J. Group Theory **13** (2010), 873–890
DOI 10.1515/JGT.2010.029

Journal of Group Theory

Lower bounds for representation growth

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(Communicated by J. S. Wilson)

Abstract. This article examines lower bounds for the representation growth of finitely generated (particularly profinite and pro-p) groups. It also considers the related question of understanding the maximal multiplicities of character degrees in finite groups, and in particular simple groups.

1 Introduction

The representation growth of residually finite (particularly profinite) groups is a relatively new area of research ([15], [11]), not yet as rich as the study of subgroup growth [16]. In this work we study lower bounds for the representation growth of pro-p and profinite groups, and the connected topic of character degree multiplicities for finite groups.

Let G be a finitely generated, residually finite group, and let $r_n(G)$ be the number of inequivalent, complex irreducible representations of G of dimension n, whose kernels have finite index. If G is a finitely generated profinite group, $r_n(G)$ is the number of *continuous* complex irreducible representations of dimension n. When we say 'finitely generated, residually finite group', we allow the case where G is a finitely generated profinite group. The main aim of representation growth is to relate the arithmetic properties of the sequence $(r_n(G))$ with algebraic properties of the group. A group G is *FAb* (short for 'finite abelianizations') if all finite-index subgroups of G have finite abelianizations; it is well known [1, Proposition 2] that all of the $r_n(G)$ are finite if and only if G is FAb.

We begin with two contrasting theorems on bounding from below the sequences $(r_n(G))$ and $(R_n(G))$, the latter being the partial sums of the sequence $(r_n(G))$.

Theorem A. *There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $f(x) \to \infty$ as $x \to \infty$ with the following property: for any infinite, finitely generated, residually finite group G with FAb, $r_n(G)$ is greater than $f(n)$ infinitely often. In particular, for any such G, the sequence $(r_n(G))$ contains a subsequence that tends to infinity.*

Theorem B. *Let f be a divergent, non-decreasing function. There exists an infinite, finitely generated profinite group G such that $R_n(G) < f(n)$ for infinitely many n. In* *other words, there is no divergent, non-decreasing function f such that $f(n) < R_n(G)$ for every infinite, finitely generated profinite group G, and for all sufficiently large n.*

These theorems suggest that requiring $r_n(G)$ (or $R_n(G)$) to be less than a function f infinitely often should be the 'correct' concept of a lower bound for representation growth.

Our next result concerns groups G for which at least one, but only finitely many, of the numbers $r_n(G)$ are infinite.

Theorem C. *Let G be an infinite, finitely generated, residually finite group. Let $I(G)$ be the set of all natural numbers i such that $r_i(G) = \infty$. The following are equivalent*:

(1) *$I(G)$ is finite and non-empty*;

(2) *$r_i(G) = 0$ for all sufficiently large i; and*

(3) *G is virtually abelian.*

Using Theorems A and C, we get the following tetrachotomy for finitely generated, residually finite groups.

Corollary D. *Let G be a finitely generated, residually finite group. Let $I(G)$ be the set of all natural numbers i such that $r_i(G) = \infty$. Exactly one of the following possibilities holds*:

(1) *$I(G) = \varnothing$ and only finitely many of the $r_i(G)$ are non-zero*;

(2) *$I(G) = \varnothing$ and the sequence $(r_i(G))$ contains a subsequence that tends to infinity*;

(3) *$0 < |I(G)| < \infty$, and only finitely many of the $r_i(G)$ are non-zero*; *and*

(4) *$I(G)$ is infinite.*

In the first case, G is finite, in the second, G has FAb and is infinite, and in the third case G is infinite and virtually abelian.

In the final case, using simply the sequence $(r_n(G))$, and its associated objects like zeta functions, it seems unlikely that very much can be said. (We should note that if G is a finitely generated nilpotent group, we can study so-called *twist isoclasses*: see [14, Theorem 6.6].)

We move on to symmetric groups; the process in [3] to generate irreducible characters with the same degree is constructive, and in Section 4 we derive an explicit bound, proving the following result.

Theorem E. *Let n be an integer, and let $X(n)$ denote the multiset of the degrees of the irreducible characters of the symmetric group S_n. Let $m(n)$ denote the largest of the multiplicities of the elements of $X(n)$. For all sufficiently large n, $m(n) \geqslant n^{1/7}$.*

In Section 4 we derive a more complicated explicit bound. We conjecture that a similar upper bound holds.

Conjecture F. Let n be an integer, and let $X(n)$ denote the multiset of the degrees of the irreducible characters of the symmetric group S_n. Let $m(n)$ denote the largest of the multiplicities of the elements of $X(n)$. There are positive constants ε_1 and ε_2 such that, for all sufficiently large n,

$$n^{\varepsilon_1} < m(n) < n^{\varepsilon_2}.$$

When one considers the *order* of the symmetric group instead of the *degree* of the symmetric group, then all the functions n^{ε} become functions of the same order; this conjecture would yield the exact rate of growth of $m(n)$ in terms of $|S_n| = n!$. In Section 5, we compare this growth (or rather, that of the alternating groups A_n) to that of the other finite simple groups, and prove that the alternating groups are the finite simple groups G for which the maximum $m(G)$ of the $r_n(G)$ grows slowest relative to the order of the group G.

