C$, an impossibility by **C4**. Hence there are no consistent chains for $\Sigma_{2,x}$. □

We now make an important remark. For any $n \geqslant 3$ there are too many consistent chains with which to reasonably work. To illustrate, we mention that there are 91 consistent chains for $\Sigma_{3,x}$. Fortunately, as the following result shows, we do not need to consider all consistent chains.

Proposition 13.4. *Every consistent chain for $\Sigma_{3,x}$ contains $\{\beta_2, \beta_2^x\}$.*

Proof. By Example 13.1 we have:

Word $= w$	Coefficient $= \lambda_w$
xyx	$\beta_1\beta_2^x\beta_3^{xy}$
yx	$\alpha_1\beta_2\beta_3^y$
xy	$\beta_1\beta_2^x\alpha_3^{xy}$
y	$\alpha_1\beta_2\alpha_3^y$
x	$\alpha_1\alpha_2\beta_3 + \beta_1\alpha_2^x\alpha_3^x$
1	$\alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2^x\beta_3^x x^2$

Let C be a consistent chain for $\Sigma_{3,x}$. We will show in a series of three steps that $\beta_2, \beta_2^x \in C$.

Step 1. *Either $\alpha_1 \in C$ or $\beta_2 \in C$, and either $\beta_1 \in C$ or $\beta_2^x \in C$.* Considering the coefficients $\alpha_1\beta_2\beta_3^y$ of yx and $\alpha_1\beta_2\alpha_3^y$ of y, and as both α_3^y and β_3^y cannot lie in C, we must have that either $\alpha_1 \in C$ or $\beta_2 \in C$. Similarly, considering the coefficients $\alpha_1\beta_2\beta_3^y$ of yx and $\beta_1\beta_2^x\beta_3^{xy}$ of xyx, we see that either $\beta_1 \in C$ or $\beta_2^x \in C$.

Notice that since α_1 and β_1 cannot lie in C, if $\alpha_1 \in C$ then $\beta_2^x \in C$, and similarly if $\beta_1 \in C$ then $\beta_2 \in C$.

Step 2. *α_1 does not lie in C, and so $\beta_2 \in C$.* Suppose by way of contradiction that $\alpha_1 \in C$. Since this means that $\beta_2^x \in C$, we must have that α_2^x does not lie in C. Considering the coefficients $\lambda_x = \alpha_1\alpha_2\beta_3 + \beta_1\alpha_2^x\alpha_3^x$ and $\lambda_1 = \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2^x\beta_3^x x^2$, we see that the first terms in both cases contain an element from C, and so the remaining terms, $\beta_1\alpha_2^x\alpha_3^x$ and $\beta_1\alpha_2^x\beta_3^x a$ of each, contain an element from C. This yields a contradiction, since β_1 and α_2^x do not lie in C. Hence α_1 does not lie in C, so by Step 1, $\beta_2 \in C$.

Step 3. *β_1 does not lie in C, and so $\beta_2^x \in C$.* Suppose by way of contradiction that $\beta_1 \in C$. Since this means that $\beta_2 \in C$, we must have that α_2 does not lie in C. Considering the coefficients $\lambda_x = \alpha_1\alpha_2\beta_3 + \beta_1\alpha_2^x\alpha_3^x$ and $\lambda_1 = \alpha_1\alpha_2\alpha_3 + \beta_1\alpha_2^x\beta_3^x x^2$, we see that the second terms in both cases contain an element from C, and so the first terms, $\alpha_1\alpha_2\beta_3$ and $\alpha_1\alpha_2\alpha_3$ of each, contain an element from C. This yields a contradiction, since α_1 and α_2 do not lie in C. Hence β_1 does not lie in C, so by Step 1, $\beta_2^x \in C$. □

The theory of consistent chains applies to the study of units in $K\Gamma$ in the obvious way. Indeed if C is a consistent chain for $\Sigma_{n,x}$, and if p is a KN-prime, we say that *p divides C*, denoted $p \mid C$, if p divides each element of C in KN. We then have the following:

Theorem 13.5. *Assume that σ is a unit of L-length $n \geqslant 3$ in $K\Gamma$ with unique maximal word beginning with x. If $(\varepsilon, \bar{\sigma})$ is a left-reduced split-form for σ and p is a KN-prime, then $p \mid \varepsilon$ implies $p \mid C$ for some consistent chain C for $\bar{\sigma}$.*

Proof. By definition of left-reduced split-form we have

$$\varepsilon\bar{\sigma} = (\alpha_1 + \beta_1 u_1) + \cdots + (\alpha_n + \beta_n u_n) = \sum_{w \in W} \lambda_w w$$

with $u_1 = x$, $\bar{\sigma}$ reduced of L-length $n \geqslant 3$, $\gcd(\alpha_i, \beta_i) = 1$, and for unique $\lambda_w \in KN \setminus \{0\}$. Assume p is a KN-prime such that $p \mid \varepsilon$. By Proposition 10.2 we have $p \mid \lambda_w$ for all $w \in W$. Define λ_w to be *p-admissible* if p divides each of the terms of λ_w. Observe that if p divides a term t_w in some λ_w, then by unique factorisation in KN, t_w contains a p-divisible conjugate of the form α_i^u or β_j^u, for some $u \in \{1, x, y, xy\}$.

