The Number of $t$-Cores of Size $n$

David A. Craven, University of Oxford

August 2006

In this article we will prove two theorems regarding the number $c_t(n)$ of $t$-cores of size $n$: the first describes the behaviour of the function $c_t(n)$ as $n \to \infty$ and $t$ is a proportion of $n$; and the second describes the behaviour of $c_t(n)$ where $t > n/2$. We will also derive bounds on the quotient $p(n+1)/p(n)$, where $p(n)$ denotes the partition function.

1 Introduction

The main focus of this article is on the function $c_t(n)$, which measures the number of $t$-cores of size $n$ for each natural number $n$ and $t$. A famous result of Granville and Ono [2] states that, for all $n$ and for all $t \geq 4$, we have that $c_t(n) > 0$. Numerical evidence led Stanton in [4] to state the (in his words ‘possibly rash’) monotonicity conjecture, namely that for all $n$ and for all $t \geq 4$,

$$c_{t+1}(n) \geq c_t(n).$$

Indeed, it was slightly rash since in its stated form it is technically not true: in fact, $c_{n-1}(n) = p(n) - (n-1)$, where $p(n)$ is the partition function, and $c_n(n) = p(n) - n$. However, one can restrict the range of $t$ to get the following conjecture.

**Conjecture 1.1 (Monotonicity Conjecture)** Suppose that $n$ and $t$ are natural numbers, and that $4 \leq t < n - 1$. Then

$$c_{t+1}(n) \geq c_t(n).$$

One of the results in this article is that this conjecture is true for $t > n/2$, and in this case the inequality is strict. The method of proving this is elementary, although it might be able to be applied to smaller values of $t$, at least in the range $\sqrt{n} \leq t \leq n/2$.

The first theorem proved here is concerned with the proportion of $t$-cores of $n$, as $n$ tends to infinity. If $t$ is defined to be a proportion of $n$, say $t = n/3$, then in the limit the ratio of $t$-cores of $n$ by all partitions of $n$ tends to 1. More generally, we have the following theorem.

**Theorem 1.2** Suppose that $q$ is a real number lying strictly between 0 and 1. Then, as $n \to \infty$, we have

$$\frac{c_{\lfloor qn \rfloor}(n)}{p(n)} \to 1.$$
**Theorem 1.3** Suppose that $n$ is an integer, and let $t$ be an integer such that $t \geq 4$, and $n/2 < t < n - 1$. Then $c_t(n) < c_{t+1}(n)$.

## 2 The Asymptotics of the Number of Cores

In this section we will derive two related formulae for the number of $t$-cores of $n$, and apply one of them to prove Theorem 1.2. Let us start by recalling some facts about the weights of partitions.

Suppose that $n$ and $t$ are natural numbers, and write $n = td + r$, where $0 \leq r < t$. Let $\lambda$ be a partition of $n$, with $t$-core $\lambda'$. Then $|\lambda| - |\lambda'|$ is a multiple of $t$, and this multiple $x$ is referred to as the weight of a partition. The number of partitions $\lambda$ of $n$ with the same $t$-core $\lambda'$ is given by the number of multipartitions of $x$ into exactly $t$ partitions. (For example, if $x = 1$, then this number is $t$, and if $x = 2$ then this number is $(t^2 + 3t)/2$.) In particular, the number of partitions with a given $t$-core is independent of the shape of the core, and depends only on the weight of the partitions involved (the weight is the same for each partition) and the number $t$. (See [3, 2.7.17].) Denote by $w_t(x)$ the number of partitions of weight $x$ with the same $t$-core, or equivalently, the number of multipartitions of $x$ into $t$ partitions. The following result is therefore obvious.

**Lemma 2.1** Let $n$ and $t$ be natural numbers, and let $\lfloor n/t \rfloor = d$. Then

$$c_t(n) = p(n) - \sum_{x=1}^{d} w_t(x)c_t(n - xt).$$

We desire a formula that expresses the quantity $c_t(n)$ as a combination of partition functions only, not of the number of $t$-cores of other integers. Before we state this formula, suppose that $\lambda$ is a composition of $n$; by this we mean a sequence of positive integers $(\lambda_1, \ldots, \lambda_r)$ such that their sum is equal to $n$. Then define $w_t(\lambda)$ by the equation

$$w_t(\lambda) = (-1)^r w_t(\lambda_1)w_t(\lambda_2)\ldots w_t(\lambda_r).$$

We write $\lambda \mid n$ to mean that $\lambda$ is a composition of $n$.