In Section 6 we turn our attention to p-groups, using known results on conjugacy classes of p-groups to derive bounds for the growths of $r_n(G)$ and the partial sums $R_n(G)$. After considering $r_n(G)$ and $R_n(G)$ for powerful pro-p groups and the Nottingham group, we consider all pro-p groups. The strongest result that we derive here is a consequence of a remarkable recent theorem of Jaikin-Zapirain [8] (stated here as Theorem 6.5), which resolves a problem first posed by Pyber in [19].

Theorem G. *There exists a constant c such that, if G is an infinite, finitely generated pro-p group, then for all sufficiently large n,*

$$R_{p^n}(G) \geqslant cn\frac{\log_p n}{\log_p \log_p n}.$$

and for infinitely many n,

$$r_{p^n}(G) \geqslant 2c\frac{\log_p n}{\log_p \log_p n}.$$

A weaker result than the second part of Theorem G first appeared in [7].

2 Proof of Theorem A

The main tool for the proof of Theorem A is the following result from [3], itself depending on results from [7] and [17].

Theorem 2.1 ([3, Corollary 1.3]). *There exists a function f such that, if G is a finite group, and m(G) denotes the maximum of the set* $\{r_n(G) : n \in \mathbb{N}\}$, *then* $|G| \leqslant f(m(G))$.

We will briefly mention how this theorem is proved. It relies on two special cases of this result, for p-groups and for finite simple groups. For p-groups, this is Theorem

G, and for simple groups Sections 4 and 5 give the result (see also [17] and [3]). Using these two explicit computations, we can firstly give a bound for soluble groups, and then extend this to all finite groups using the generalized Fitting subgroup.

Write $\ell : \mathbb{N} \to \mathbb{N}$ for the function

$$\ell(n) = \min_{|G| \geqslant n} m(G).$$

By Theorem 2.1, this function is well defined and non-decreasing. Also, for every finite group G of order n, we have $r_i(G) \geqslant \ell(n)$ for some $i < \sqrt{n}$. In fact, since $|G| \geqslant r_j(G) j^2$ for all j, we see that

$$i \leqslant \sqrt{\frac{n}{\ell(n)}}.$$

Thus if H is a quotient of G, and H has order n, then there exists $i \leqslant \sqrt{n/\ell(n)}$ such that $r_i(G) \geqslant \ell(n)$.

Now let G be an infinite, finitely generated, residually finite group, and suppose that G is FAb. There is an infinite sequence $H_1, H_2, \dots$ of finite quotients of G with $|H_i| < |H_{i+1}|$ for all i. For H_i of order n_i, let j_i denote the natural number, at most $\sqrt{n_i/\ell(n_i)}$, such that $r_{j_i}(H_i) \geqslant \ell(n_i)$. Therefore

$$r_{j_i}(G) \geqslant \ell(n_i),$$

and thus the $r_{j_i}(G)$ form a subsequence of the $r_m(G)$ that tends to infinity, bounded below by $\bar{\ell}(n_i)$, proving Theorem A.

This theorem proves that the sequence $(r_n(G))$ contains a subsequence that tends to infinity, but for the $R_n(G)$, we can get reasonable growth bounds that are likely to be close to sharp. Using a theorem of Pyber from [19], it is easy to show that

$$R_n(G) \geqslant c \frac{\log n}{(\log \log n)^8}$$

for infinitely many n, since he proves that for all finite groups G of order at least 4, we have $k(G) \geqslant c \log|G| / (\log \log|G|)^8$ for some $c > 0$, where $k(G)$ is the number of conjugacy classes of G. However, a more involved argument, due to Andrei Jaikin-Zapirain, proves something considerably better.

Proposition 2.2. *There is a constant $c > 0$ such that if G is an infinite, finitely generated profinite group with FAb, then there are infinitely many integers n for which*

$$R_n(G) \geqslant c \log n (\log \log n)^{1-\varepsilon},$$

for any $\varepsilon > 0$.

Proof. Let G be an infinite, finitely generated profinite group. If G possesses infinitely many maximal subgroups, then G maps onto infinitely many finite groups with trivial Frattini subgroup, in particular onto such groups with arbitrarily large order. At the end of the proof of [19, Theorem A], Pyber proves that if H is a finite group with trivial Frattini subgroup, then

$$k(H) \geqslant 2^{c(\log|H|)^{1/8}}$$

for some constant $c > 0$; in particular, all quotients H with trivial Frattini subgroup satisfy $R_{|H|}(H) \geqslant 2^{c(\log|H|)^{1/8}}$, and so therefore does G for infinitely many $n = |H|$. This is well above the bound needed.