Assume λ_w is p-admissible. For each term t of λ_w, select one p-divisible conjugate of α_i or β_j contained in t and label it γ_t. Define C_w to be the set of such γ_t, one for each term t of λ_w, and let

$$C = \bigcup C_w$$

where the union is indexed by $w \in W$ such that λ_w is p-admissible.

Then C is a consistent chain for the reduced splitting $\bar{\sigma}$. Indeed, we first observe that by construction, each member of C is divisible by p. Now certainly condition **C1** holds since the elements of C are conjugates α_i^u or β_j^u, $u \in \{1, x, y, xy\}$, contained in terms of various λ_w. If λ_w has precisely one term t_w then by construction there exists a conjugate α_i^u or β_j^u, divisible by p, contained in t_w and lying in C. Therefore **C2** holds. Now assume that λ_w has precisely $k \geqslant 2$ terms, $t_1, \ldots, t_k$. Suppose $t_1, \ldots, t_{k-1}$ contain some conjugate α_i^u or β_j^u (depending on the term) lying in C. Then because these conjugates are p-divisible, there exists, by construction, a conjugate lying in C that is p-divisible and contained in t_k. Thus **C3** holds. Finally if $\alpha_i^u \in C$ then $p \mid \alpha_i^u$, so $p \nmid \beta_i^u$, as $\gcd(\alpha_i^u, \beta_i^u) = 1$, and therefore β_i^u does not lie in C, as all elements of C are p-divisible. Thus **C4** holds, and C is a consistent chain. □

The foregoing result, together with Proposition 13.4, explain the essential ingredient in the proof of Theorem 11.2: If σ is a unit of L-length 3 in $K\Gamma$, with left-reduced split-form $(\varepsilon, \bar{\sigma})$ such that

$$\bar{\sigma} = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)(\alpha_3 + \beta_3 x),$$

then for any KN-prime p:

$$p \mid \varepsilon \quad \Longrightarrow \quad p \bigm| \{\beta_2, \beta_2^x\}.$$

For $n > 3$, one cannot expect a single set to lie in all consistent chains for $\Sigma_{n,x}$. Thus to generalise the role played by $\{\beta_2, \beta_2^x\}$, we seek a suitably nice collection of finite sets such that any consistent chain for $\Sigma_{n,x}$ contains a member from the collection. This motivates the idea of *minimal chains*, which we now develop, recursively.

Let $n \geqslant 3$, and let

$$\Sigma_{n,x} = (\alpha_1 + \beta_1 u_1)(\alpha_2 + \beta_2 u_2) \dots (\alpha_n + \beta_n u_n).$$

Define $U_{n,x}$ to be the set consisting of elements β_2, β_3^y, β_4^{xy}, β_5^x, β_6, repeating this sequence until arriving at the appropriate conjugate of β_{n-1}. Define $U_{n,y}$ to be the set obtained from $U_{n,x}$, with x interchanged by y. Let $M_{n,x}$ be the collection of all sets, called *minimal chains for* $\Sigma_{n,x}$, defined recursively by $M_{3,x} = \{\beta_2, \beta_2^x\}$, and for $n > 3$ as:

M1. Sets $\{\beta_1\} \cup C$ for $C \in M_{n-1,y}$ (with indices for α_i and β_i in C incremented by 1);
M2. Sets $\{\alpha_1\} \cup C^x$ for $C \in M_{n-1,y}$ (with indices for α_i and β_i in C incremented by 1);
M3. All pairs $\{\lambda, \mu^x\}$, with λ and μ appearing in the list $U_{n,x}$.

For $n > 3$, $M_{n-1,y}$ is obtained from $M_{n-1,x}$ by interchanging the roles of x and y.

A minimal chain for $\Sigma_{n,x}$ shall also be referred to as a *minimal chain from* $M_{n,x}$. The key result is the following:

Theorem 13.6. *Let $n \geqslant 3$ and let*

$$\Sigma_{n,x} = (\alpha_1 + \beta_1 u_1)(\alpha_2 + \beta_2 u_2) \dots (\alpha_n + \beta_n u_n).$$

Then every consistent chain for $\Sigma_{n,x}$ contains a minimal chain from $M_{n,x}$.

