A New Maximal Subgroup of E_8 in Characteristic 3

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Abstract

We prove the existence and uniqueness of a new maximal subgroup of the algebraic group of type E_8 in characteristic 3. This has type F_4, and was missing from previous lists of maximal subgroups produced by Seitz and Liebeck–Seitz. We also prove a result about the finite group $H = {}^3D_4(2)$, that if H embeds in E_8 (in any characteristic p) and has two composition factors on the adjoint module then $p = 3$ and H lies in this new maximal F_4 subgroup.

1 Introduction

The classification of the maximal subgroups of positive dimension of exceptional algebraic groups [13] is a cornerstone of group theory. In the course of understanding subgroups of the finite groups $E_8(q)$ in [3], the first author ran into a configuration that should not occur according to the tables in [13].

We elicit a previously undiscovered maximal subgroup of type F_4 of the algebraic group E_8 over an algebraically closed field of characteristic 3. This discovery corrects the tables in [13], and the original source [17] on which it depends.

Theorem 1.1. *Let* $\mathbf{G}$ *be a simple algebraic group of type* E_8 *over an algebraically closed field of characteristic* 3*. Then* $\mathbf{G}$ *contains a unique conjugacy class of simple maximal subgroups of type* F_4*.*

If $\mathbf{X}$ *is in this class, then the restriction of the adjoint module* $L(E_8)$ *to* $\mathbf{X}$ *is isomorphic to* $L_{\mathbf{X}}(1000) \oplus L_{\mathbf{X}}(0010)$*, where the first factor is the adjoint module for* $\mathbf{X}$ *of dimension* 52 *and the second is a simple module of dimension* 196 *for* $\mathbf{X}$*.*

The classification from [13] states that the maximal subgroups are maximal-rank or parabolic subgroups, or one of a short list of reductive subgroups that exist for all but a few small primes, together with G_2 inside F_4 for $p = 7$. This last case arises from a generic embedding of G_2 in E_6, which falls into F_4 on reduction modulo the prime $p = 7$ only. This new E_8 subgroup is therefore the only example of a maximal subgroup that exists for a single prime, whose embedding cannot be explained using generic phenomena.

The structure of this note is as follows. The existence and uniqueness up to conjugacy of a maximal Lie subalgebra $\mathfrak{f}_4 \subset \mathfrak{e}_8$ is first established. From this we are able to write down the root elements of $\mathfrak{f}_4$; exponentiation gives expressions for the root groups of $\mathbf{X}$ in terms of the root groups of E_8, providing an explicit construction of $\mathbf{X}$.

In the final section we determine various results providing extra details on this new class of maximal subgroups. For each unipotent class in $\mathbf{X}$ we determine the corresponding unipotent class in E_8 that contains it, and we do the same for nilpotent orbits of the corresponding Lie algebras. We also consider the maximal connected subgroups of $\mathbf{X}$. The maximal parabolic subgroups of $\mathbf{X}$ will be contained in parabolic subgroups of E_8 by the Borel–Tits Theorem. There are four classes of reductive maximal connected subgroups of $\mathbf{X}$ when $p = 3$ with types $B_4, A_1C_3, A_1G_2, A_2A_2$. We show that all of these classes are contained in other maximal connected subgroups of E_8 and we specify such an overgroup. Moreover, we determine that the first three classes are E_8-irreducible but the last class is not. (A subgroup is $\mathbf{G}$-irreducible if it is not contained in any proper parabolic subgroup of $\mathbf{G}$.)

In establishing the existence of $\mathbf{X}$, we prove the following extra result, of use in the project to classify maximal subgroups of the finite exceptional groups of Lie type.

Proposition 1.2. *Let H be the group ${}^3D_4(2)$, let p be a prime, and suppose that H embeds in the algebraic group E_8 in characteristic p. If the composition factors of the action of H on the adjoint module $L(E_8)$ have dimensions 52 and 196, then $p = 3$ and H is contained in a maximal subgroup $\mathbf{X}$ of type F_4; furthermore, H and $\mathbf{X}$ stabilize the same subspaces of $L(E_8)$.*

2 From the Thompson group to F_4

One path to a construction of the F_4 subgroup of E_8 starts with the Thompson group, which contains a copy of $H \cong {}^3D_4(2)$, acting on $L(E_8)$ with composition factors of dimensions 52 and 196. In fact, every H-invariant alternating bilinear form on the 248-dimensional module is invariant under a suitable copy of $F_4 \leq \mathrm{GL}_{248}(k)$.

