The Number of t-Cores of Size n

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In this article we will prove two theorems regarding the number $c_t(n)$ of t-cores of size n: the first describes the behaviour of the function $c_t(n)$ as $n \to \infty$ and t is a proportion of n; and the second describes the behaviour of $c_t(n)$ where $t > n/2$. We will also derive bounds on the quotient $p(n+1)/p(n)$, where $p(n)$ denotes the partition function.

1 Introduction

The main focus of this article is on the function $c_t(n)$, which measures the number of t-cores of size n for each natural number n and t. A famous result of Granville and Ono [2] states that, for all n and for all $t \geqslant 4$, we have that $c_t(n) > 0$. Numerical evidence led Stanton in [4] to state the (in his words 'possibly rash') monotonicity conjecture, namely that for all n and for all $t \geqslant 4$,

$$c_{t+1}(n) \geqslant c_t(n).$$

Indeed, it was slightly rash since in its stated form it is technically not true: in fact, $c_{n-1}(n) = p(n) - (n-1)$, where $p(n)$ is the partition function, and $c_n(n) = p(n) - n$. However, one can restrict the range of t to get the following conjecture.

Conjecture 1.1 (Monotonicity Conjecture) Suppose that n and t are natural numbers, and that $4 \leqslant t < n-1$. Then

$$c_{t+1}(n) \geqslant c_t(n).$$

One of the results in this article is that this conjecture is true for $t > n/2$, and in this case the inequality is strict. The method of proving this is elementary, although it might be able to be applied to smaller values of t, at least in the range $\sqrt{n} \leqslant t \leqslant n/2$.

The first theorem proved here is concerned with the proportion of t-cores of n, as n tends to infinity. If t is defined to be a proportion of n, say $t = n/3$, then in the limit the ratio of t-cores of n by all partitions of n tends to 1. More generally, we have the following theorem.

Theorem 1.2 Suppose that q is a real number lying strictly between 0 and 1. Then, as $n \to \infty$, we have

$$\frac{c_{\lfloor qn \rfloor}(n)}{p(n)} \to 1.$$

Turning to the question of the monotonicity conjecture, we introduce a function from the set of all partitions of n to the set of all partitions of $n+1$. Crude bounds for the factors involved in this function yields a (crude) lower bound on the quantity $p(n+1)/p(n)$, which is nevertheless enough to prove half of the monotonicity conjecture.

Theorem 1.3 Suppose that n is an integer, and let t be an integer such that $t \geqslant 4$, and $n/2 < t < n-1$. Then $c_t(n) < c_{t+1}(n)$.

2 The Asymptotics of the Number of Cores

In this section we will derive two related formulae for the number of t-cores of n, and apply one of them to prove Theorem 1.2. Let us start by recalling some facts about the weights of partitions.

Suppose that n and t are natural numbers, and write $n = td + r$, where $0 \leqslant r < t$. Let λ be a partition of n, with t-core λ'. Then $|\lambda| - |\lambda'|$ is a multiple of t, and this multiple x is referred to as the *weight* of a partition. The number of partitions λ of n with the same t-core λ' is given by the number of multipartitions of x into exactly t partitions. (For example, if $x = 1$, then this number is t, and if $x = 2$ then this number is $(t^2 + 3t)/2$.) In particular, the number of partitions with a given t-core is independent of the shape of the core, and depends only on the weight of the partitions involved (the weight is the same for each partition) and the number t. (See [3, 2.7.17].) Denote by $w_t(x)$ the number of partitions of weight x with the same t-core, or equivalently, the number of multipartitions of x into t partitions. The following result is therefore obvious.

Lemma 2.1 Let n and t be natural numbers, and let $\lfloor n/t \rfloor = d$. Then

$$c_t(n) = p(n) - \sum_{x=1}^{d} w_t(x) c_t(n - xt).$$

We desire a formula that expresses the quantity $c_t(n)$ as a combination of partition functions only, not of the number of t-cores of other integers. Before we state this formula, suppose that λ is a composition of n; by this we mean a sequence of positive integers $(\lambda_1, \dots, \lambda_r)$ such that their sum is equal to n. Then define $w_t(\lambda)$ by the equation

$$w_t(\lambda) = (-1)^r w_t(\lambda_1) w_t(\lambda_2) \dots w_t(\lambda_r).$$

We write $\lambda \models n$ to mean that λ is a composition of n.

Proposition 2.2 Let n and t be natural numbers, and let $\lfloor n/t \rfloor = d$. Then

$$c_t(n) = p(n) + \sum_{x=1}^{d} \sum_{\lambda \models x} w_t(\lambda) p(n - xt).$$

Proof: We proceed by induction on d, noting that in the case where $d = 1$, we get

$$c_t(n) = p(n) - w_t(1) p(n - t),$$

as in Lemma 2.1.