Thus we may assume that G possesses only finitely many maximal subgroups, in which case G is virtually pro-nilpotent, with pro-nilpotent subgroup H of finite index. In this case, Theorem G proves that $R_n(H)$ grows at least as quickly as $c \log n (\log\log n)^{1-\varepsilon}$ for any $\varepsilon > 0$, and hence so does $R_n(G)$, as required. (To move between $R_n(H)$ and $R_n(G)$, we note that if N is a normal subgroup of a finite group G, then $k(G) \geqslant k(N)/|G:N|$ (see e.g. [19, Lemma 2.1 (ii)]), so if $|G:H|$ is fixed, $R_n(G)$ grows with the same order as $R_n(H)$.) □

It seems that pro-nilpotent groups (or pro-p groups) are the bounding case in this result. In particular, if G is an infinite, finitely generated group with many simple quotients, then $2^{c(\log n)^{1/8}}$ is the slowest that $R_n(G)$ can grow, at least for infinitely many n. We will return to this concept of only being able to bound infinitely many $R_n(G)$ from below in Section 7.

3 Proof of Theorem C

We start with a lemma, which gives us extra information in the case where $r_1(G) = \infty$.

Lemma 3.1. *Let G be a finitely generated group. If $r_1(G) = \infty$ then for all n, either $r_n(G) = 0$ or $r_n(G) = \infty$.*

Proof. Suppose that G has a representation ϕ of degree n, with kernel K, and let ψ be a 1-dimensional representation, with kernel H. If $\phi \otimes \psi = \phi$, then it must be that H contains K, since otherwise the kernel of $\phi \otimes \psi$ would not be K. Thus for each representation ϕ of degree n, there are only finitely many 1-dimensional representations ψ such that $\phi \otimes \psi = \phi$, and so $r_n(G) = \infty$, as claimed. □

We now prove Theorem C. Firstly, if G is infinite and virtually abelian, then it has some infinite abelian subgroup H of index n. If ρ is a representation of G, then an irreducible constituent ϕ of $\rho\downarrow_H$ is 1-dimensional, and so by Frobenius reciprocity ρ has dimension at most n, giving (iii) implies (ii); that (ii) implies (i) is clear.

Suppose that $I(G)$ is finite and non-empty; since G is not FAb, choose a subgroup H of index n with $|H/H'|$ infinite. Suppose that there are infinitely many i such that $r_i(H) = \infty$, and write $I = I(H)$. Let $J = I(G)$ be the corresponding (finite) set for G. We will derive a contradiction, proving that I is finite.

Let ρ be an irreducible representation of H, of dimension m. Since $(\rho \uparrow^G) \downarrow_H$ has ρ as a constituent, there is some constituent ψ of $\rho \uparrow^G$ such that ρ is a constituent of $\psi \downarrow_H$; thus $m \leqslant \dim \psi$. Note that $\rho \uparrow^G$ has dimension nm, and so $\dim \psi \leqslant nm$.

Let $a \in I$ be greater than any element of J. There are infinitely many (inequivalent) representations ρ_i of H of dimension a, and thus there must be infinitely many representations ψ_i such that ρ_i is a constituent of $\psi_i \downarrow_H$ and $\dim \psi_i$ lies between a and am. Since there are only finitely many constituents of a given representation, this implies that $r_c(G)$ is infinite for some $a \leqslant c \leqslant am$, a contradiction. Thus $I(H)$ is finite, and so by Lemma 3.1 we have $r_i(H) \neq 0$ for only finitely many i; let c be the largest dimension of an irreducible representation of H. By a well-known theorem of Jordan [9], for each finite quotient H/K of H, there is an abelian normal subgroup W/K such that $|H/W|$ is bounded by $d = f(c)$ for some non-decreasing function f. Since H is finitely generated, there are only finitely many subgroups of index at most d, so let A be the intersection of all such subgroups, necessarily a normal subgroup of finite index in H. We claim that A is residually (finite abelian), and is hence abelian. This proves that H, and hence G, is virtually abelian, as required.

Let

$$H = H_1 \geqslant H_2 \geqslant H_3 \geqslant \cdots$$

be a descending sequence of normal subgroups of finite index of H such that $\bigcap H_i = 1$. Let $A_i = A \cap H_i$, and note that the A_i is a descending sequence of normal subgroups of finite index of A such that $\bigcap A_i = 1$. Since H/A_i is a finite group, there is some abelian subgroup B/A_i of index at most d, and by construction $B \geqslant A$, so A/A_i is abelian, as required.

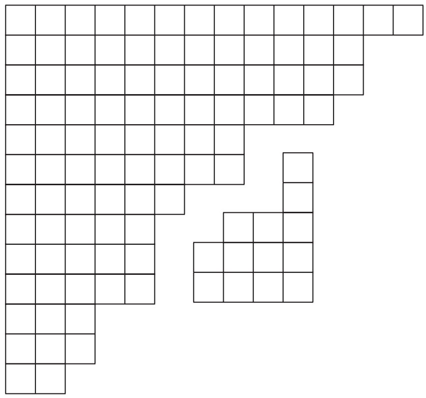
Corollary D follows from Theorems A and C.