Proof. We proceed by induction on the L-length $n \geqslant 3$ of the reduced splitting $\Sigma_{n,x}$. The case $n = 3$ holds by Proposition 13.4. Assume $n > 3$. Let C be a consistent chain for $\Sigma_{n,x}$, and suppose first that $\beta_1 \in C$. We may remove all of the terms from $V_{n,x}$ that start with β_1 to get a set $V_{n,x}^*$, and by considering

$$\Sigma_{n,x} = (\alpha_1 + \beta_1 u_1)(\alpha_2 + \beta_2 u_2) \dots (\alpha_n + \beta_n u_n), \tag{1}$$

we clearly see that

$$V_{n,x}^* = \{\alpha_1 w \mid w \in V'_{n-1,y}\},$$

where the prime denotes incrementing the indices of the α_i and β_i by 1. Since $\alpha_1 \notin C$, we may remove the α_1 from the start of the words in $V_{n,x}^*$, and so $C \setminus \{\beta_1\}$ must be a consistent chain for the reduced splitting

$$\Sigma_{n-1,y} = (\alpha_2 + \beta_2 y) \dots (\alpha_n + \beta_n u_n)$$

(with indices shifted by 1), as C is a consistent chain for $\Sigma_{n,x}$. Since $\Sigma_{n-1,y}$ is reduced, its L-length is $n-1 < n$ and thus by induction (with x and y interchanged), this case is covered by **M1** in the theorem, so we may assume that β_1 does not lie in C.

Similarly, suppose that α_1 lies in C. In this case we may remove all of the terms from $V_{n,x}$ that start with α_1 to get a set $V_{n,x}^*$, and we see that

$$V_{n,x}^* = \{\beta_1 w^x \mid w \in V'_{n-1,y}\},$$

where the prime again denotes incrementing the indices of the α_i and β_i by 1. As above, the elements $C \setminus \{\alpha_1\}$ conjugated by x form a consistent chain for $\Sigma_{n-1,y}$ (with indices shifted by 1), and by induction this case is also covered by **M2** in the theorem. Hence we may assume that neither α_1 nor β_1 lie in C.

We now note that, when expanding (1), there are four elements of $V_{n,x}$ that are monomials, namely the coefficients of the words of lengths n, $n-1$, and the word of length $n-2$ starting in y: two of these words start with x, and two start with y. If a_1 and a_2 are the two monomial coefficients of the words starting in x, then

$$a_1 = \beta_1\beta_2^x\beta_3^{xy}\dots\beta_n^u, \qquad a_2 = \beta_1\beta_2^x\beta_3^{xy}\dots\alpha_n^u$$

(where $u \in \{1, x, y, xy\}$, and for the rest of the proof will also denote one of these four). Since a_1 and a_2 differ only in the last element, if C is a consistent chain then C must contain at least one of the terms β_i^u for $1 < i < n$. Similarly, if b_1 and b_2 denote the two monomial coefficients of the words starting in y, then

$$b_1 = \alpha_1\beta_2\beta_3^y\dots\beta_n^u, \qquad b_2 = \alpha_1\beta_2\beta_3^y\dots\alpha_n^u.$$

Again, b_1 and b_2 differ only in the last element, so if C is a consistent chain then C must contain at least one of the terms β_i^u for $1 < i < n$. It remains to note that the β_i^u, for $1 < i < n$, of the b_j constitute $U_{n,x}$, and the β_i^u, for $1 < i < n$, of the a_j are the elements of $U_{n,x}$ conjugated by x. Thus C contains $\{\lambda, \mu^x\}$, where $\lambda, \mu \in U_{n,x}$, as claimed by **M3** the theorem, and the result follows. □

As an application, we derive the sets in $M_{3,x}$ and $M_{4,x}$.

Proposition 13.7. *The only set in $M_{3,x}$ is*:

$$\{\beta_2, \beta_2^x\}.$$

Proof. Here $U_{3,x} = \{\beta_2\}$. Since $M_{2,y}$ is not defined, conditions **M1** and **M2** do not apply, and therefore the only minimal chains are those given by **M3**. These are of the form $\{\lambda, \mu^x\}$ for $\lambda, \mu \in U_{3,x}$, and the result follows. □

Proposition 13.8. *The sets in $M_{4,x}$ are*:

$$\{\beta_2, \beta_2^x\},\ \{\beta_2, \beta_3^{xy}\},\ \{\beta_2^x, \beta_3^y\},\ \{\beta_3^y, \beta_3^{xy}\},\ \{\beta_1, \beta_3, \beta_3^y\},\ \{\alpha_1, \beta_3^x, \beta_3^{xy}\}.$$

Proof. Here $U_{4,x} = \{\beta_2, \beta_3^y\}$ and $M_{2,y} = \{\beta_2, \beta_2^y\}$.

The sets given by **M1** are:

$$\{\beta_1, \beta_3, \beta_3^y\}.$$

The sets given by **M2** are:

$$\{\alpha_1, \beta_3^x, \beta_3^{xy}\}.$$

The sets given by **M3** are:

$$\{\beta_2, \beta_2^x\},\ \{\beta_2, \beta_3^{xy}\},\ \{\beta_3^y, \beta_3^{xy}\},\ \{\beta_3^y, \beta_2^x\}.$$

Several important remarks are now in order:

The minimal chain $\{\beta_2, \beta_2^x\}$ is in fact a consistent chain, meeting conditions **C1**–**C4**. A careful inspection shows, similarly, that every minimal chain from $M_{4,x}$ is also a consistent chain, meeting conditions of **C1**–**C4**. However, for $n = 5$, the minimal chain $\{\beta_2, \beta_2^x\}$ lies in $M_{5,x}$ but is not a consistent chain for $\Sigma_{5,x}$, and therefore is only a proper subset of some consistent chain. Thus the name *minimal chain* is derived from a slightly more subtle property. Loosely speaking a minimal chain is a subset M of a chain, having the property that each coefficient λ_w consisting of a single monomial contains only *one* member from M. The recursive procedure, given above, produces a suitable class of subsets of consistent chains with this property. Moreover by Theorems 13.5 and 13.6, it follows that if σ is a unit of L-length $n \geqslant 3$ in $K\Gamma$, with reduced split-form $(\varepsilon, \bar{\sigma})$, then for any KN-prime $p \mid \varepsilon$, there exists a minimal chain $M \in M_{n,x}$ such that $p \mid M$. From this we can now retrieve the key ingredient of the proof that there are no non-trivial units of L-length 3 in $K\Gamma$. Indeed we have the following:

Theorem 13.9. *Assume σ is a unit of L-length $n \geqslant 3$ in $K\Gamma$, with reduced split-form $(\varepsilon, \bar{\sigma})$. Assume that each minimal chain for $\bar{\sigma}$ contains an element that is relatively prime to ε in KN. Then ε is a unit in KN, and therefore σ is a trivial unit in $K\Gamma$.*

Proof. Assume by way of contradiction that ε is not a unit of KN. Then ε has a KN-prime factor p. By Theorems 13.5 and 13.6, there exists a minimal chain M such that $p \mid M$. By hypothesis M contains a unit of KN, and this yields the desired contradiction. The result now follows by Proposition 10.3. □

Upon closer inspection, the actual proof showing that there are no non-trivial units of L-length 3 in $K\Gamma$, is a special case of the following:

Corollary 13.10. *Assume σ is a unit of L-length $n \geqslant 3$ in $K\Gamma$, with reduced split-form $(\varepsilon, \bar{\sigma})$. Assume that each minimal chain for $\bar{\sigma}$ contains a unit of KN. Then ε is a unit in KN, and therefore σ is a trivial unit in $K\Gamma$.*

Proof. Immediate by Theorem 13.9. □

To apply Theorem 13.9 as a possible strategy in analysing units of L-length $n \geqslant 4$, it is convenient to recall by Proposition 13.8 that the minimal chains from $M_{4,x}$ are:

$$\{\beta_2, \beta_2^x\}, \ \{\beta_2, \beta_3^{xy}\}, \ \{\beta_2^x, \beta_3^y\}, \ \{\beta_3^y, \beta_3^{xy}\}, \ \{\beta_1, \beta_3, \beta_3^y\}, \ \{\alpha_1, \beta_3^x, \beta_3^{xy}\}.$$

If we assume that each minimal chain contains an element relatively prime to ε, then we see that we can eliminate from consideration those minimal chains consisting of three elements and replace them by certain subsets. This is because some minimal chains are related to subsets of other minimal chains by applying automorphisms.

Indeed, to see this assume $\phi : \Gamma \to \Gamma$ is an automorphism that is extended K-linearly to an automorphism $\phi : K\Gamma \to K\Gamma$. Let σ be a unit in $K\Gamma$ with left-reduced split-form $(\varepsilon, \bar{\sigma})$. Then it is easy to see that $(\varepsilon^\phi, \bar{\sigma}^\phi)$ is a left-reduced split-form for the unit σ^ϕ in $K\Gamma$, noting by Proposition 7.5 that $K\Gamma$ is a $(\phi(X), \phi(Y), \phi(N))$-group algebra with corresponding set of words $\phi(W)$.

Using automorphisms we can use suitably chosen subsets of minimal chains for $\Sigma_{n,x}$ to arrive at a *chain diagram* $\mathscr{C}_n$, separated into *components*. We illustrate chain diagrams $\mathscr{C}_2$, $\mathscr{C}_3$ in the specific cases $n = 3$ and $n = 4$, respectively.

For $n = 3$, the chain diagram $\mathscr{C}_3$ has only one *component* consisting of one *vertex* and no *edges*:

$$\{\beta_2, \beta_2^x\}.$$

To better understand the case $n = 4$, suppose

$$\Sigma_{3,x} = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)(\alpha_3 + \beta_3 x)(\alpha_4 + \beta_4 y).$$

Then by our remarks above, applying the automorphism interchanging x with y yields

$$\Sigma_{3,x}^{\psi} = \left(\alpha_1^{\psi} + \beta_1^{\psi} y\right)\left(\alpha_2^{\psi} + \beta_2^{\psi} x\right)\left(\alpha_3^{\psi} + \beta_3^{\psi} y\right)\left(\alpha_4^{\psi} + \beta_4^{\psi} x\right).$$