We cannot quite show this without a computer. Splitting $L(E_8)$ up as the sum of 52 and 196 fragments the space of alternating forms into six components. For five of these six we can show that the H-invariant maps are F_4-invariant, but for the sixth we cannot do so without a computer. With a computer we can check that this sixth component is at least $F_4(9)$-invariant, and thus every subgroup H of E_8 is contained in a copy of $F_4(9)$. But $F_4(9)$ contains elements of order 6562, and thus there is an F_4 subgroup of E_8 containing it, via [12, Proposition 2].

We then show that this F_4 subgroup is unique up to E_8-conjugacy, obtaining as a by-product that H is unique up to E_8-conjugacy.

We start with a copy J of the Thompson sporadic simple group. This has a 248-dimensional simple module M over $\mathbb{C}$, and it remains simple upon reduction modulo all primes. From [1, p.176], we see that there are elements of order 9 with Brauer character value 5 on M. The only integers that are the traces of semisimple elements of order 9 in E_8 (on the adjoint module) are -1, 2, 8 and 29. Thus these elements cannot be semisimple, and in particular, $p \mid 9$. Thus we see that if J embeds in the algebraic group E_8 in any characteristic p, then $p = 3$. It is a famous result that J does indeed embed in $E_8(3)$, and is unique up to conjugacy.

It is well known that J contains a subgroup H isomorphic to ${}^3D_4(2)$. From [1, p.90] and [6, pp.251–253], we see that in characteristic not 2, the restriction of M to H is the direct sum of a 52-dimensional simple module M_1 and a 196-dimensional simple module M_2. However, all elements of order 9 in H act on $M_1 \oplus M_2$ with trace 2, so we cannot use the previous method to show that H cannot embed in E_8 in characteristic $p \neq 2, 3$ acting on $L(E_8)$ as $M_1 \oplus M_2$.

Lemma 2.1. *Let p be an odd prime, let H denote the group ${}^3D_4(2)$, and suppose that H embeds in the algebraic group E_8 in characteristic p, acting on the adjoint module with composition factors of dimensions 52 and 196. Then the submodule 52 carries the structure of a Lie algebra of type $\mathfrak{f}_4$, and in particular $p = 3$. Furthermore, such an $\mathfrak{f}_4$ subalgebra of $\mathfrak{e}_8$ does exist for $p = 3$.*

Proof. Let M_1 denote the 52-dimensional kH-submodule of the adjoint and M_2 the 196-dimensional submodule. Note that $|H| = 2^{12} \cdot 3^4 \cdot 7^2 \cdot 13$, so either $p \nmid |H|$ and we are essentially in characteristic 0, or $p = 7, 13$, or $p = 3$.

From [4], there is a unique conjugacy class of subgroups H in F_4 for any odd characteristic p, acting irreducibly on the minimal and adjoint modules. In particular, we see from [4] that $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1)$ is 1-dimensional for all odd primes p. From an ordinary character calculation, we see that

$$\Lambda^2(\chi_{52}) = \chi_{52} + \chi_{1274},$$

where χ_i is the irreducible character of degree i. For both 7 and 13, the reduction modulo p of χ_{1274} is irreducible, hence the reduction modulo p is irreducible modulo p for all $p > 3$ (see [6, pp.252–253]). If $p = 3$ then χ_{1274} has Brauer character constituents of degrees 52 and 1222 from [6, p.251]. Since

$\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) = k$, we see that the exterior square is uniserial, with layers of dimensions 52, 1222 and 52. Moreover,

$$\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_2) = 0.$$

Thus M_1 forms an H-invariant subalgebra, which must be non-zero, since $\mathfrak{e}_8$ contains no abelian subspace of dimension 52 by [5, Proposition 2.3]. Furthermore, as M_1 is irreducible for H, the restriction of the Lie bracket to M_1 furnishes it with the structure of a semisimple Lie algebra. Since $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) = k$, there is at most one isomorphism class of such, but as the algebraic k-group F_4 does contain a subgroup isomorphic to H, acting as χ_{52} on its adjoint module, it follows that $M_1 \cong \mathfrak{f}_4$. In particular, this means that the $\mathfrak{f}_4$ Lie algebra must have a simple module of dimension 196, as it acts H-equivariantly on $L(E_8)$.

Such a simple module must be restricted: if not the p-closure L_p of the image L of $\mathfrak{f}_4$ in $\mathfrak{e}_8$ will contain a centre [19, 2.5.8(2)]; but L has no 1-dimensional submodules on M. By Curtis's theorem, M_2 arises by differentiation of a restricted representation for the algebraic group F_4, whence $p = 3$, from the tables in [14].