4 Degree multiplicity for symmetric groups

This section relies on work of the author in [3], and we will briefly recall what is involved there. There is a standard bijection between the irreducible characters of the symmetric group of degree n and the partitions of n, with the degree of a particular character calculable from the corresponding partition, via *hook numbers*. We presume that the reader is familiar with this method, and we will pause to fix notation only.

If λ is a partition, write $|\lambda|$ for the number of which λ is a partition, and λ' for the conjugate of λ. Let $t(\lambda)$ be the sum of the number of rows of λ and the number of columns of it. Let $H(\lambda)$ denote the multiset of all hook numbers of λ. If $H(\lambda) = H(\mu)$, then the characters corresponding to λ and μ have the same degree.

To any partition, one may associate the *enveloping partition* $E(\lambda)$, which is constructed in [3], and is illustrated here by example. One takes a square of length $t(\lambda)$, appends a copy of λ to the left and to the bottom of the square, and removes a reflected copy of λ from the bottom-right portion of the square. This is the enveloping partition of $(5,3,3,2)$:



If λ is a partition, and $t = t(\lambda)$, then write $E(\lambda)_i$ for the partition got from $E(\lambda)$ by incrementing the first t rows of $E(\lambda)$ by i. In [3], the following theorem is proved.

Theorem 4.1 ([3, Theorem 7.1]). *Suppose that λ and μ are partitions, and that $H(\lambda) = H(\mu)$. Write t for the sum of the number of rows and the number of columns of λ. (This is the same as that for μ.) Then*

$$H(E(\lambda)_i) = H(E(\mu)_i).$$

If we start with a partition λ that is not self-conjugate, then the partitions λ and $\lambda' \neq \lambda$ have the same hook numbers. From these two partitions, we may construct four partitions with the same hook numbers, namely

$$E(\lambda)_1,\ E(\lambda')_1,\ E(\lambda)'_1,\ \text{and}\ E(\lambda')'_1.$$

If $|\lambda| = n$ and $t = t(\lambda)$, then all four of these partitions are partitions of $n + t^2 + t$. This procedure can be iterated, to produce, given a partition λ, a set of 2^i partitions with the same hook numbers. Here we will calculate the smallest integer N such that it can be guaranteed using this procedure that for all $n \geqslant N$, there are 2^i different partitions of n, each of which has the same hook numbers.

Firstly, given a partition λ with $|\lambda| = n$ and $t(\lambda) = t$, we need to calculate the size of the partition got by applying the above procedure of taking $\mu \mapsto E(\mu)_1$ a number, say i, of times. Let f denote the function on the set of all partitions given by $f(\mu) = E(\mu)_1$, and write $n_1 = n$ and $t_1 = t$. It is clear that

$$|E(\lambda)_1| = n + t + t^2 \quad \text{and} \quad t(E(\lambda)_1) = 3t + 1.$$

Therefore, if n_i and t_i denote the size and row and column sum of $f^{(i-1)}(\lambda)$ (i.e., f applied $i-1$ times to λ), we see that

$$t_i = 3t_{i-1} + 1, \quad n_i = n_{i-1} + t_{i-1} + t_{i-1}^2.$$

The first recurrence is easily solved to get

$$t_i = 3^{i-1}t + \frac{3^{i-1}-1}{2},$$

and solving the second recurrence relation yields

$$n_i = n + \frac{(4t^2+4t+1)(9^{i-1}-1)}{32} - \frac{i-1}{4}.$$

The equations above imply that given a partition λ that is not self-conjugate with $|\lambda| = n$ and $t(\lambda) = t$, one may construct 2^i partitions with row and column sum t_i, and by extending the first t_{i-1} rows by j each, they may be taken to have sizes $n_i + jt_{i-1}$ for all $j \geqslant 0$.

The idea is to find t_{i-1} partitions, each of which has the same row and column sum t, and whose sizes cover the t_{i-1} congruence classes modulo t_{i-1}. Therefore for some integer N we would have found 2^i partitions of size N with the same hook numbers, and for all subsequent integers as well.

Suppose that a partition λ has $t(\lambda) = t$. Furthermore, suppose that t is odd (so that λ is definitely not self-conjugate), and write $t = 2r+1$. Then the largest that $|\lambda|$ can be is $(t^2-1)/4$ (which is a rectangle of sides r and $r+1$), and the smallest that $|\lambda|$ can be is $t-1$ (which is a hook). Furthermore, it is easy to see that every possible size between these two can be given by a partition that is not self-conjugate. Thus given a row and column sum t, there are $(t^2-4t+7)/4$ different possibilities for n, and these possibilities form an interval.

