Strong Closure and p-Normality for Fusion Systems

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Grün's second theorem appears in most textbooks on the fusion theory of finite groups (see for example [3, Theorem 7.5.2]). Since the advent of fusion systems, much of this theory has been generalized and expanded, and in this short note we extend Grün's second theorem to fusion systems, as well as give a generalization of it that also applies to finite groups. We begin by giving our general result.

Theorem A Let $\mathcal{F}$ be a saturated fusion system on the finite p-group P. If Z is a weakly $\mathcal{F}$-closed subgroup of $\mathrm{Z}(P)$, then $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Z)$ and so in particular $[\mathcal{F}, P] = [\mathrm{N}_{\mathcal{F}}(Z), P]$, where for any subsystem $\mathcal{E}$ and subgroup Q,

$$[\mathcal{E}, Q] = \langle x^{-1}(x\phi) \mid x \in Q,\ \phi \in \mathrm{Hom}_{\mathcal{E}}(\langle x\rangle, Q)\rangle.$$

[The focal subgroup theorem states that, for $\mathcal{F}$ the fusion system of a finite group G with Sylow p-subgroup P, we have $[\mathcal{F}, P] = G' \cap P$.] This has the following corollary for finite groups, generalizing Grün's second theorem (which is only for the case $Z = \mathrm{Z}(P)$).

Corollary B Let G be a finite group with Sylow p-subgroup P. If Z is a central subgroup of P, weakly closed in P with respect to G, then

$$G' \cap P = \mathrm{N}_G(Z)' \cap P.$$

In the next section we prove Corollary B without reference to fusion systems, and in the section after that we give a short, self-contained proof of Theorem A.

1 Group-Theoretic Proof

We firstly need to show that weakly and strongly closed mean the same thing for central subgroups.

Theorem 1.1 Let G be a finite group with Sylow p-subgroup P. Suppose that a central subgroup Z is weakly closed in P with respect to G. Then Z is strongly closed in P with respect to G.

Proof: Let X and Y be subgroups of P, and suppose that there is an element g of G such that $X^g \leqslant Y$. Suppose that Y is an extremal conjugate of X, in the sense that $\mathrm{C}_P(Y)$ is a Sylow

p-subgroup of $\mathrm{C}_G(Y)$, and that one of X and Y lies inside Z. Firstly, we claim that g may be chosen so that $Z^g \leqslant P$. Suppose that this is true; the fact that Z is weakly closed in P means that $Z^g = Z$, and since one of X and Y lies inside Z, the other must, since $Z^g = Z$ and $X^g = Y$. If $X \leqslant Z$ and Y is a fixed extremal G-conjugate of X, then $Y \leqslant Z$ by the argument above. If A is any other subgroup of P that is G-conjugate to X, then there is an element $k \in G$ such that $A^k = Y$, and so $A \leqslant Z$ by another application of the argument above. Thus all G-conjugates of X lie inside Z, and so Z is strongly closed in P with respect to G, as claimed.

It suffices to show that if $X^g = Y$, and Y is extremal, then there is $h \in \mathrm{C}_G(Y)$ such that $Z^{gh} \leqslant P$. To see this, notice that $Z \leqslant \mathrm{C}_G(X)$, and so $Z^g \leqslant \mathrm{C}_G(Y)$, whence there is an element $h \in \mathrm{C}_G(Y)$ such that $(Z^g)^h \leqslant \mathrm{C}_P(Y)$, since $\mathrm{C}_P(Y)$ is a Sylow p-subgroup of $\mathrm{C}_G(Y)$, and in particular $Z^{gh} \leqslant P$, as needed. $\square$

Recall that a group is said to be *p-normal* if $\mathrm{Z}(P)$ is weakly closed in P with respect to G. Theorem 1.1 implies that this is equivalent to $\mathrm{Z}(P)$ being strongly closed in P with respect to G.

To prove Grün's result, we need a result of Glauberman and the focal subgroup theorem.

Proposition 1.2 (Glauberman [2, Theorem 6.1]) Let G be a finite group and P a Sylow p-subgroup of G. If A is an abelian subgroup of P, strongly closed in P with respect to G, then $\mathrm{N}_G(A)$ controls G-fusion in P.

Theorem 1.3 (Focal subgroup theorem) Let G be a finite group and let P be a Sylow p-subgroup of G. Then

$$G' \cap P = \langle g^{-1}g^x \mid g \in P, x \in G,\ g^x \in P \rangle.$$

In particular, if H controls G-fusion in P, then $G' \cap P = H' \cap P$. The preceding three results immediately imply Corollary B.