The embedding and subalgebra do exist for $p = 3$ via the Thompson group, as seen above. □

We will prove that the $\mathfrak{f}_4$-subalgebra is the Lie algebra of an F_4 algebraic subgroup of E_8. To do so, we will actually prove that every H-invariant alternating product on the kH-module $M = M_1 \oplus M_2$ for $p = 3$ is also F_4-invariant, for the unique $F_4 \leq \mathrm{GL}_{52}(k)$ containing H. To do so, we need to understand the space

$$\mathrm{Hom}_{kH}(\Lambda^2(M), M)$$

of alternating products on M. Using $M = M_1 \oplus M_2$, and the formula

$$\Lambda^2(A \oplus B) \cong \Lambda^2(A) \oplus \Lambda^2(B) \oplus A \otimes B$$

we split the space of products up into six components. The next result gives the dimensions of these components.

Proposition 2.2. *We have*

$$\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) = k, \qquad \mathrm{Hom}_{kH}(\Lambda^2(M_1), M_2) = 0,$$
$$\mathrm{Hom}_{kH}(\Lambda^2(M_2), M_1) = k, \qquad \mathrm{Hom}_{kH}(\Lambda^2(M_2), M_2) = k,$$
$$\mathrm{Hom}_{kH}(M_1 \otimes M_2, M_1) = 0, \qquad \mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2) = k.$$

Proof. One may use a computer to check these with ease. Some may be checked easily by hand as well, using the ordinary character table and the 3-decomposition matrix for H. For example, using those two tables, $M_1 \otimes M_2$ does not possess a composition factor M_1, and thus

$$\mathrm{Hom}_{kH}(M_1 \otimes M_2, M_1) = \mathrm{Hom}_{kH}(\Lambda^2(M_1), M_2) = 0.$$

The statement that $\mathrm{Hom}_{kH}(\Lambda^2(M_1), M_1) = k$ appears in [4, Section 4.3.4] (where it is proved by computer). □

At least the existence of two of the three remaining maps is clear from the fact that H embeds in $E_8(3)$ with representation $M_1 \oplus M_2$. If $\mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2) = 0$ then M_1 would be an ideal of the Lie algebra, which is not possible. A character calculation shows that $S^2(M_2)$ does not have a composition factor M_1, so

$$\mathrm{Hom}_{kH}(\Lambda^2(M_2), M_1) = \mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2).$$

It is only $\mathrm{Hom}_{kH}(\Lambda^2(M_2), M_2)$ that cannot easily be seen. Indeed, this space will cause us a problem later on.

We now prove that five of the six Hom-spaces extend to the algebraic group $\mathbf{X} = F_4$, with only $\mathrm{Hom}_{kH}(\Lambda^2(M_2), M_2)$ missing. If one is happy to use a computer for all of this, one simply checks that all H-invariant maps are $F_4(3)$- and even $F_4(9)$-invariant, and thus one does not need to prove the next proposition.

Proposition 2.3. *Let* $\mathbf{X}$ *be an algebraic* k*-group of type* F_4*. We have*

$$\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(1000)), L(1000)) = k,$$

$$\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(1000)), L(0010)) = \mathrm{Hom}_{\mathbf{X}}(L(1000) \otimes L(0010), L(1000)) = 0, \textit{ and}$$

$$\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(0010)), L(1000)) = \mathrm{Hom}_{\mathbf{X}}(L(1000) \otimes L(0010), L(0010)) = k.$$

Proof. Note that each of these spaces must have dimension at most the dimension of the corresponding space for H. This yields the two 0-dimensional spaces, and that the other spaces have dimension at most 1. The first statement holds because $\mathbf{X}$ is a non-abelian Lie algebra, and therefore the space is non-zero.

For the last statement, since $S^2(M_2)$ has no composition factor isomorphic to M_1, certainly $S^2(L(0010))$ has no composition factor isomorphic to $L(1000)$. Thus the two Hom-spaces are isomorphic, so it remains to find a non-zero map in the latter space.

The composition factors of the kH-module $M_1 \otimes M_2$ are of dimensions

$$25, 196, 196, 441, 1963, 2457, 2457, 2457.$$

The highest-weight module $L(1010)$, which must appear as a composition factor in $L(1000) \otimes L(0010)$, has dimension 7371, and must restrict to kH to be the sum of the three (non-isomorphic) modules of dimension 2457. The rest of the composition factors, in total, have dimension 2821, so there must be a composition factor of dimension between 1963 and 2821. Consulting [14, Appendix A.50], we find exactly one such module: $L(0011)$ of dimension $2404 = 1963 + 441$. The remaining kH-modules, 25, 196 and 196, must be the other composition factors for F_4, because F_4 has no simple modules of dimension $25 + 196$, $196 + 196$, or $25 + 196 + 196$.