Finally, since $t_{i-1} = 3^{i-2}t + (3^{i-2}-1)/2$, we see that there are enough partitions if

$$t^2 - 4t + 7 \geqslant 4 \cdot 3^{i-2}t + 2(3^{i-2}-1).$$

Using the quadratic formula, we get the exact solution

$$t = 2(1+3^{i-2}) \pm \sqrt{4(1+3^{i-2})^2 + 2 \cdot 3^{i-2} - 9},$$

and we take the approximate solution

$$t = 5 + 4 \cdot 3^{i-2},$$

which guarantees that there are enough partitions. Notice that the smallest value of n is $t-1$, and therefore substituting these values into the equation for n_i gives

$$n_i = 4 + 4 \cdot 3^{i-2} + \frac{(4(5+4\cdot 3^{i-2})^2 + 4(5+4\cdot 3^{i-2})+1)(9^{i-1}-1)}{32} - \frac{i-1}{4}.$$

Thus we have proved that the symmetric group S_n has 2^i irreducible characters of the same degree if

$$n \geqslant \frac{15 - 16 \cdot 3^{i-1} + 1025 \cdot 9^{i-2} + 1584 \cdot 27^{i-2} + 576 \cdot 81^{i-2} - 8i}{32},$$

as required.

This is far from optimal. In [3], it was shown that for all $n \geqslant 22$ there are four partitions with the same hook numbers, whereas this strategy proves it only for $n \geqslant 98$. For eight partitions, this method requires $n \geqslant 3078$, and while the real bound is not known precisely, it is known to be true for $n \geqslant 200$. In general, however, there appears to be no easy improvement on the method above.

5 The other finite simple groups

Apart from the alternating groups, the finite simple groups are the sixteen classes of groups of Lie type, together with the twenty-six sporadic simple groups. In terms of asymptotic group theory, the sporadic groups are unimportant, but we briefly mention the maximal degree multiplicities of the sporadic groups in a table, derived from the information in [2].

G	$m(G)$	G	$m(G)$	G	$m(G)$
M_{11}	3	Co_3	3	B	2
M_{12}	3	McL	2	M	3
M_{22}	2	Suz	3	J_1	3
M_{23}	3	He	3	ON	3
M_{24}	3	HN	3	J_3	3
HS	3	Th	2	Ru	3
J_2	2	Fi_{22}	4	J_4	3
Co_1	2	Fi_{23}	3	Ly	5
Co_2	3	Fi'_{24}	2	T	2

(We include the Tits group $T = {}^2F_4(2)'$ here, since it is 'semi-sporadic', and not really one of the Ree groups ${}^2F_4(2^{2n+1})$.) What is interesting here is that, with the exception of the Lyons and smallest Fischer groups, all of the sporadic groups have maximal multiplicity either 2 or 3. In particular, if G is a finite simple group and $m(G) = 2$, then $|G|$ is at most that of the Baby Monster, and if $m(G) = 3$, then $|G|$ is at most that of the Monster.

For the alternating groups, it is easy to see that

$$\frac{2}{5} m(S_n) \leqslant m(A_n) \leqslant \frac{5}{2} m(S_n),$$

using Clifford theory.

Lemma 5.1. *Let G be a finite p-group, and let N be a normal subgroup of index p^n. We have*

$$\left(\frac{p}{p^2+1}\right)^n m(G) \leqslant m(N) \leqslant \left(\frac{p^2+1}{p}\right)^n m(G).$$

Proof. Firstly assume that $n = 1$. Suppose that there are m characters of G of the same degree. There are pi of them that restrict to i irreducible characters ψ of N (as p of them restrict to each such ψ), and $m - pi$ of them that do not restrict to an irreducible character, and instead restrict to $p(m - pi)$ characters of N (with the same degree).

The case where there are the fewest characters of the same degree in N is when $i = p(m - pi)$, and so $i = mp/(p^2 + 1)$. Hence $m(N) \geqslant m(G) \cdot p/(p^2 + 1)$. However, by Frobenius reciprocity the situation is exactly the same for induction from N to G, and so $m(G) \geqslant m(N) \cdot p/(p^2 + 1)$. A simple induction completes the proof. □

Of course, this leaves only the groups of Lie type, so fix a Lie-type group $G = G(q)$. It is known [12, Theorem 1.7] that the orders, character degrees, and character degree multiplicities, of G are polynomials in q (dependent on the type, but just the Lie rank of the group determines a lot). For the exceptional groups, these polynomials are known, and have been collated by Lübeck; they are currently available on his website [13]. However, these data are only for the adjoint and simply connected versions of the groups, and so the simple group is not given for ${}^{\varepsilon}E_6(q)$ and $E_7(q)$ (for certain values of q). Using some elementary Clifford theory and the tables of character degrees, it is possible to still get the maximal multiplicities for the simple groups.

For the general group of Lie type, the polynomials are in the order q of the finite field over which the group lies. For the Suzuki and Ree groups ${}^2B_2(q)$, ${}^2G_2(q)$, and ${}^2F_4(q)$ we use the notation $q^2 = p^{2n+1}$, where p is either 2 or 3. In [12], Liebeck and Shalev prove that if the Lie rank of $G(q)$ is ℓ, then $m(G(q))$ is a polynomial with degree ℓ; in the tables below, we reproduce the *exact* polynomial for $m(G(q))$ (and the character degree at which it is attained) for each of the exceptional groups, and describe afterwards the small values of q for which the table is incorrect.

For the *simple* group $G(q)$, with q odd, the multiplicities are given in Table 5.1, and for q even, the multiplicities are given in Table 5.2.

