Contrasting the Definitions of Normal Subsystems in Fusion Systems

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In this article we examine the two competing concepts of normal subsystem, and prove that Aschbacher's notion appears to have significant advantages over the definition of Puig.

1 Introduction

In recent years there has been some debate over which of the two (for now) notions of normal subsystem is the 'correct' one. The original definition of a normal subsystem was defined by Puig, but see also [6] for a description in the modern language. We will call this *weakly normal.* In [2], Aschbacher provided an alternative, stronger definition for a normal subsystem, which we will call *strongly normal*, and proved that it had a nice property that weak normality lacks: namely, that for a constrained fusion system, the strongly normal subsystems are in one-to-one correspondence with the normal subgroups of the (unique) model of the fusion system, but this is not true for weak normality. Using this result, Aschbacher is able to produce a large corpus of results about fusion systems, mimicking the local theory of finite groups; for example, he is able to define the generalized Fitting subsystem of a fusion system, and show that it is the central product of the components and $\mathrm{O}_p(\mathcal{F})$. These results are in a long manuscript [1], which to the author's knowledge has not (yet) been published.

In this article we will attack the problem of which definition is best from another direction, by examining direct and central products.

Theorem A Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ and strongly normal subsystems of $\mathcal{F}$, with $\mathcal{E}_1$ based on Q_1 and $\mathcal{E}_2$ based on Q_2. If $Q_1 \cap Q_2 = 1$, then there exists a strongly normal subsystem $\mathcal{E}$ of $\mathcal{F}$ such that $\mathcal{E} \cong \mathcal{E}_1 \times \mathcal{E}_2$. If the $\mathcal{E}_i$ are weakly normal then there need not be such a subsystem present.

What we are saying is that if one finds trivially intersecting normal subsystems in a fusion system, then one wants their direct product to be in it: this is true for strong normality but *false* for weak normality, a fact that appears to show that strong normality is superior.

We now move on to central products. If $\mathcal{E}$ and $\mathcal{F}$ are fusion systems, a *central product* of $\mathcal{E}$ and $\mathcal{F}$ is a fusion system of the form $(\mathcal{E} \times \mathcal{F})/Z$, where Z is a subgroup of $\mathrm{Z}(\mathcal{E} \times \mathcal{F})$ with $Z \cap \mathrm{Z}(\mathcal{E}) = Z \cap \mathrm{Z}(\mathcal{F}) = 1$. The following theorem is proved in [1].

Theorem B (Aschbacher [1]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be strongly normal subsystems of $\mathcal{F}$, with $\mathcal{E}_1$ based on Q_1 and $\mathcal{E}_2$ based on Q_2. If $Q_1 \cap Q_2 \leqslant \mathrm{Z}(\mathcal{E}_1) \cap \mathrm{Z}(\mathcal{E}_2)$, then there exists a strongly normal subsystem $\mathcal{E}$ of $\mathcal{F}$ containing $\mathcal{E}_1$ and $\mathcal{E}_2$, such that $\mathcal{E}$ is isomorphic to a central product of $\mathcal{E}_1$ and $\mathcal{E}_2$.

Using this theorem, Aschbacher goes on to provide an analogue of the generalized Fitting subsystem: a *quasisimple* fusion system $\mathcal{F}$ is one for which $\mathcal{F}/\mathrm{Z}(\mathcal{F})$ is simple and $\mathrm{O}^p(\mathcal{F}) = \mathcal{F}$, and a *component* is a subnormal quasisimpe subsystem. Denote by $\mathrm{Comp}(\mathcal{F})$ the set of components of $\mathcal{F}$.