Using the expression from Lemma 2.1 and induction, we get

$$\begin{aligned}
c_t(n) &= p(n) - \sum_{x=1}^{d} w_t(x) c_t(n - xt) \\
&= p(n) - \sum_{x=1}^{d} w_t(x) \left(p(n - xt) + \sum_{y=1}^{d-x} \sum_{\lambda \models y} w_t(\lambda) p(n - (x+y)t) \right) \\
&= p(n) + \left(\sum_{x=1}^{d} \sum_{y=1}^{d-x} \sum_{\lambda \models y} (-1) w_t(x) w_t(\lambda) p(n - (x+y)t) \right) + \left(\sum_{x=1}^{d} (-1) w_t(x) p(n - xt) \right) \\
&= p(n) + \sum_{x=1}^{d} \sum_{\lambda \models x} w_t(\lambda) p(n - xt).
\end{aligned}$$

This proves the result. $\square$

We now extend the partition function $p(n)$ so that $p(r) = p(\lfloor r \rfloor)$ in the case where r is a real number. We can now state the following result, which is Theorem 1.2 from the introduction.

Theorem 2.3 Suppose that $0 < q < 1$ is a real number. Extending our notation, for any integer n, write $c_{qn}(n)$ for the number of $\lfloor qn \rfloor$-cores of n. Then as n tends to infinity, we have

$$\frac{c_{qn}(n)}{p(n)} \to 1;$$

that is, for any $0 < q < 1$, the number of partitions of n that are not $\lfloor qn \rfloor$-cores becomes negligible as n tends to infinity.

To prove this, notice that it will follow from Proposition 2.2 if we can show that $w_{qn}(\lambda)p(qn)/p(n)$ tends to 0 as n tends to infinity. To prove this, we need two lemmas.

Lemma 2.4 The functions $w_t(x)$ are polynomials in t of degree x, whenever $t > x$.

Proof: Let $M(x,t)$ denote the set of all multipartitions of x into exactly t partititons. Since $t > x$, a multipartition of x into t partitions must have at least $t - x$ zero partitions. We will count the number of multipartitions lying in $M(x,t)$ that have exactly a non-zero partitions.

Firstly, let us count the size of $M'(x,a)$, the multipartitions of x into a partitions, none of which is zero. If $a = 1$ then this number is $p(x)$, if $a = 2$ then this number is

$$\sum_{i=1}^{x-1} p(i)p(x-i),$$

and in general this quantity is

$$\sum_{\lambda} p(\lambda_1)p(\lambda_2)\dots p(\lambda_a),$$

where the sum ranges over all compositions of x into a parts. Notice that this quantity $|M'(x,a)|$ is independent of t.

Finally, for each element λ' of $M'(x,a)$, there are $t!/a!(t-a)!$ different elements of $M(x,t)$ whose non-zero partitions are the same as those of λ' *in that order*. Thus

$$|M(x,t)| = \binom{t}{1}|M'(x,1)| + \binom{t}{2}|M'(x,2)| + \cdots + \binom{t}{x}|M'(x,x)|.$$

This is clearly a polynomial in t, of degree x. $\square$

Lemma 2.5 Let $0 < q < 1$ be a real number. Let $f(x)$ denote any polynomial. Then, as n tends to infinity, we have

$$f(qn)\frac{p(qn)}{p(n)} \to 0.$$

Proof: To prove this, we simply need to show that $p(qn)/p(n)$ tends to zero faster than any reciprocal of a polynomial. Recall [1, 5.12] the Hardy–Ramanujan asymptotic formula for $p(n)$,

$$p(n) \sim \frac{\mathrm{e}^{c\sqrt{n}}}{bn},$$

where c and b are positive constants. Then, for any $\varepsilon > 0$, we have, for all sufficiently large n,

$$\frac{\mathrm{e}^{c\sqrt{n}}}{bn}(1-\varepsilon) < p(n) < \frac{\mathrm{e}^{c\sqrt{n}}}{bn}(1+\varepsilon).$$

Hence

$$\begin{aligned}\frac{p(qn)}{p(n)} &\leqslant \frac{\mathrm{e}^{c\sqrt{qn}}(1+\varepsilon)/bqn}{\mathrm{e}^{c\sqrt{n}}(1-\varepsilon)/bn} \\ &= \mathrm{e}^{c\sqrt{q-1}\sqrt{n}}q\left(\frac{1+\varepsilon}{1-\varepsilon}\right),\end{aligned}$$

which tends to zero faster than any reciprocal of a polynomial, since $\mathrm{e}^{\sqrt{n}}$ grows faster than any polynomial. □

This therefore proves Theorem 2.3.