We should describe briefly how to determine the values in the table for ${}^{\varepsilon}E_6(q)$ and $E_7(q)$ when there is a non-trivial centre. For $E_6(q)$ for example, there are $m = (q^4 - 1)(q^2 - 1)/8$ characters ϕ of the degree given in the table for the adjoint group $E_6(q).3$, and since there are no characters of degree $(\deg \phi)/3$ for the simply connected group $3 \cdot E_6(q)$, there are $m/3$ characters of degree $\deg \phi$ for the simple group $E_6(q)$. Also, if ψ is a character of $3 \cdot E_6(q)$ then there are no characters of degree $\deg \psi/3$ for $3 \cdot E_6(q)$, and so it suffices to consider those character degrees for $E_6(q).3$ whose multiplicities exceed $m/3$; in all cases it is easy to see that one gets fewer than m characters with the same degree for the simple group. The technique is similar for ${}^2E_6(q)$ and $E_7(q)$.

Group	Degree	Multiplicity
$G_2(q)$	Y_6	$(q-1)^2/2$
${}^2G_2(q)$	q^6+1	$(q^2-3)/2$
${}^3D_4(q)$	$Y_6(q^4-q^2+1)(q^2-q+1)$	$(q^4-2q+1)/4$
$F_4(q)$	$Y_{12}Y_8Y_2(q^2-q+1)$	$q^2(q^2-1)/6$
$E_6(q)$	$Y_{12}Y_9Y_6Y_5Y_4$	$(q^4-1)(q^2-1)/8\gcd(q-1,3)$
${}^2E_6(q)$	$Y_{18}Y_{12}Y_{10}Y_6Y_4/Y_9Y_5$	$(q^4-1)(q^2-1)/8\gcd(q+1,3)$
$E_7(q)$	$Y_{18}Y_{12}Y_{10}Y_8Y_7Y_6Y_2$ $Y_{18}Y_{14}Y_{12}Y_{10}Y_8Y_6Y_2/Y_7$	$q(q^6-1)/28$
$E_8(q)$	$Y_{30}Y_{24}Y_{20}Y_{18}Y_{14}Y_{12}Y_2$	$(q^4-1)(5q^4-2q^3-7)/64$

Table 5.1. Multiplicities of character degrees for exceptional groups of Lie type for odd q. (Here, $Y_i = q^i - 1$.)

Group	Degree	Multiplicity
${}^2B_2(q)$	q^4+1	$(q^2-2)/2$
$G_2(q)$	Y_6	$q(q-2)/2$
${}^3D_4(q)$	$Y_6(q^4-q^2+1)(q^2-q+1)$	$q(q^3-2)/4$
$F_4(q)$	$Y_{12}Y_8Y_2(q^2-q+1)$	$q^2(q^2-1)/6$
${}^2F_4(q)$	$Y_{24}(q^4+1)/(q^4+q^2+1)$	$q^2(q^2-2)/4$
$E_6(q)$	$Y_{12}Y_9Y_6Y_5Y_4$	$q^4(q^2-1)/8\gcd(q-1,3)$
${}^2E_6(q)$	$Y_{18}Y_{12}Y_{10}Y_6Y_4/Y_9Y_5$	$q^4(q^2-1)/8\gcd(q+1,3)$
$E_7(q)$	$Y_{18}Y_{12}Y_{10}Y_8Y_7Y_6Y_2$ $Y_{18}Y_{14}Y_{12}Y_{10}Y_8Y_6Y_2/Y_7$	$q(q^6-1)/14$
$E_8(q)$	$Y_{30}Y_{24}Y_{20}Y_{18}Y_{14}Y_{12}Y_2$	$q^4(5q^4-2q^3-8)/64$

Table 5.2. Multiplicities of character degrees for exceptional groups of Lie type for even q. (Here, $Y_i = q^i - 1$.)

Note that the maximal multiplicity of character degrees for $E_7(q)$ is realized by two different sets of characters, as suggested in the table: the first one has the smaller degree, and is also more naturally expressed as a product of polynomials of the form $(q^i - 1)$.

There are obviously some small exceptions, and these are summarized in Table 5.3. The only unresolved case is $E_7(3)$, for which the multiplicity lies between 78 and 80. Naïve Clifford theory and the information for the adjoint and simply connected versions of $E_7(3)$ appears to be not enough to determine the multiplicities.

For the classical groups, there is no known general formula for the maximal multiplicity of the character degrees, and so we use the lower bounds given in [17], as displayed in Table 5.4. (The choice of d in the table below is influenced by the requirement that the numerator in each multiplicity should be the same.)

Group	Degree	Multiplicity	Group	Degree	Multiplicity
$E_6(2)$	42826799925	8	$G_2(2)'$	7	3
$E_6(3)$	127752132719411200	84	$G_2(3)$	91	3
${}^2E_6(2)$	27498621150	5	$G_2(4)$	819	7
$E_7(2)$	5070690584338804425	9	${}^2B_2(8)$	35	3
$F_4(2)$	541450	4	${}^3D_4(2)$	351	3

Table 5.3. Exceptions in $m(G)$ for exceptional groups of Lie type for small q.