Theorem C (Aschbacher [1]) Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $E(\mathcal{F})$ denote the subsystem generated by $\mathrm{Comp}(\mathcal{F})$. Each element of $\mathrm{Comp}(\mathcal{F})$ is strongly normal in $E(\mathcal{F})$, and $E(\mathcal{F})$ is the central product of the elements of $\mathrm{Comp}(\mathcal{F})$. If $F^*(\mathcal{F})$ denotes the subsystem generated by $E(\mathcal{F})$ and $\mathrm{O}_p(\mathcal{F})$, then $F^*(\mathcal{F})$ is a central product of $E(\mathcal{F})$ and $\mathrm{O}_p(\mathcal{F})$.

We will prove that there is no way to define $E(\mathcal{F})$ for weak normality that gives it analogous properties to $E(G)$ for a finite group G, which are given here.

In this article, we will prove very few new things; our examples are considerably easier tto construct and analyze, however.

The next section will recall the various definitions that we need, and construct the direct product of two fusion systems. We prove here that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are strongly normal subsystems of a fusion system whose subgroups intersect trivially, then the direct product of $\mathcal{E}_1$ and $\mathcal{E}_2$ is also a strongly normal subsystem of the fusion system. In the succeeding section we provide examples to show how, using weak normality, we fail to get the nice results mentioned earlier in this section. In the final section we mention some open questions regarding both kinds of normality.

2 Definitions and Strong Normality

We assume that the reader is familiar with the definitions of fusion systems, saturated subsystems, and so on.

Definition 2.1 Let $\mathcal{F}$ be a fusion system on a finite p-group P, and let $\mathcal{E}$ be a subsystem on a subgroup Q of P, where Q is strongly $\mathcal{F}$-closed. We say that $\mathcal{E}$ is *$\mathcal{F}$-invariant* if, for each $R \leqslant S \leqslant Q$, $\phi \in \mathrm{Hom}_{\mathcal{E}}(R, S)$, and $\psi \in \mathrm{Hom}_{\mathcal{F}}(S, P)$, we have that $\psi^{-1}\phi\psi$ is a morphism in $\mathrm{Hom}_{\mathcal{E}}(R\psi, Q)$. If in addition $\mathcal{E}$ is saturated, we say that $\mathcal{E}$ is *weakly normal* in $\mathcal{F}$. We denote weak normality by $\mathcal{E} \prec \mathcal{F}$.

This definition of normality seems to be the most natural, but it suffers from certain problems, such as the following: if one takes a constrained fusion system $\mathcal{F}$, then it has a model (i.e., a group

G for which $\mathcal{F} = \mathcal{F}_P(G)$) that is essentially unique (G is unique subjected to being p-constrained and satisfying $\mathrm{O}_{p'}(G) = 1$). To any normal subgroup of G there is a corresponding weakly normal subsystem, but not vice versa.

Definition 2.2 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{E}$ be a subsystem of $\mathcal{F}$ on the subgroup Q. We say that $\mathcal{E}$ is *strongly normal* if $\mathcal{E}$ is weakly normal, and each $\phi \in \mathrm{Aut}_{\mathcal{E}}(Q)$ extends to $\bar\phi \in \mathrm{Aut}_{\mathcal{F}}(Q\,\mathrm{C}_P(Q))$ such that $[\bar\phi, \mathrm{C}_P(Q)] \leqslant \mathrm{Z}(Q)$. Write $\mathcal{E} \prec\!\!\prec \mathcal{F}$ if $\mathcal{E}$ is a strongly normal subsystem of $\mathcal{F}$. (This extra condition on weakly normal subsystems will be called the *(N1) property.*)

Let us prove that normal subgroups yield strongly normal subsystems. It is obvious that they yield saturated subsystems, and $\mathcal{F}$-invariance is easy, and so they yield weakly normal subsystems.