**Proposition 2.2** Let $n$ and $t$ be natural numbers, and let $\lfloor n/t \rfloor = d$. Then

$$c_t(n) = p(n) + \sum_{x=1}^{d} \sum_{\lambda=x}^{d} w_t(\lambda)p(n - xt).$$

**Proof:** We proceed by induction on $d$, noting that in the case where $d = 1$, we get

$$c_t(n) = p(n) - w_t(1)p(n - t),$$

as in Lemma 2.1.
Using the expression from Lemma 2.1 and induction, we get
\[ c_t(n) = p(n) - \sum_{x=1}^{d} w_t(x)c_t(n - xt) \]
\[ = p(n) - \sum_{x=1}^{d} w_t(x) \left( p(n - xt) + \sum_{y=1}^{d-x} \sum_{\lambda=\gamma} w_t(\lambda)p(n - (x + y)t) \right) \]
\[ = p(n) + \left( \sum_{x=1}^{d} \sum_{y=1}^{d-x} \sum_{\lambda=\gamma} (-1)^{y} w_t(x)w_t(\lambda)p(n - (x + y)t) \right) + \left( \sum_{x=1}^{d} (-1)^{x} w_t(x)p(n - xt) \right) \]
\[ = p(n) + \sum_{x=1}^{d} \sum_{\lambda=\gamma} w_t(\lambda)p(n - xt). \]
This proves the result. \( \square \)

We now extend the partition function \( p(n) \) so that \( p(r) = p([r]) \) in the case where \( r \) is a real number. We can now state the following result, which is Theorem 1.2 from the introduction.

**Theorem 2.3** Suppose that \( 0 < q < 1 \) is a real number. Extending our notation, for any integer \( n \), write \( c_{qn}(n) \) for the number of \( [qn]\)-cores of \( n \). Then as \( n \) tends to infinity, we have
\[ \frac{c_{qn}(n)}{p(n)} \to 1; \]
that is, for any \( 0 < q < 1 \), the number of partitions of \( n \) that are not \( [qn]\)-cores becomes negligible as \( n \) tends to infinity.

To prove this, notice that it will follow from Proposition 2.2 if we can show that \( w_{qn}(\lambda)p(qn)/p(n) \) tends to 0 as \( n \) tends to infinity. To prove this, we need two lemmas.

**Lemma 2.4** The functions \( w_t(x) \) are polynomials in \( t \) of degree \( x \), whenever \( t > x \).

**Proof:** Let \( M(x,t) \) denote the set of all multipartitions of \( x \) into exactly \( t \) partitions. Since \( t > x \), a multipartition of \( x \) into \( t \) partitions must have at least \( t - x \) zero partitions. We will count the number of multipartitions lying in \( M(x,t) \) that have exactly \( a \) non-zero partitions.

Firstly, let us count the size of \( M'(x,a) \), the multipartitions of \( x \) into \( a \) partitions, none of which is zero. If \( a = 1 \) then this number is \( p(x) \), if \( a = 2 \) then this number is
\[ \sum_{i=1}^{x-1} p(i)p(x - i), \]
and in general this quantity is
\[ \sum_{\lambda} p(\lambda_1)p(\lambda_2)\ldots p(\lambda_a), \]
where the sum ranges over all compositions of \( x \) into \( a \) parts. Notice that this quantity \( |M'(x,a)| \) is independent of \( t \).

Finally, for each element \( \lambda' \) of \( M'(x,a) \), there are \( t!/a!(t-a)! \) different elements of \( M(x,t) \) whose non-zero partitions are the same as those of \( \lambda' \) in that order. Thus
\[ |M(x,t)| = \binom{t}{1}|M'(x,1)| + \binom{t}{2}|M'(x,2)| + \cdots + \binom{t}{x}|M'(x,x)|. \]
This is clearly a polynomial in \( t \), of degree \( x \). \( \square \)
Lemma 2.5 Let $0 < q < 1$ be a real number. Let $f(x)$ denote any polynomial. Then, as $n$ tends to infinity, we have

$$f(q^n)\frac{p(qn)}{p(n)} \to 0.$$ 

**Proof:** To prove this, we simply need to show that $p(qn)/p(n)$ tends to zero faster than any reciprocal of a polynomial. Recall [1, 5.12] the Hardy–Ramanujan asymptotic formula for $p(n)$,

$$p(n) \sim e^{\sqrt{n}} \frac{1}{bn},$$

where $c$ and $b$ are positive constants. Then, for any $\varepsilon > 0$, we have, for all sufficiently large $n$,

$$\frac{e^{\sqrt{n}}}{bn} (1 - \varepsilon) < p(n) < \frac{e^{\sqrt{n}}}{bn} (1 + \varepsilon).$$

Hence

$$\frac{p(qn)}{p(n)} \leq \frac{e^{\sqrt{n} q \sqrt{n}} (1 + \varepsilon)/bqn}{e^{\sqrt{n} (1 - \varepsilon)/bn}} = e^{\sqrt{n} q - \sqrt{n}} q \left( \frac{1 + \varepsilon}{1 - \varepsilon} \right),$$

which tends to zero faster than any reciprocal of a polynomial, since $e^{\sqrt{n}}$ grows faster than any polynomial.