We think of this splitting expressed as

$$\Sigma_{3,y} = (\alpha_1 + \beta_1 y)(\alpha_2 + \beta_2 x)(\alpha_3 + \beta_3 y)(\alpha_4 + \beta_4 x)$$

for possibly new α_i, β_i, $i = 1, 2, 3, 4$. Therefore the automorphism $\psi : K\Gamma \to K\Gamma$ induces a map

$$M(\psi) : \left\{\beta_3^{y}, \beta_3^{xy}\right\} \to \left\{\beta_3^{x}, \beta_3^{xy}\right\}.$$

For $n = 4$, the chain diagram $\mathscr{C}_4$ has three *components*:

$$\begin{tikzcd}[arrows=dash]
 & & & \{\beta_3^{y}, \beta_3^{xy}\} \arrow[d, "M(\psi)"'] \arrow[dr, "M(\psi\circ c_y)"] & \\
\{\beta_2, \beta_2^{x}\} & \{\beta_2, \beta_3^{xy}\} \arrow[r, "M(c_x)"] & \{\beta_2^{x}, \beta_3^{y}\} & \{\beta_3^{x}, \beta_3^{xy}\} \arrow[r, "M(c_x)"] & \{\beta_3, \beta_3^{y}\}.
\end{tikzcd}$$

(Note: Not every set appearing in this diagram is a minimal chain for $\Sigma_{4,x}$. Moreover, the three-element minimal chains are not included since they contain sets in this diagram.)

We now proceed to make the foregoing precise. Let $\phi : K\Gamma \to K\Gamma$ be any of the four following automorphisms on Γ: the identity map id; the map ψ, which interchanges x with y; conjugation by x, denoted c_x; conjugation by y, denoted c_y. For any such automorphism, let $\phi : K\Gamma \to K\Gamma$ also denote its K-linear extension to an automorphism on $K\Gamma$. For any $\nu^u \in KN$ such that $u \in \{1, x, y, xy\}$, let

$$M(\phi)\left(\nu^u\right) = \nu^{\phi(u)}.$$

Specifically, then, we have

$$M(\mathrm{id})\left(\nu^u\right) = \nu^u, \qquad M(\psi)\left(\nu^u\right) = \nu^{\psi(u)}, \qquad M(c_x)\left(\nu^u\right) = \nu^{xu}, \qquad M(c_y)\left(\nu^u\right) = \nu^{yu}.$$

We note that for any $\phi_1, \phi_2 \in \{\mathrm{id}, \psi, c_x, c_y\}$ we have

$$M(\phi_1 \circ \phi_2) = M(\phi_1) \circ M(\phi_2).$$

Let $\Phi = \langle \psi, x, y \rangle$ be the group of $K\Gamma$-automorphisms generated by ψ, x, y.

Define for any minimal chain $C \in M_{n,x}$:

$$M(\phi)(C) = \left\{ M(\phi)(\nu) \,\middle|\, \nu \in C \right\}.$$

Let $\mathscr{M}_n$ be the set of subsets $D_0 \subset D \in M_{n,x}$ such that

$$M(\phi)(C) = D_0$$

for some minimal chain $C \in M_{n,x}$ and for some automorphism $\phi \in \Phi$.

It is easy to see that the set $\mathscr{M}_n$ forms a lattice [1] under set-theoretic inclusion. An element $A \in \mathscr{M}_n$ is said to be an *atom*, in the usual way, if for any $B \in \mathscr{M}_n$,

$$B \subset A \quad \Longrightarrow \quad B = A.$$

The *chain diagram* $\mathscr{C}_n$, $n \geqslant 3$, is the non-directed simple graph whose vertices are labelled by the atoms of $\mathscr{M}_n$, with an (unlabelled) edge between vertices A and B if

$$M(\phi)(A) = B$$

for some $\phi \in \Phi$. (We leave out loops given by the identity id and as a matter of convenience we sometimes label an edge by a *connecting* automorphism, though technically speaking the label is not part of the graph.) A *component* of $\mathscr{C}_n$ is as a component of the non-directed simple graph.

Thus the chain diagram $\mathscr{C}_4$ is:

$$\begin{tikzcd}
 & & & \{\beta_3^y, \beta_3^{xy}\} \arrow[d, dash] \arrow[dr, dash] & \\
\{\beta_2, \beta_2^x\} & \{\beta_2, \beta_3^{xy}\} \arrow[r, dash] & \{\beta_2^x, \beta_3^y\} & \{\beta_3^x, \beta_3^{xy}\} \arrow[r, dash] & \{\beta_3, \beta_3^y\}
\end{tikzcd}$$

though we may view it as:

$$\begin{tikzcd}
 & & & \{\beta_3^y, \beta_3^{xy}\} \arrow[d, dash, "M(\psi)"'] \arrow[dr, dash, "M(\psi \circ c_y)"] & \\
\{\beta_2, \beta_2^x\} & \{\beta_2, \beta_3^{xy}\} \arrow[r, dash, "M(c_x)"] & \{\beta_2^x, \beta_3^y\} & \{\beta_3^x, \beta_3^{xy}\} \arrow[r, dash, "M(c_x)"] & \{\beta_3, \beta_3^y\}
\end{tikzcd}$$

keeping in mind that there may be more than one way to label a particular edge (for example in $\mathscr{C}_4$, the label $M(\psi \circ c_y)$ can be replaced by the label $M(c_y \circ \psi)$).