Lemma 2.3 Let G be a finite group with Sylow p-subgroup P. Let H be a normal subgroup of G with Sylow p-subgroup $Q = P \cap H$. If $\phi \in \mathrm{Aut}_H(Q)$ then there is an $h \in \mathrm{N}_H(Q)$ inducing ϕ and normalizing $Q\,\mathrm{C}_P(Q)$, such that

$$[h, \mathrm{C}_P(Q)] \leqslant \mathrm{Z}(Q).$$

Proof: Let x be any element of $\mathrm{N}_H(Q)$; then $\mathrm{C}_H(Q) \trianglelefteq \mathrm{N}_H(Q)$, and since $\mathrm{C}_P(Q)$ is a Sylow p-subgroup of $\mathrm{C}_H(Q)$, the Frattini argument gives

$$\mathrm{N}_H(Q) = \mathrm{C}_H(Q)\,\mathrm{N}_{\mathrm{N}_H(Q)}(\mathrm{C}_P(Q)).$$

Thus x may be written as gh, where $g \in \mathrm{C}_H(Q)$ and h normalizes $\mathrm{C}_P(Q)$. Since h differs from x by an element centralizing Q, h normalizes Q and induces the automorphism ϕ on Q. It remains to show that $[h, \mathrm{C}_P(Q)] \leqslant \mathrm{Z}(Q)$. Since h normalizes $\mathrm{C}_P(Q)$, we have that $[h, \mathrm{C}_P(Q)] \leqslant \mathrm{C}_P(Q)$, and since $H \trianglelefteq G$ it lies in $H \cap \mathrm{C}_P(Q) = \mathrm{Z}(Q)$. □

For *constrained* fusion systems, this correspondence is bijective.

Theorem 2.4 (Aschbacher [2, Theorem 1]) Let $\mathcal{F}$ be a constrained, saturated fusion system on a finite p-group P, and let G denote its unique model. There is a one-to-one correspondence between the normal subgroups of G and the strongly normal subsystems of $\mathcal{F}$.

If G is not the model of $\mathcal{F}$ but some other group with the same fusion system (e.g., $\mathcal{F} = \mathcal{F}_{V_4}(A_4)$ and $G = A_5$) then there need not be a one-to-one correspondence (and in this case, it is obviously not so). Moreover, if $\mathcal{F}$ is not constrained then there need not be a correspondence; for example, the fusion system of J_4 at the prime 3 has a strongly normal subsystem on the same 3-group as the fusion system, but since J_4 is simple there is no corresponding normal subgroup.

Next, we need the definition of direct and central products of fusion systems.

Definition 2.5 Let $\mathcal{E}$ and $\mathcal{F}$ be saturated fusion systems on the finite p-groups Q and R respectively. By the *direct product*, $\mathcal{E} \times \mathcal{F}$, of $\mathcal{E}$ and $\mathcal{F}$, we mean the fusion system on the p-group

$P = Q \times R$ given by all maps $(\phi, \psi)|_S$ such that ϕ is a map in $\mathcal{E}$ with domain S_1 and ψ is a map in $\mathcal{F}$ with domain S_2, and $S \leqslant S_1 \times S_2$. A *central product* of $\mathcal{E}$ and $\mathcal{F}$ is a fusion system of the form $(\mathcal{E} \times \mathcal{F})/Z$, where Z is a central subgroup of $\mathcal{E} \times \mathcal{F}$ such that $Z \nleqslant Q$ and $Z \nleqslant R$.

Central products (and in particular direct products) are saturated fusion systems by [3, Lemma 1.5]. We clearly have that $\mathcal{F}_P(G) \times \mathcal{F}_Q(H) \cong \mathcal{F}_{P\times Q}(G \times H)$.

We now prove the following result.

Theorem 2.6 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, let Q_1 and Q_2 be strongly $\mathcal{F}$-closed subgroups with $Q_1 \cap Q_2 = 1$, and write $Q = Q_1Q_2$. If $\mathcal{E}_i$ is a strongly normal subsystem on Q_i, then there is a strongly normal subsystem $\mathcal{E}$ on Q such that $\mathcal{E} \cong \mathcal{E}_1 \times \mathcal{E}_2$.