This therefore proves Theorem 2.3.

3 The Monotonicity Conjecture

Suppose that $n$ is a natural number, and that $n/2 < t < n - 1$. We wish to prove that $c_t(n) < c_{t+1}(n)$. By Lemma 2.1, the quantity $c_{t+1}(n) - c_t(n)$ is given by

$$p(n-t)w_t(1) - p(n-t-1)w_{t+1}(1).$$

Recall that $w_t(1) = t$. Then the monotonicity conjecture, for $t$ in this range, becomes

$$\frac{p(n-t)}{p(n-t-1)} > \frac{t + 1}{t}.$$ 

We will in fact show that $p(n+1)/p(n) > (n+1)/n$ if $n \geq 3$, which, since $n - t < t + 1$ and $t \geq 4$, yields the result.

To prove the growth condition on $p(n)$ given above, we will give a function from the set of all partitions of $n$ to those of $n+1$. This yields an exact expression for $p(n+1)/p(n)$, and by inserting the crudest reasonable bounds for the terms involved, we get $p(n+1)/p(n) > (n+1)/n$.

Let $P(n)$ denote the set of all partitions of $n$: we will decompose the set $P(n)$ into the disjoint union of three subsets, $A(n)$, $B(n)$ and $C(n)$. The first, $A(n)$, consists of all partitions whose last part is 1. The set $B(n)$ consists of all partitions $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ for which $\lambda_r > 1$ and $\lambda_1 > \lambda_2$, together with the partition $(n)$. The third set, $C(n)$, consists of all partitions $(\lambda_1, \ldots, \lambda_r)$ for which $\lambda_r > 1$ and $\lambda_1 = \lambda_2$. Write $a(n) = |A(n)|$, and so on.

Firstly, notice that $a(n) = p(n-1)$, since the function between $P(n-1)$ and $A(n)$ obtained by adding 1 to the end of each partition is clearly bijective. Thus we have

$$p(n) = p(n-1) + b(n) + c(n).$$
Next, we describe a function from $P(n)$ to $B(n+1)$: this function is surjective, but obviously not injective. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of $n$, with the last $k$ parts of $\lambda$ equal to 1. Then define $f(\lambda)$ by

$$f(\lambda) = (\lambda_1 + 1 + k, \lambda_2, \ldots, \lambda_{r-k});$$

that is, by removing the parts of size 1 from the bottom of $\lambda$, attaching them to the first row of $\lambda$, and then increasing it by 1. It is obvious that $f(\lambda) \in B(n+1)$, and that every partition in $B(n+1)$ can be written as $f(\lambda)$ for some $\lambda$.

Suppose that $\lambda' \in B(n)$, and that $\lambda_1 - \lambda_2 = k + 1$, where $\lambda_2$ is taken to be 0 if $\lambda = (n)$. Then it is reasonably obvious that there are exactly $k$ different partitions of $n - 1$ whose image under $f$ is equal to $\lambda'$. Write $s(n)$ for the mean value of $k$, as the partition $\lambda'$ ranges over all elements of $B(n)$. Finally, write $r(n) = c(n)/p(n-1)$. Then we see that

$$p(n) = p(n-1) + \frac{p(n-1)}{s(n)} + r(n)p(n-1) = p(n-1)\left(1 + \frac{1}{s(n)} + r(n)\right).$$

Now, to get a lower bound on $p(n)/p(n-1)$, we simply need a lower bound on $r(n)$ and an upper bound on $s(n)$. A perfectly good lower bound on $r(n)$ is zero, and from the discussion in the previous paragraph, an upper bound on $s(n)$ is $n - 1$. Thus

$$p(n) \geq p(n-1)\left(1 + \frac{1}{n-1}\right).$$

If $n \geq 8$ then $r(n) > 0$, and so for $n \geq 8$, this inequality is strict. In fact, $s(n) < n - 1$ if $n \geq 4$, and so, for $n \geq 4$, we have

$$\frac{p(n)}{p(n-1)} > \frac{n}{n-1},$$

as we needed.

The bounds on $r(n)$ and $s(n)$ given above are extremely crude. With more work, one can get sharper bounds, although this does not seem to generate a proof of smaller cases of the monotonicity conjecture.

References