The definition of a chain diagram $\mathscr{C}_n$ is purely formal and does not depend on any specific reduced splitting $\Sigma_{n,x}$ in the same way that the recursive definition of minimal chain is purely formal and provides a formula for deriving minimal chains for $\Sigma_{n,x}$. Thus we may think of a chain diagram $\mathscr{C}_n$ as the graph consisting of formal symbols involving *conjugates* of the *symbols* α_i, β_j. It is clear then that two chain diagrams $\mathscr{C}_m$ and $\mathscr{C}_n$ are isomorphic as simple unlabelled graphs if and only if $m = n$. To each vertex C of $\mathscr{C}_n$ and each reduced splitting $\Sigma_{n,x}$, we may associate a set $C^{\Sigma_{n,x}} \subset KN$, simply by replacing each formal element of C by its corresponding element in KN, labelled within $\Sigma_{n,x}$. We refer to $C^{\Sigma_{n,x}}$ as the *evaluation of* C *in* $\Sigma_{n,x}$ and denote it simply by C if the context is clear. It is then easy to show that the evaluation of every vertex is a subset of a minimal chain for $\Sigma_{n,x}$, and conversely that every minimal chain for $\Sigma_{n,x}$ contains a subset that is the evaluation of some vertex from the chain diagram $\mathscr{C}_n$. Finally we remark that an automorphism $\phi : \Gamma \to \Gamma$, extending K-linearly to an automorphism $\phi : K\Gamma \to K\Gamma$, sends a reduced splitting $\Sigma_{n,x}$ to the reduced splitting $\phi(\Sigma_{n,x})$, and in so doing yields a chain diagram $\mathscr{C}_n^{\phi}$ for this splitting. It is then clear that the map ϕ induces a natural graph-isomorphism

$$\mathscr{C}_n \approx \mathscr{C}_n^{\phi}.$$

The concept of a chain diagram reduces the number of minimal chains to be considered and suggests that the action of automorphisms on reduced split-forms may prove useful in analysing the structure of units in $K\Gamma$. For example one possible strategy is the following.

If p is a KN-prime and C a subset of a minimal chain for some reduced splitting of L-length n, then we say that p *divides* C, denoted $p \mid C$, if p divides each member of C in KN. We denote by C^{ϕ} the image of C under ϕ. Given an automorphism $\phi \in \Phi$, and a vertex C within a chain diagram $\mathscr{C}_n$, we say that a KN-prime p *divides* $M(\phi)(C)$, denoted $p \mid M(\phi)(C)$, if there exists a reduced splitting with evaluation C such that $p \mid C^{\phi}$.

Theorem 13.11. *Suppose $\mathscr{C}_n$, $n \geqslant 3$, is a chain diagram, and let $C_1, \dots, C_r$ be vertices, one from each component of $\mathscr{C}_n$. Then all units of L-length $n \geqslant 3$ in $K\Gamma$ are trivial if and only if no prime divisor of ε divides the evaluations $C_1, \dots, C_r$ in $\bar{\sigma}$, for any left-reduced split-form $(\varepsilon, \bar{\sigma})$ of any unit σ of L-length $n \geqslant 3$ in $K\Gamma$.*

Proof. If σ is a trivial unit of L-length $n \geqslant 3$ in $K\Gamma$, with left-reduced split-form $(\varepsilon, \bar{\sigma})$, then ε is a unit of KN by Proposition 10.3. Thus if all units of L-length $n \geqslant 3$ in $K\Gamma$ are trivial then it is vacuously true that no prime divisor of ε divides the evaluations $C_1, \dots, C_r$ in $\bar{\sigma}$, for any left-reduced split-form $(\varepsilon, \bar{\sigma})$ of any unit σ of L-length $n \geqslant 3$ in $K\Gamma$. Conversely, suppose by way of contradiction that there exists a non-trivial unit σ of L-length $n \geqslant 3$ in $K\Gamma$. Let $(\varepsilon, \bar{\sigma})$ be a left-reduced split-form for σ. By Proposition 10.3, ε is a non-unit of KN, and therefore has a KN-prime factor p. By Theorem 13.5, there exists a minimal chain C for $\bar{\sigma}$ such that $p \mid C$. The minimal chain C contains a subset D_0 which is the evaluation of a vertex (though possibly of a non-uniquely defined vertex). Select one such vertex. Then this vertex, denoted D_0, lies in the same component as some vertex C_i. Thus $M(\phi)(D_0) = C_i$, for some automorphism $\phi \in \Phi$. By our remarks above, σ^{ϕ} is a unit of L-length n in $K\Gamma$, with reduced split-form $(\varepsilon^{\phi}, \bar{\sigma}^{\phi})$. The KN-prime p^{ϕ} divides ε^{ϕ}, and, moreover, $p^{\phi} \mid C^{\phi}$ so $p^{\phi} \mid D_0^{\phi}$. Hence, by definition, the KN-prime p^{ϕ} divides the vertex $M(\phi)(D_0) = C_i$, and this contradicts our assumption. Thus all units of L-length n in $K\Gamma$ are trivial as desired. □

Along the same lines of reasoning we can use chain diagrams to reduce the number of minimal chains to be considered in Theorem 13.9 and Corollary 13.10.