Proof: Firstly, Q is strongly $\mathcal{F}$-closed by [1] (see also [5]). We will show that if ϕ_1 is a morphism in $\mathcal{E}_1$ and ϕ_2 is a morphism in $\mathcal{E}_2$, then (ϕ_1, ϕ_2) is in $\mathcal{F}$. If we can show that $(\phi_1, \mathrm{id}_{Q_2})$ and $(\mathrm{id}_{Q_1}, \phi_2)$ are in $\mathcal{F}$ then we are done. Since $\mathcal{E}_1$ and $\mathcal{E}_2$ are saturated, the morphisms ϕ_i may be written as (the restriction of) a product of automorphisms of $\mathcal{E}_i$-centric subgroups, so to show that $(\phi_1, \mathrm{id}_{Q_2})$ (which we will abbreviate to (ϕ_1, id)) is in $\mathcal{F}$ it will suffice to assume that ϕ_1 is an automorphism of an $\mathcal{E}_1$-centric subgroup.

By [2, Lemma 8.10(3)], there is an extension $\bar{\phi}_1$ of ϕ_1 to R_1Q_2 in $\mathcal{F}$ such that $\bar{\phi}_1$ acts trivially on Q_2, and so (ϕ_1, id) lies in $\mathcal{F}$, and so indeed we have that $\mathcal{E} = \mathcal{E}_1 \times \mathcal{E}_2 \leqslant \mathcal{F}$.

We know that $\mathcal{E}$ is saturated by the remark before the statement of this theorem, and so it remains to show that it is $\mathcal{F}$-invariant and satisfies the (N1) property. It is clearly $\mathcal{F}$-invariant since both $\mathcal{E}_1$ and $\mathcal{E}_2$ are, and so $\mathcal{E} \prec \mathcal{F}$. To check the (N1) property, suppose that $\phi \in \mathrm{Aut}_{\mathcal{E}}(Q)$; then $\phi = (\phi_1, \phi_2)$, with $\phi_i \in \mathrm{Aut}_{\mathcal{E}_i}(Q_i)$. Since $\mathcal{E}_i \prec\!\!\prec \mathcal{F}$, we have that ϕ_i extends to a map $\bar{\phi}_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i\,\mathrm{C}_P(Q_i))$ such that $[\bar{\phi}_i, \mathrm{C}_P(Q_i)] \leqslant \mathrm{Z}(Q_i)$.

If Q_i is strongly $\mathcal{F}$-closed then $Q_i\,\mathrm{C}_P(Q_i)$ is strongly $\mathcal{F}$-closed for each i. Since Q_1 and Q_2 commute, we see that $Q\,\mathrm{C}_P(Q) \leqslant Q_i(\mathrm{C}_P(Q_i)$ and so we have that

$$\hat{\phi}_i = \bar{\phi}_i|_{Q\,\mathrm{C}_P(Q)} \in \mathrm{Aut}_P(Q\,\mathrm{C}_P(Q).$$

Since $[\bar{\phi}_i, \mathrm{C}_P(Q_i)] \leqslant \mathrm{Z}(Q_i)$, we see that $[\bar{\phi}_1\bar{\phi}_2, \mathrm{C}_P(Q)] \leqslant \mathrm{Z}(Q_1)\mathrm{Z}(Q_2) = \mathrm{Z}(Q)$. Lastly, we need to show that $\bar{\phi}_1\bar{\phi}_2$ is an extension of ϕ, but for this we simply note that $[\bar{\phi}_i, Q_{3-i}] \leqslant \mathrm{Z}(Q_i) \cap Q_{3-i} = 1$, so that $\bar{\phi}_i$ acts trivially on Q_{3-1}, as needed. □

This is enough for our purposes, but it is possible with some more careful choices to construct a strongly normal subsystem with the condition being that $[Q_1, Q_2] = 1$, rather than $Q_1 \cap Q_2 = 1$ as we have used here.

3 Examples of Bad Behaviour

Our first example of bad behaviour is a very small group, of order only 18.