Thus the composition factors of $L(1000) \otimes L(0010)$ have dimensions 25, 196, 196, 2404 and 7371. Since $L(0010)$ is the unique module to appear more than once, and the tensor product is self-dual, $L(0010)$ must be a submodule, and the maps in $\mathrm{Hom}_{kH}(M_1 \otimes M_2, M_2)$ extend to $\mathbf{X}$. □

The last remaining Hom-space to check is $\mathrm{Hom}_{\mathbf{X}}(\Lambda^2(L(0010)), L(0010))$. This seems difficult to do by hand, and we resort to a computer. There are two ways to proceed. The first is to prove that that there is an $F_4(3)$-invariant map in the space (this takes a couple of minutes), and thus the H is contained in a copy of $F_4(3)$ in E_8. We then apply [3, Proposition 6.8], which states that $F_4(3)$ is contained in a positive-dimensional subgroup stabilizing the same subspaces of $L(E_8)$, which are M_1 and M_2. This must be a copy of F_4 (as it stabilizes an $\mathfrak{f}_4$-Lie subalgebra), and we are done. Alternatively, we prove the same statement for $F_4(9)$ (which contains elements of order $9^4 + 1 = 6562$, this takes about half an hour on one of the first author's computers) and then apply [12, Proposition 2], which yields the same positive-dimensional subgroup.

In either case, we obtain the following.

Proposition 2.4. *Let* $p = 3$ *and let* H *be a subgroup* ${}^3D_4(2)$ *of* E_8*, acting on* $L(E_8)$ *with composition factors of dimensions* 52 *and* 196*. Then* H *is contained in a positive-dimensional subgroup of type* F_4*, stabilizing exactly the same subspaces of* $L(E_8)$ *that are stabilized by* H*.*

It suffices to ascertain the uniqueness up to conjugacy of the hypothesized subgroup of type F_4, and as a by-product we also obtain uniqueness of H up to conjugacy.

Let t be an involution in $\mathbf{X} = F_4$ with centralizer B_4. The trace of t on $L(E_8)$ is -8, and so the centralizer of t in E_8 is D_8, which acts with composition factors $L(\lambda_2)$ and $L(\lambda_7)$, of dimensions 120 and 128 respectively. The restriction of the $k\mathbf{X}$-module $L(1000)$ to B_4 is $L(0001) \oplus L(0100)$, with dimensions 16 and 36 respectively. The restriction of $L(0010)$ to B_4 is $L(0010) \oplus L(1001)$, of dimensions 84 and 112 respectively. (This can be checked using weights or quickly on a computer for $F_4(3)$.) From [20, Table 60] we see that this B_4 is subgroup $E_8(\#45)$ and is unique up to conjugacy. But clearly there is a unique way to assemble the numbers 16, 36, 84 and 112 to make 52 and 196. Thus given any subgroup of type B_4 there exists at most one F_4 containing it, which must stabilize the submodule $L(0001) \oplus L(0100)$. Thus we obtain the result that $\mathbf{X}$ is unique up to conjugacy.

This completes the proof of uniqueness of $\mathbf{X}$, and thus Theorem 1.1 is proved.

3 Presentation of F_4

For their application to future explicit computations, we give expressions for the root elements $x_{\pm\beta_i}(t)$ for β_i a root of the maximal subgroup F_4 as products of root elements of E_8; see [2, §4.4] for notation. We also provide the elements $h_{\beta_i}(t)$ written in terms of those of E_8. Note that since the coefficients t_j we exhibit are elements of GF(3), it follows that the subgroup $\mathbf{Y}$ we produce is defined over GF(3). As the factors of $x_{\beta_i}(t) = \prod_j x_{\alpha_j}(c_j t)$ commute we get easily that $x_{-\beta_i}(t) = \prod_j x_{-\alpha_j}(c_j t)$. For this reason we only list the root group elements for positive roots. Note also that the element e_{β_i} of a Chevalley basis for $\mathfrak{h}$ is recovered from such an expression as $\sum c_j e_{\alpha_j}$.