2 Fusion System Proof

For the terminology of fusion systems, we refer to, for example, [1]. We start by giving a proof of the generalization of Glauberman's Proposition 1.2 to all saturated fusion systems.

Proposition 2.1 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a strongly $\mathcal{F}$-closed subgroup. If Q is abelian then $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$.

Proof: Let T be any fully normalized, $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroup. Suppose that $Q \not\leqslant T$, and set $R = Q \cap T$ and $S = \mathrm{N}_Q(T)$; as $Q \not\leqslant T$, we have that $R < S$. Since S normalizes T and $Q \trianglelefteqslant P$, we have that $[T, S] \leqslant R$, and S centralizes R since $S, R \leqslant Q$ and Q is abelian. Since Q is strongly $\mathcal{F}$-closed, $\mathrm{Aut}_{\mathcal{F}}(T)$ acts on $R = Q \cap T$. Hence there is an $\mathrm{Aut}_{\mathcal{F}}(T)$-invariant series $1 \leqslant R \leqslant T$, with S centralizing each factor. The set of all such automorphisms of T is clearly a normal subgroup, and is a p-subgroup by [3, Corollary 5.3.3]. Therefore, $\mathrm{Aut}_S(T)$ is contained in a normal p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(T)$, and so in particular $\mathrm{Aut}_S(T) \leqslant \mathrm{Inn}(T)$ since T is $\mathcal{F}$-radical. Thus $S = \mathrm{N}_Q(T) \leqslant T\,\mathrm{C}_P(T) = T$ since T is $\mathcal{F}$-centric. Therefore, $Q \cap T \geqslant S > R = Q \cap T$, a contradiction. Hence Q is contained in every $\mathcal{F}$-centric, $\mathcal{F}$-radical subgroup.

By Alperin's fusion theorem, any morphism in $\mathcal{F}$ is the composition of restrictions of automorphisms of fully normalized, $\mathcal{F}$-radical, $\mathcal{F}$-centric subgroups, so if ϕ is a morphism in $\mathcal{F}$, we may write $\phi = \psi_1\psi_2\dots\psi_n$, where $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(T_i)$. By the above paragraphs, $Q \leqslant T_i$ for each i and therefore we may extend the domain of ϕ to include ϕ. Hence $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$, as claimed. □

We now prove Theorem A with a lemma and a (very) short argument.

Lemma 2.2 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a strongly $\mathcal{F}$-closed subgroup of P. If Z is a weakly $\mathcal{F}$-closed, central subgroup of Q, then Z is strongly $\mathcal{F}$-closed.

Proof: Let $\phi : X \to Y$ be an $\mathcal{F}$-isomorphism with either X or Y contained in Z, and Y fully normalized. Since Q is strongly $\mathcal{F}$-closed, both X and Y lie inside Q. The subgroup Z is central in Q, and so $Z \leqslant \mathrm{C}_P(X)$; hence ϕ extends to a morphism $\psi : ZX \to Q$ which, as Z is weakly $\mathcal{F}$-closed, restricts to an automorphism $\psi|_Z : Z \to Z$. Therefore if either X or Y lies in Z, then the other also does.

To see how this proves our result, suppose that X is a subgroup of Z, and that $\phi : X \to Y$ is an isomorphism with Y fully normalized. Then Y is contained in Z by the argument above. Now let $\psi : Y \to W$ be any $\mathcal{F}$-isomorphism; then by the argument above applied to the *inverse* of ψ, we see that W lies in Z as well, proving our result. □

The combination of Proposition 2.1 and Lemma 2.2 proves that $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Z)$ for any weakly $\mathcal{F}$-closed, central subgroup Z, and the focal subgroup theorem states that $P \cap G' = [\mathcal{F}_P(G), P]$, proving Corollary B. In fact, we do not even need the full strength of Proposition 2.1, since our subgroup is central, and in this case it is obviously contained in every $\mathcal{F}$-centric subgroup so a direct application of Alperin's fusion theorem suffices.

We cannot extend Lemma 2.2 to abelian subgroups, since for example in the group fusion system of $\mathrm{PSL}_2(7)$ at the prime 2, the two subgroups isomorphic with the Klein four group V_4 are weakly closed but not strongly closed; they also do not have the property that $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$.

References

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[2] George Glauberman, *A sufficient condition for p-stability*, Proc. London Math. Soc. **25** (1972), 253–287.

[3] Daniel Gorenstein, *Finite groups*, second ed., Chelsea Publishing Co., New York, 1980.