Example 3.1 Let G be the group generated by $x = (1, 2, 3)$, $y = (4, 5, 6)$, and $(1, 2)(4, 5)$. This is really an elementary abelian group of order 9 with a diagonal action of C_2 on the two factors, making each into an S_3. Let $\mathcal{F} = \mathcal{F}_P(G)$, where P is the Sylow 3-subgroup. Let $\mathcal{E}_1$ denote the subsystem on $X = \langle x \rangle$ consisting of id_X and the map $\psi_x : x \mapsto x^{-1}$, and let $\mathcal{E}_2$ denote the corresponding subsystem on $Y = \langle y \rangle$ (with ψ_y inverting the elements of Y).

Firstly, we notice that $\mathcal{E}_1$ and $\mathcal{E}_2$ are both saturated subsystems, being the fusion system of S_3. Secondly, we have $\mathcal{E}_i \prec \mathcal{F}$: to see this, consider what the elements of $\mathcal{F}$ are. There are identity maps, the inversion maps on X and Y, and the map ϕ that inverts all elements of P. To prove normality, we merely need to show that $\phi^{-1}\psi_x\phi = \psi_x$, which it clearly is, and $\phi^{-1}\psi_y\phi = \psi_y$, which is similar.

However, we clearly do not have the entire direct product $\mathcal{E}_1 \times \mathcal{E}_2$ inside $\mathcal{F}$, since we do not have the map that acts as ψ_x on X and the identity on Y, for example. The non-existence of this map is the precise reason why $\mathcal{E}_1 \not\prec\!\!\prec \mathcal{F}$: we would need an extension $\hat{\psi}$ of ψ_x to $X\,\mathrm{C}_P(X) = P$ such that $[\hat{\psi}, \mathrm{C}_P(X)] = [\hat{\psi}, P] \leqslant \mathrm{Z}(X) = X$. However, since Y is strongly closed, we would have $[\hat{\psi}, Y] \leqslant Y \cap X = 1$, and so $\hat{\psi}|_Y = \mathrm{id}Y$.

This example is a 2-soluble fusion system (in the sense of [5]), so in particular its generalized Fitting subsystem is just $\mathrm{O}_2(\mathcal{F})$. Therefore this example does not show that the generalized Fitting subsystem cannot be defined for weakly normal subsystems. A very similar example will suffice, however.

To construct this example, we need a simple group G whose fusion system, $\mathcal{F}_P(G)$, is also simple, where P is a Sylow p-subgroup of G. Furthermore, we need a p'-element ϕ of $\mathrm{Out}(G)$ that induces an automorphism in $\mathrm{Out}(\mathcal{F})$. The finite group $(G \times G)\langle\phi\rangle$, where ϕ acts diagonally on both factors (as a subgroup of $\mathrm{Aut}(G) \times \mathrm{Aut}(G)$), will produce the same type of example as the previous one. An example of such a G and p is the Suzuki simple group at the prime 3.

Example 3.2 Let G be the Suzuki simple group. From the ATLAS [4] one can see that G has two conjugacy classes of elements of order 9, has a Sylow 3-subgroup P of order 3^7, and an outer automorphism group of order 2. The group $\mathrm{Aut}(G)$ has only one conjugacy class of elements of order 9, and so $\mathcal{F}_P(G) \neq \mathcal{F}_P(\mathrm{Aut}(G))$. Therefore the outer automorphism of G induces an outer automorphism of $\mathcal{F} = \mathcal{F}_P(G)$. Let H be the subgroup of index 2 in $\mathrm{Aut}(G) \times \mathrm{Aut}(G)$ that is not isomorphic with $\mathrm{Aut}(G) \times G$ (i.e., the 'diagonal' subgroup). The fusion system $\mathcal{F}_{P\times P}(H)$ has the same properties as before; i.e., that there are two trivially intersecting, weakly normal subsystems $\mathcal{F}_1$ and $\mathcal{F}_2$ (both isomorphic with $\mathcal{F}$) that are not strongly normal. As before, the direct product of the two subsystems is not inside $\mathcal{F}_{P\times P}(H)$, and so the generalized Fitting subsystem *cannot* be defined in a standard way for weak normality, since it could not contain $\mathcal{F}_1 \times \mathcal{F}_2$.