$x_{1000}(t) = x_{00010000}(t)x_{00000100}(t)$

$x_{0100}(t) = x_{00100000}(-t)x_{00000010}(t)$

$x_{0010}(t) = x_{10000000}(-t)x_{00000001}(t)x_{00011000}(-t)x_{00001100}(-t)$

$x_{0001}(t) = x_{11110000}(t)x_{01121000}(-t)x_{01111100}(-t)x_{01011110}(t)$

$x_{1100}(t) = x_{00110000}(-t)x_{00000110}(-t)$

$x_{0110}(t) = x_{10100000}(t)x_{00000011}(-t)x_{00111000}(-t)x_{00001110}(-t)$

$x_{0011}(t) = x_{11121000}(t)x_{11111100}(t)x_{01011111}(t)x_{01122100}(-t)$

$x_{1110}(t) = x_{10110000}(t)x_{00000111}(t)x_{00111100}(-t)x_{00011110}(t)$

$x_{0120}(t) = x_{10111000}(-t)x_{00001111}(-t)$

$x_{0111}(t) = x_{11221000}(t)x_{11111110}(t)x_{01111111}(t)x_{01122110}(-t)$

$x_{1120}(t) = x_{10111100}(-t)x_{00011111}(t)$

$x_{1111}(t) = x_{11221100}(t)x_{11121110}(-t)x_{01121111}(-t)x_{01122210}(t)$

$x_{0121}(t) = x_{11111111}(-t)x_{11222100}(-t)x_{11122110}(t)x_{01122111}(t)$

$x_{1220}(t) = x_{10111110}(t)x_{00111111}(-t)$

$x_{1121}(t) = x_{11121111}(t)x_{11232100}(t)x_{11122210}(-t)x_{01122211}(-t)$

$x_{0122}(t) = x_{12232111}(-t)x_{12233210}(t)$

$x_{1221}(t) = x_{11221111}(-t)x_{11232110}(-t)x_{11222210}(t)x_{01122221}(-t)$

$x_{1122}(t) = x_{12232211}(t)x_{12243210}(-t)$

$x_{1231}(t) = x_{11232111}(t)x_{11222211}(-t)x_{11122221}(-t)x_{11233210}(t)$

$x_{1222}(t) = x_{12232221}(t)x_{12343210}(t)$

$x_{1232}(t) = x_{22343210}(t)x_{12343211}(t)x_{12243221}(t)x_{12233321}(-t)$

$x_{1242}(t) = x_{22343211}(t)x_{12244321}(t)$

$x_{1342}(t) = x_{22343221}(-t)x_{12344321}(t)$

$x_{2342}(t) = x_{22343321}(t)x_{12354321}(-t)$

$h_{1000}(t) = h_{00010000}(t)h_{00000100}(t)$

$h_{0100}(t) = h_{00100000}(t)h_{00000010}(t)$

$h_{0010}(t) = h_{10000000}(t)h_{00010000}(t)h_{00001000}(t^2)h_{00000100}(t)h_{00000001}(t)$

$h_{0001}(t) = h_{1000000}(t)h_{01000000}(t^4)h_{00100000}(t^3)h_{00010000}(t^5)h_{00001000}(t^3)h_{00000100}(t^2)h_{00000010}(t)$

Remark 3.1. *We note that the* 24*-dimensional unipotent subgroup generated by the positive root elements of* F_4 *is contained in the* 120*-dimensional unipotent subgroup generated by the positive roots of* E_8*. The given presentation has the slightly unfortunate property that the structure constants are not the same as those used in Magma, a standard choice. However, one can if one wishes rectify this by choosing a* *different base: let $\tilde{x}_{\alpha_1}(t) = x_{-0100}(t)$, $\tilde{x}_{\alpha_2}(t) = x_{-1242}(t)$, $\tilde{x}_{\alpha_3}(t) = x_{1232}(t)$, $\tilde{x}_{\alpha_4}(t) = x_{-0001}(t)$. Then we generate the same maximal subgroup F_4 but this time the structure constants do agree with those in Magma.*

3.1 How did we find this presentation?

We start with the 248-dimensional module M for $H \cong {}^3D_4(2)$. We use the Ryba space, as used in [4] (see also the description as the 'Lie product method' in [5]), to construct an explicit Lie product on M that turns M into a copy of $\mathfrak{e}_8$. This gives us explicit structure constants. The module M splits as the sum of 52- and 196-dimensional H-stable submodules. As explained above, the first of these subspaces is forced to be a subalgebra $\mathfrak{h}$ of $\mathfrak{e}_8$ isomorphic to $\mathfrak{f}_4$ and so Magma could write down a basis for it in terms of a Chevalley basis of $\mathfrak{e}_8$. However, this process left us with basis elements for $\mathfrak{h}$ with around 120 non-zero coefficients in terms of a basis of $\mathfrak{e}_8$.