Group	$\mathrm{O}(\lvert G\rvert)$	Multiplicity
$\mathrm{PSL}_d(q)$	$\frac{q^{d^2-1}}{\gcd(q-1,d)}$	$\frac{\phi(q^d-1)}{d^2(q-1)}$
$\mathrm{PSU}_{2d}(q)$	$\frac{q^{4d^2-1}}{\gcd(q+1,2d)}$	$\frac{\phi(q^d-1)}{4d^2}$
$\mathrm{PSU}_{2d+1}(q)$	$\frac{q^{4d(d+1)}}{\gcd(q+1,2d+1)}$	$\frac{\phi(q^d-1)}{(2d+1)^2}$
$\mathrm{PSp}_{2d}(q)$	$\frac{q^{2d^2+d}}{\gcd(2,q-1)}$	$\frac{\phi(q^d-1)}{4d}$
$\mathrm{P}\Omega_{2d+1}(q)$	$\frac{q^{2d^2+d}}{\gcd(2,q-1)}$	$\frac{\phi(q^d-1)}{4d+2}$
$\mathrm{P}\Omega^+_{2d}(q)$	$\frac{q^{2d^2-d}}{\gcd(4,q^d-1)}$	$\frac{\phi(q^d-1)}{4d}$
$\mathrm{P}\Omega^-_{2d+2}(q)$	$\frac{q^{2d^2+d+1}}{\gcd(4,q^{d+1}+1)}$	$\frac{\phi(q^d-1)}{4d+4}$

Table 5.4. Lower bounds for $m(G)$ for classical groups.

We aim to prove that each of these grows faster (in terms of $|G|$) than the symmetric groups can possibly do, proving that the symmetric groups are definitely the simple groups with the slowest-growing function $m(G)$ in terms of $|G|$. We will prove that, for sufficiently large $|G|$, for symmetric groups,

$$\log(\log m(G) + \log\log|G|) < (\log\log|G|)/2,$$

whereas for groups of Lie type the opposite inequality holds. This shows that the growth of $m(G)$ with respect to $|G|$ is slower for the symmetric groups than for the groups of Lie type, proving our claim.

The number of partitions of m is asymptotically

$$p(m) \sim \frac{\mathrm{e}^{a\sqrt{m}}}{bm},$$

where $a = \pi\sqrt{2/3}$ and $b = 4\sqrt{3}$, by the famous Hardy–Ramanujan asymptotic formula [5, (1.41)]. This is the number of irreducible characters of S_m, and so certainly $m(S_m)$ is bounded by this number. Written as a function of $|S_m| = m! = n$, this becomes (of the order of)

$$\frac{e^{a\sqrt{f(n)}}}{b \cdot f(n)},$$

where $f(n) = \log n/\log\log n$. By removing b from the denominator, we get a function that is definitely larger than $m(S_m)$ for sufficiently large m. Taking logarithms yields

$$\begin{aligned}\log m(S_m) &\leqslant a\sqrt{f(n)} - \log f(n)\\ &= a\sqrt{\log n/\log\log n} - \log(\log n/\log\log n)\\ &\leqslant a\sqrt{\log n/\log\log n} - \log\log n.\end{aligned}$$

Thus

$$\begin{aligned}\log(\log m(S_m) + \log\log n) &\leqslant \log(a\sqrt{\log n/\log\log n})\\ &= \log a + \frac{1}{2}(\log\log n - \log\log\log n)\\ &< \frac{1}{2}\log\log n\end{aligned}$$

for sufficiently large n. This proves the first assertion.

Moving on to the groups of Lie type, let G be a group of Lie type of the form in the table above, and let m be the dimension of the natural module for G (so that $m = d$ for $\mathrm{SL}_d(q)$, $m = 2d+1$ for $\mathrm{P}\Omega_{2d+1}(q)$, and so on). If G is untwisted, write $n = q^{m^2}$, and if G is special unitary, write $n = (q^2)^{m^2}$. In all cases, $n > |G|$ since n is equal to the total number of $m \times m$ matrices over $\mathbb{F}_q$ (or $\mathbb{F}_{q^2}$ in the twisted case).

Let us firstly consider the groups $G = \mathrm{PSL}_d(q)$, with $n = q^{d^2}$. By [6, Theorem 327], $\phi(a) \geqslant a^\delta$ for any $\delta < 1$ and all sufficiently large a. Therefore, for all sufficiently large n,

$$m(G) \geqslant \frac{\phi(q^d - 1)}{d^2(q-1)} \geqslant \frac{(q^d-1)^\delta}{qd^2\log q} \approx \frac{q^{d\delta-1}}{\log n}.$$

(The middle inequality holds for all q (even $q = 2$) since $q\log q/(q-1) > 1$ for all $q \geqslant 2$.) Taking logarithms gives

$$\log m(G) \geqslant \log\left(\frac{q^{d\delta-1}}{\log n}\right) = (d\delta - 1)\log q - \log\log n,$$

and thus

$$\begin{aligned}\log(\log m(G) + \log\log n) &\geqslant \log((d\delta - 1)\log q)\\ &> \log d + \log\delta - 1 + \log\log q\\ &> \frac{1}{2}\log\log n + \log\delta - 1,\end{aligned}$$

since $\log\log n = 2\log d + \log\log q$. The same argument works for the other classical groups, completing the proof of our claim.