Theorem 13.12. *Suppose $\mathscr{C}_n$, $n \geqslant 3$, is a chain diagram, and let $C_1, \dots, C_r$ be vertices, one from each component of $\mathscr{C}_n$. Assume that σ is a unit of L-length $n \geqslant 3$ in $K\Gamma$, with left-reduced split-form $(\varepsilon, \bar{\sigma})$. Assume that each of the evaluations $C_1, \dots, C_r$ in $\bar{\sigma}$ contains an element that is relatively prime to ε in KN. Then ε is a unit in KN, and therefore σ is a trivial unit in $K\Gamma$.*

Proof. We will show that ε is a unit of KN and therefore conclude by Proposition 10.3 that σ is a trivial unit of $K\Gamma$. If p is a KN-prime dividing ε, then by Theorem 13.5, there exists a minimal chain C for $\bar{\sigma}$ such that $p \mid C$. The minimal chain C contains a subset D_0 which is the evaluation of a vertex (though possibly of a non-uniquely defined vertex). Select one such vertex. Then this vertex, denoted D_0, lies in the same component as some vertex C_i. Thus $M(\phi)(D_0) = C_i$, for some automorphism $\phi \in \Phi$. By our remarks above, σ^{ϕ} is a unit of L-length n in $K\Gamma$, with reduced split-form $(\varepsilon^{\phi}, \bar{\sigma}^{\phi})$. The KN-prime p^{ϕ} divides ε^{ϕ}, and, moreover, $p^{\phi} \mid C^{\phi}$ so $p^{\phi} \mid D_0^{\phi}$. Thus p^{ϕ} divides the evaluation C_i^{ϕ}. Since the evaluation C_i contains an element that is relatively prime to ε in KN, it follows that C_i^{ϕ} contains an element relatively prime to ε^{ϕ} in KN, and this yields the desired contradiction. □

Corollary 13.13. *Suppose $\mathscr{C}_n$, $n \geqslant 3$, is a chain diagram, and let $C_1, \dots, C_r$ be vertices, one from each component of $\mathscr{C}_n$. Assume that σ is a unit of L-length $n \geqslant 3$ in $K\Gamma$, with left-reduced split-form $(\varepsilon, \bar{\sigma})$. Assume that each of the evaluations $C_1, \dots, C_r$ in $\bar{\sigma}$ contains a unit in KN. Then ε is a unit in KN, and therefore σ is a trivial unit in $K\Gamma$.*

Proof. Immediate by Theorem 13.12. □

The foregoing result leads to a nice reduction.

Theorem 13.14. *Assume σ is a unit of L-length 4 in $K\Gamma$, with left-reduced split-form $(\varepsilon, \bar{\sigma})$ such that*

$$\bar{\sigma} = (\alpha_1 + \beta_1 x)(\alpha_2 + \beta_2 y)(\alpha_3 + \beta_3 x)(\alpha_4 + \beta_4 y).$$

If β_2 and β_3 are units in KN then σ is a trivial unit of $K\Gamma$.

Proof. For any $\gamma \in KN$ and $\phi \in \{x, y, xy\}$, we have γ is a unit in KN if and only if γ^{ϕ} is a unit in KN. Since the vertices in $\mathscr{C}_4$ only involve conjugates of β_2 and β_3, the result now follows by Corollary 13.13. □

The theory of consistent chains therefore provides a strategy for analysing units of higher L-length in $K\Gamma$, while also showing that there is a large jump in complexity from units of L-length 3 to units of L-length $\geqslant 4$. Some minimal chains are related via automorphisms, as we have seen, and this viewpoint may help reduce the number of minimal chains needed to be considered. Nevertheless as we shall see in the next section, the supports of units and their inverses are very closely related within torsion-free supersoluble group algebras, and in particular within $K\Gamma$.

14. Bounding units

Let K be a field and G a group. Roughly speaking, a group algebra KG has property (U) if the support of a unit determines a *finite bound* on the support of its inverse. More precisely, *KG has property* (U) if for each finite set $X \subset G$ there is a finite set $Y(X) \subset G$ such that for each unit $\sigma \in KG$:

$$\operatorname{Supp}\sigma \subset X \quad \Longrightarrow \quad \operatorname{Supp}\sigma^{-1} \subset Y(X).$$

Let X be a non-empty finite subset of G. Then we say that X *has property* (U) in KG if there is a finite subset $Y(X) \subset G$ such that for each unit $\sigma \in KG$:

$$\operatorname{Supp}\sigma \subset X \quad \Longrightarrow \quad \operatorname{Supp}\sigma^{-1} \subset Y(X).$$

Property (U) was introduced in [12] in connection with the semi-primitivity problem for group algebras, and later studied in [7]. It is clear that KG has property (U) if G is finite. Furthermore by [13, Lemma 1.1.4] $\operatorname{Supp}\sigma^{-1} \subset \langle \operatorname{Supp}\sigma \rangle$, so that KG has property (U) if G is locally finite. With a little more work one can then show KG has property (U) if G contains an abelian subgroup of finite index [7]. Finally we remark that KG has property (U) if all units of KG are trivial. Thus if KG satisfies the unit conjecture, then it has property (U).