We found that four of the basis vectors for $\mathfrak{h}$ were toral and commuted with each other, thus spanning a maximal toral subalgebra $\mathfrak{t}$. We then searched for a $\mathbf{G}$-conjugate of $\mathfrak{h}$ such that the corresponding conjugate of $\mathfrak{t}$ was contained in the standard toral subalgebra of $\mathfrak{g}$. To do this we used the inbuilt `InnerAutomorphism` function to construct the automorphisms of $\mathfrak{g}$ corresponding to $x_\gamma(\pm 1)$ for all roots γ in the root system of E_8.

Our strategy was to implement a naive hill climb for the first basis element t_1 of $\mathfrak{t}$. Indeed, we searched through all 480 possible conjugating elements and selected the one that yielded the largest number of zero coefficients when expressing t_1^g in terms of the basis of $\mathfrak{g}$. We remembered the elements we used at each step. This meant that when we could no longer increase the number of zero coefficients we could trace our steps back and take the next best conjugating element and continue the process. This lead to a significant increase in the number of zero coefficients but nowhere near the 240 we needed.

We then slightly upgraded our hill climb algorithm to include using a random conjugating element at fixed intervals. Every 100 steps we chose a random conjugating element and used this, regardless of what it did to the number of zero coefficients. This method was not optimized; it could be that a better choice would have been every 5 steps, or 500 steps. But this hill climb was enough for us; it quickly led us to a conjugating element g_0 which took t_1 into the standard toral subalgebra of $\mathfrak{g}$. At this point, the remaining three basis elements t_2, t_3 and t_4 were not sent by g_0 to something in the standard toral subalgebra of $\mathfrak{g}$, but they had significantly fewer non-zero coefficients. We then repeated the algorithm looking to increase the total number of zero coefficients in $t_1^g, \ldots, t_4^g$ and this quickly converged, yielding a conjugating element g_1 which sent $t_1, \ldots, t_4$ to the four toral elements corresponding to the generators of the maximal torus given in the presentation. From the toral subalgebra of $\mathfrak{h}^{g_1}$ it was then routine to take a Cartan decomposition and find the corresponding root elements.

It turned out that the root elements of $\mathfrak{h}$ were expressed as a sum of commuting root elements of $\mathfrak{e}_8$; in fact long root elements of $\mathfrak{h}$ were of type $2A_1$ in $\mathfrak{e}_8$, whereas short root elements were of type $4A_1$. For pairwise commuting root elements $e_{\alpha_1}, \ldots, e_{\alpha_k}$ of $\mathfrak{e}_8$, the operators $\operatorname{ad} e_{\alpha_i}$ pairwise commute and so one has

$$\exp(t_1 \operatorname{ad} e_{\alpha_1} + \cdots + t_j \operatorname{ad} e_{\alpha_k}) = \exp(t_1 e_{\alpha_1}) \ldots \exp(t_k e_{\alpha_k}).$$

Thus if $e_{\beta_i} = \sum t_j e_{\alpha_j} \in \mathfrak{h}$ is of this form, then evidently the left-hand side of the displayed equation normalizes $\mathfrak{h}$ in the group $\mathrm{GL}_{248}(k)$, and the right-hand side belongs to $E_8(k)$, hence for $t \in k$, the elements $x_{\beta_i}(t)$ generate a connected smooth subgroup $\mathbf{Y}$ of E_8 for which $\mathfrak{h} \subseteq \mathrm{Lie}(\mathbf{Y})$. But now maximality of $\mathfrak{h}$ forces $\mathrm{Lie}(\mathbf{Y}) = \mathfrak{h}$. The only connected smooth affine k-group whose Lie algebra is a simple Lie algebra of type F_4 is a group of type F_4 itself and using the maximality of $\mathfrak{f}_4$ we conclude that $\mathbf{Y}$ must be a maximal connected subgroup.

4 Consequences

We extend the results of [10] to this new maximal subgroup, determining which unipotent classes of E_8 meet the F_4 non-trivially.