3 The Monotonicity Conjecture

Suppose that n is a natural number, and that $n/2 < t < n-1$. We wish to prove that $c_t(n) < c_{t+1}(n)$. By Lemma 2.1, the quantity $c_{t+1}(n) - c_t(n)$ is given by

$$p(n-t)w_t(1) - p(n-t-1)w_{t+1}(1).$$

Recall that $w_t(1) = t$. Then the monotonicity conjecture, for t in this range, becomes

$$\frac{p(n-t)}{p(n-t-1)} > \frac{t+1}{t}.$$

We will in fact show that $p(n+1)/p(n) > (n+1)/n$ if $n \geqslant 3$, which, since $n-t < t+1$ and $t \geqslant 4$, yields the result.

To prove the growth condition on $p(n)$ given above, we will give a function from the set of all partitions of n to those of $n+1$. This yields an exact expression for $p(n+1)/p(n)$, and by inserting the crudest reasonable bounds for the terms involved, we get $p(n+1)/p(n) > (n+1)/n$.

Let $P(n)$ denote the set of all partitions of n: we will decompose the set $P(n)$ into the disjoint union of three subsets, $A(n)$, $B(n)$ and $C(n)$. The first, $A(n)$, consists of all partitions whose last part is 1. The set $B(n)$ consists of all partitions $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ for which $\lambda_r > 1$ and $\lambda_1 > \lambda_2$, together with the partition (n). The third set, $C(n)$, consists of all partitions $(\lambda_1, \ldots, \lambda_r)$ for which $\lambda_r > 1$ and $\lambda_1 = \lambda_2$. Write $a(n) = |A(n)|$, and so on.

Firstly, notice that $a(n) = p(n-1)$, since the function between $P(n-1)$ and $A(n)$ obtained by adding 1 to the end of each partition is clearly bijective. Thus we have

$$p(n) = p(n-1) + b(n) + c(n).$$

Next, we describe a function from $P(n)$ to $B(n+1)$: this function is surjective, but obviously not injective. Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be a partition of n, with the last k parts of λ equal to 1. Then define $f(\lambda)$ by

$$f(\lambda) = (\lambda_1 + 1 + k, \lambda_2, \ldots, \lambda_{r-k});$$

that is, by removing the parts of size 1 from the bottom of λ, attaching them to the first row of λ, and then increasing it by 1. It is obvious that $f(\lambda) \in B(n+1)$, and that every partition in $B(n+1)$ can be written as $f(\lambda)$ for some λ.

Suppose that $\lambda' \in B(n)$, and that $\lambda_1 - \lambda_2 = k + 1$, where λ_2 is taken to be 0 if $\lambda = (n)$. Then it is reasonably obvious that there are exactly k different partitions of $n - 1$ whose image under f is equal to λ'. Write $s(n)$ for the mean value of k, as the partition λ' ranges over all elements of $B(n)$. Finally, write $r(n) = c(n)/p(n-1)$. Then we see that

$$p(n) = p(n-1) + \frac{p(n-1)}{s(n)} + r(n)p(n-1) = p(n-1)\left(1 + \frac{1}{s(n)} + r(n)\right).$$

Now, to get a lower bound on $p(n)/p(n-1)$, we simply need a lower bound on $r(n)$ and an upper bound on $s(n)$. A perfectly good lower bound on $r(n)$ is zero, and from the discussion in the previous paragraph, an upper bound on $s(n)$ is $n - 1$. Thus

$$p(n) \geqslant p(n-1)\left(1 + \frac{1}{n-1}\right).$$

If $n \geqslant 8$ then $r(n) > 0$, and so for $n \geqslant 8$, this inequality is strict. In fact, $s(n) < n - 1$ if $n \geqslant 4$, and so, for $n \geqslant 4$, we have

$$\frac{p(n)}{p(n-1)} > \frac{n}{n-1},$$

as we needed.

The bounds on $r(n)$ and $s(n)$ given above are extremely crude. With more work, one can get sharper bounds, although this does not seem to generate a proof of smaller cases of the monotonicity conjecture.

References

[1] Andrews, George, The theory of partitions, Encyclopedia of Mathematics and its Applications, 2, Addison-Wesley Publishing Co., Reading, Mass., 1976.

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[4] Stanton, Dennis, *Open Positivity Conjectures for Integer Partitions*, Trends Math. **2** (1999) 19–25 (electronic).