On the other hand, Theorem 4.10 shows that if KG is an (X, Y, N)-group algebra with length function L, then $L(\sigma) = L(\sigma^{-1})$ for every $\sigma \in UKG$. This leads us to the following general result.

Theorem 14.1. *Let K be a field and G a torsion-free supersoluble group. If $\sigma \in UKG$, then there exists $H \leqslant G$ such that*

$$\sigma = \alpha g$$

for some $\alpha \in UKH$, $g \in G$, satisfying either

(i) $\operatorname{Supp}\alpha$ *has property* (U) *in KH or*
(ii) $L(\alpha) = L(\alpha^{-1})$ *for some length function $L : KH \to \mathbb{N} \cup \{-\infty\}$, induced from a surjective homomorphism of H onto the infinite dihedral group.*

Proof. We proceed by induction on the Hirsch number $h(G)$. Assume $\sigma \in UKG$. If $h(G) = 0$ then G is finite (hence trivial) so that KG has property (U). Thus

$$\sigma = \sigma \cdot 1$$

satisfies condition (i) above. Assume $h(G) > 0$ and therefore that G is infinite. Then by [13, Lemma 13.3.8] G has a normal subgroup N such that G/N is either infinite dihedral or infinite cyclic. By [6], KG has no proper divisors of zero. Thus if G/N is infinite dihedral then by [13, Lemma 13.3.6(iii)] KN is an Ore domain so that KG is an (X, Y, N)-group algebra with corresponding length function L. In this case, Theorem 4.10 yields $L(\sigma) = L(\sigma^{-1})$ so that

$$\sigma = \sigma \cdot 1$$

satisfies (ii) above with $H = N$. If G/N is cyclic fix $z \in G$ so that $G/N = \langle zN \rangle$. Then

$$\sigma = \alpha z^i$$

with $\alpha \in UKN$. Since $h(N) < h(G)$ it follows by induction that there exists $N' \leqslant N$ with

$$\alpha = \alpha' n$$

with $\alpha' \in KN'$, $n \in N$ satisfying either

(i) $\operatorname{Supp} \alpha'$ has property (U) in KN' or
(ii) $L(\alpha') = L(\alpha'^{-1})$ for some length function $L : KN' \to \mathbb{N} \cup \{-\infty\}$, induced from a surjective homomorphism of N' onto the infinite dihedral group.

Therefore

$$\sigma = \alpha z^i = \bigl(\alpha' n\bigr) z^i.$$

If $\operatorname{Supp} \alpha'$ has property (U) in KN' then $\operatorname{Supp} \alpha' n = \operatorname{Supp} \alpha$ has property (U) in KN, and therefore (i) holds with $H = N$. If $L(\alpha') = L(\alpha'^{-1})$ for some length function $L : KN' \to \mathbb{N} \cup \{-\infty\}$, induced from a surjective homomorphism of N' onto the infinite dihedral group, then (ii) holds with $H = N'$ and $g = nz^i$. □

In the case where KG is a virtually abelian (X, Y, N)-group algebra, we have the following stronger version of the previous result.

Theorem 14.2. *Let KG be a virtually abelian (X, Y, N)-group algebra with corresponding length function L. If $\sigma \in UKG$ then* $\operatorname{Supp} \sigma$ *has property* (U) *and* $L(\sigma) = L(\sigma^{-1})$.

Proof. This follows from [7] and Theorem 4.10. □

Theorem 14.3. *Let Γ be the fours group. If $\sigma \in UK\Gamma$ then* $\operatorname{Supp} \sigma$ *has property* (U) *and* $L(\sigma) = L(\sigma^{-1})$.

Proof. This follows from Theorem 8.1 and Theorem 14.2. □

15. Concluding remarks

If the unit conjecture for group algebras of torsion-free supersoluble groups is false then the natural candidate in which to locate a counterexample is $K\Gamma$. In this case, a reasonable approach is to work with small K and use the determinant condition afforded by Theorem 8.5 together with the specific regular embedding of Theorem 8.6. By studying minimal chains and reduced split-forms, along the lines of the length-3 case, one might be able to gain a better understanding at which L-length $n \geqslant 4$ a potential counterexample might exist. On the other hand, if no counterexample exists within $K\Gamma$, then Theorem 13.12 or Corollary 13.13 may provide a possible strategy for establishing this fact. Finally, in the previous section, we observed that $K\Gamma$ satisfies property (U). The question remains open for which group algebras property (U) holds, and whether KG satisfies property (U) if G is torsion-free.

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