Class in F_4	Class in E_8
A_1	$2A_1$
$\tilde{A}_1$	$4A_1$
$A_1+\tilde{A}_1$	A_2+2A_1
A_2	$2A_2$
$\tilde{A}_2$	$2A_2$
$A_2+\tilde{A}_1$	$2A_2+A_1$
$\tilde{A}_2+A_1$	$2A_2+2A_1$
B_2	$2A_3$
$C_3(a_1)$	A_4+2A_1
$F_4(a_3)$	A_4+A_2
B_3	A_6
C_3	$D_6(a_1)$
$F_4(a_2)$	D_5+A_2
$F_4(a_1)$	$E_8(b_6)$
F_4	$E_8(b_4)$

Table 4.1: Fusion of unipotent classes of maximal F_4 into E_8. (Horizontal lines separate elements of different orders.)

Proposition 4.1. *If u is a unipotent element of the maximal F_4 subgroup of E_8, then the class of u in F_4 and E_8 is given in Table 4.1.*

Proof. The proof is a fast computer check. Randomly generate elements u of orders 3, 9 and 27 in $F_4(3)$ until we hit each class. (The class to which u belongs can be deduced from [8, Tables 3 and 4].) The Jordan blocks of the action of u on the sum of $L(1000)$ and $L(0010)$ are trivial to compute then. From [8] we obtain the class in E_8 to which u belongs.

However, note that there is an error in [8], due to an error in [16], which leads to a single class having the wrong Jordan blocks in characteristic 3. This is corrected in [9], and it concerns exactly the class $E_8(b_6)$ in the table. It has Jordan blocks $9^{26}, 7, 3^2, 1$ on $L(E_8)$, not $9^{25}, 8^2, 2^2, 1^3$ as stated in [8]. With this correction, the Jordan block structure of u on $L(E_8)$ determines the unipotent class to which u belongs, and thus we are done. □

For completeness we do the same thing for nilpotent orbits of $\mathfrak{f}_4$.

Proposition 4.2. *If x is a nilpotent element of the maximal $\mathfrak{f}_4$ subalgebra of $\mathfrak{e}_8$, then the class of x in $\mathfrak{f}_4$ and $\mathfrak{e}_8$ is given in Table 4.2.*

Proof. Using the root elements constructed for $\mathfrak{f}_4$ we find a set of orbit representatives for $\mathfrak{f}_4 \subset \mathfrak{e}_8$ using [18, Appendix]. For each representative x we use Magma to calculate the Jordan block structure for the adjoint action of x and the normalizer of each term of the derived series of $C_{\mathfrak{e}_8}(x)$. Using [7, Proposition 1.5], we then find the $\mathfrak{e}_8$ class of x. □

Proposition 4.3. *If $\mathbf{M}$ is a maximal connected reductive subgroup of the maximal F_4 subgroup $\mathbf{X}$ of the algebraic group $\mathbf{G}$ of type E_8, then $\mathbf{M}$ is conjugate to one of the following four subgroups.*

(i) $\mathbf{M}_1 = B_4 < D_8$ *embedded via the spin module $L_{B_4}(0001)$. There are two such conjugacy classes which, when p is odd, are distinguished by whether or not they are contained in a maximal subgroup of type A_8. This is the subgroup which is not contained in such an A_8. It is $\mathbf{G}$-irreducible and denoted $E_8(\#45)$ in [20].*

(ii) $\mathbf{M}_2 = A_1C_3 < D_8$ *embedded via $L_{A_1}(2) \oplus L_{C_3}(010)$. This subgroup is $\mathbf{G}$-irreducible and denoted by $E_8(\#774)$ in [20].*

(iii) $\mathbf{M}_3 = A_1G_2 < A_1E_7$ *embedded as follows: E_7 has a maximal subgroup of type A_1G_2 (when $p \neq 2$). Therefore A_1E_7 has a maximal subgroup $A_1A_1G_2$ and M_3 is embedded diagonally in this subgroup.*

Class in $\mathfrak{f}_4$	Class in $\mathfrak{e}_8$
A_1	$2A_1$
$\tilde{A}_1$	$4A_1$
$A_1 + \tilde{A}_1$	$A_2 + 2A_1$
A_2	$2A_2$
$\tilde{A}_2$	$A_2 + 3A_1$
$A_2 + \tilde{A}_1$	$2A_2 + A_1$
B_2	$2A_3$
$\tilde{A}_2 + A_1$	$2A_2 + A_1$
$C_3(a_1)$	$A_4 + 2A_1$
$F_4(a_3)$	$A_4 + A_2$
B_3	A_6
C_3	$D_5 + A_1$
$F_4(a_2)$	$E_7(a_4)$
$F_4(a_1)$	$E_6(a_1)$
F_4	E_6

Table 4.2: Fusion of nilpotent classes of maximal $\mathfrak{f}_4$ into $\mathfrak{e}_8$.