In other words, if one tries to define the generalized Fitting subsystem for weak normality, one does not get that it is the central product of its components, even in the case where they are simple and trivially intersecting.

[I have used here without proof the fact that $\mathcal{F}_P(G)$ is simple. It is.]

4 Open Problems

The area is still ripe for exploration, since most of what we should know we do not know. Here are some of the things that I at least do not know.

Question 4.1 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a strongly $\mathcal{F}$-closed subgroup. Is there a weakly normal subsystem on Q?

The corresponding question for strongly normal subsystems has a negative answer in general; I believe that the answer to this question is also negative, although the method of proof will have to change, since the counterexample for strongly normal subsystems went via a constrained fusion system and the bijection between strongly normal subsystems and normal subgroups.

Question 4.2 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be saturated subsystems on Q_1 and Q_2 respectively. If $\mathcal{E}_i \prec \mathcal{F}$, is there a weakly normal subsystem $\mathcal{E}$ on Q_1Q_2, containing $\mathcal{E}_1$ and $\mathcal{E}_2$? What if weak normality is replaced by strong normality?

This question has a positive answer for strong normality if the subgroups Q_1 and Q_2 commute, but as far as I know in general nothing is known. (Since the product of two strongly closed subgroups is strongly closed, at least the most obvious requirement is satisfied.)

Question 4.3 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let $\mathcal{E}_1$ and $\mathcal{E}_2$ be weakly normal subsystems on Q_1 and Q_2 respectively. Is there a weakly normal subsystem $\mathcal{E}$ on $Q_1 \cap Q_2$ contained within $\mathcal{E}_1 \cap \mathcal{E}_2$?

This is true for strongly normal subsystems [1, Theorem 1]; with this there is a theory of minimal normal subsystems, and without it one may do nothing. If this has a positive answer then some of the benefits of working with strongly normal subsystems disappear. For example, we have the following result.

Lemma 4.4 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P. A minimal strongly normal subsystem is either $\mathcal{F}_Q(Q)$ for some elementary abelian p-group Q or a direct product of isomorphic simple (i.e., no strongly normal subsystems) fusion systems.

The proof is the same as for finite groups. It relies entirely on the positive answer to Question 4.3 for strongly normal subsystems. Whether there is a nice theory of minimal normal subsystems mirroring that of finite groups remains to be seen, although since p-soluble fusion systems all come from finite groups, in this case something might be done.

It is also known that in general, if Q is a strongly $\mathcal{F}$-closed subgroup of a saturated fusion system $\mathcal{F}$, there is in general no one-to-one correspondence between the strongly normal subsystems of $\mathcal{F}/Q$

and the normal subsystems of $\mathcal{F}$ containing $\mathcal{F}_Q(Q)$ (we restrict here to the case where $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$, since $\mathcal{F}/Q = \mathrm{N}_{\mathcal{F}}(Q)/Q$). A notable exception to this fact is that if Q is central then there *is* a one-to-one correspondence for both weakly normal and strongly normal. In [5], we prove that the image of a saturated subsystem is saturated via the second isomorphism theorem for fusion systems.

Question 4.5 Let $\mathcal{F}$ be a saturated fusion system on a finite p-group P, and let Q be a subgroup such that $\mathcal{F} = \mathrm{N}_{\mathcal{F}}(Q)$. Suppose that $\mathcal{E}$ is a saturated subsystem of $\mathcal{F}/Q$. Is there a saturated subsystem $\mathcal{E}'$ such that the image of $\mathcal{E}'$ in $\mathcal{F}/Q$ is exactly $\mathcal{E}$?

The full preimage of $\mathcal{E}$ might well not be the correct object to consider in this case.

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