One has to twist the embedding in the first A_1 factor by the Frobenius morphism. This subgroup is again $\mathbf{G}$-irreducible and denoted by $E_8(\#967^{\{1,0\}})$ in [20].

(iv) $\mathbf{M}_4 = A_2A_2 < \bar{A}_2E_6$ *embedded as follows: E_6 has a maximal subgroup A_2G_2, and G_2 has a maximal subgroup $\tilde{A}_2$ generated by short root subgroups of the G_2 when $p = 3$. Thus $\bar{A}_2E_6$ has a subgroup $H = \bar{A}_2A_2\tilde{A}_2$ (denoted $E_8(\#1012)$ in [20]). The first A_2 factor of $\mathbf{M}_4$ is the the second A_2 factor of H and the second A_2 factor of $\mathbf{M}_4$ is diagonally embedded in the first and third factors of H (with no twisting by field or graph automorphisms). Note that $\mathbf{M}_4$ is not $\mathbf{G}$-irreducible.*

Proof. By [13, Corollary 2], F_4 has four conjugacy classes of maximal connected subgroups in characteristic 3, which are indeed of types B_4, A_1C_3, A_1G_2 and A_2A_2. The first two maximal subgroups are centralizers of involutions. It follows from the action of F_4 on the Lie algebra of E_8 that the centralizer in E_8 of both of these involutions is D_8. Thus B_4 and A_1C_3 are contained in a maximal subgroup of type D_8. By [11, Table 8.1], there are only three E_8-conjugacy classes of B_4 subgroups in D_8. Calculating the composition factors of the action of $B_4 < F_4$ on the Lie algebra of E_8 yields that it is indeed conjugate to $\mathbf{M}_1$, which is $\mathbf{G}$-irreducible by [20, Theorem 1]. Calculating the composition factors of $A_1C_3 < F_4$ on the Lie algebra of E_8, we find that it has no trivial composition factors. Therefore it must be $\mathbf{G}$-irreducible (by [20, Corollary 3.8]) and we may use the classification of irreducible subgroups determined in [20]. In particular, the composition factors on the Lie algebra of E_8 are enough to determine conjugacy and it follows that A_1C_3 is conjugate to $\mathbf{M}_2$, as required.

Another calculation shows that $A_1G_2 < F_4$ has no trivial composition factors on the Lie algebra of E_8. We can therefore use the same method as the previous case to deduce that it is conjugate to $\mathbf{M}_3$.

For $\mathbf{AB} = A_2A_2 < F_4$ we start by considering the first A_2 factor $\mathbf{A}$, which we define to be the A_2 subgroup generated by long root subgroups of F_4. This is the derived subgroup of a long A_2-Levi subgroup, and is thus a subgroup of B_4. Since B_4 is conjugate to $\mathbf{M}_1$, we find that $\mathbf{A}$ is contained in a maximal subgroup of type A_8 and acts as $L_{A_2}(10) \oplus L_{A_2}(01) \oplus L_{A_2}(00)^{\oplus 3}$ on the natural 9-dimensional module of A_8. Therefore, $\mathbf{A}$ is a diagonal subgroup (without field or graph twists) of the derived subgroup of an A_2A_2-Levi subgroup of E_8. We now claim that the connected centralizer of $\mathbf{A}$ in E_8 is $\bar{A}_2G_2$. Indeed, we use [20, Theorem 1] to see that $\mathbf{C} = \mathbf{A}\bar{A}_2G_2$ is E_8-irreducible (denoted by $E_8(\#978)$) and the only reductive connected overgroups of $\mathbf{C}$ are the maximal subgroups $\bar{A}_2E_6$ and G_2F_4. Therefore $\bar{A}_2G_2 \leq C_{E_8}(\mathbf{A})^\circ$ and it must be an equality since $\mathbf{A}C_G(\mathbf{A})^\circ$ is E_8-irreducible since $\mathbf{C}$ is.

Therefore, $\mathbf{AB}$ is contained in $\bar{A}_2E_6$ with $\mathbf{B}$ contained in $\bar{A}_2G_2$. It is straightforward to list all A_2 subgroups of $\bar{A}_2G_2$, noting that G_2 has precisely two classes of A_2 subgroups when the characteristic is 3. Computing the composition factors of $\mathbf{B}$ on the Lie algebra of E_8 shows that $\mathbf{B}$ is conjugate to the subgroup claimed and hence **AB** is conjugate to $\mathbf{M}_4$. The fact that $\mathbf{M}_4$ is **G**-reducible follows from [20, Theorem 1]. □

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