6 Representation growth of *p*-groups

For powerful p-groups (i.e., groups G for which $G' \leqslant G^p$ if p is odd and $G' \leqslant G^4$ if $p = 2$) one can get very good bounds on the number of conjugacy classes.

Lemma 6.1 ([20, Lemma 4.7 (ii)]). *If G is a powerful finite p-group, then*

$$k(G) \geqslant (1 - p^{-1})|G|^{1/d},$$

where $d = d(G)$ is the number of generators of G.

Using this, it is very easy to give a lower bound for powerful pro-p groups, and in fact a slightly larger class of pro-p groups.

Proposition 6.2. *Let G be a d-generator pro-p with FAb, and suppose that G has powerful finite images of arbitrarily large order (in particular, this holds if G is an infinite, powerful pro-p group). For all powers n of the prime p,*

$$R_n(G) \geqslant cn^{2/d},$$

where $c = (1 - p^{-1})$.

Proof. Let N be a normal subgroup such that G/N is powerful of order p^m where m is even. (Since quotients of powerful groups are powerful, we can do this for all even m.) We have $k(G/N) \geqslant c|G/N|^{1/d}$; each of the irreducible representations of G/N is of dimension less than $p^{m/2}$, and so

$$R_{p^{m/2}}(G) \geqslant ap^{m/d};$$

writing $n = p^{m/2}$ we get $R_n(G) \geqslant cn^{2/d}$. □

If one wants a result on the numbers $r_n(G)$ rather than $R_n(G)$, then this is easy now.

Corollary 6.3. *Let G be a d-generator pro-p with FAb, and suppose that G has powerful finite images of arbitrarily large order (in particular, this holds if G is an infinite, powerful pro-p group). For infinitely many powers n of the prime p,*

$$r_n(G) \geqslant \frac{2cn^{2/d}}{\log_p n},$$

where $c = 1 - p^{-1}$.

This follows simply because there are at most $(\log_p n)/2$ degrees of irreducible representations of G at most n.

A similar result can be obtained for some groups that are not powerful, like the Nottingham group.

Proposition 6.4. *Let p be an odd prime and let G be the Nottingham group over* $\mathbb{F}_q$, *where q is a power of p. For all powers n of p we have*

$$R_n(G) \geqslant cn^{2/3p},$$

where $c = c(q)$ *depends only on q.*

Proof. By [7, Theorem 1.2] we have $k(G/N) \geqslant c|G/N|^{1/3p}$ for any normal subgroup N of G, where c depends only on q. The method of proof of Proposition 6.2 now gives the result. □

For all finitely generated pro-p groups, until recently only a logarithmic bound for $R_n(G)$ was available. However, in [8], Jaikin-Zapirain proved the following theorem.

Theorem 6.5 (Jaikin-Zapirain [8]). *There is a constant* $c > 0$ *such that, for all finite p-groups G, we have*

$$k(G) > c \log_p|G| \frac{\log_p \log_p|G|}{\log_p \log_p \log_p|G|}.$$

Using this result, it is very easy to prove Theorem G, via the same methods used for Proposition 6.2.

Jaikin-Zapirain has suggested the following slight improvement to Theorem G, if one relaxes the condition that all sufficiently large n satisfy the bound to just infinitely many n: in this case, one can get

$$R_{p^n}(G) \geqslant cn \log_p n$$

for infinitely many n and some constant c (independent of the group G). To see this, suppose that G is an infinite, finitely generated pro-p group. If G is p-adic analytic, then G contains a powerful subgroup of finite index, and hence the result follows by Proposition 6.2 (for any $c > 0$). If G is not virtually powerful, then all dimension subgroups are distinct (see [4, Theorem 11.5]); write $a_n = |G : D_{2^n}(G)|$. For infinitely many n we have $a_n/a_{n-1} \geqslant a_{n-1}/2$; let $H = G/D_{2^n}$ for some such n. Note that we have

$$|D_{2^{n-1}}(H)| \geqslant |H|^{1/3}.$$

Using the proof of [8, Claim 3.5] (which states that if G is a finite p-group with maximal powerful normal subgroup P, then $k(G/\Phi(P)) \geqslant pm \log_p m/24$, where $m = d(P)$), one sees that if P is a powerful normal subgroup containing $D_{2^{n-1}}(H)$ (which is elementary abelian as $D_{2^n}(H) = 1$) then $d(P) \geqslant b/3$, where $|H| = p^b$, and so the claim is proved.

7 Constructing groups of slow representation growth

Theorem A states that the sequence $(r_n(G))$ has a subsequence that tends to infinity, and Proposition 2.2 states that the sequence $(R_n(G))$ strays above $\log n(\log \log n)^{1-\varepsilon}$ infinitely often. In some sense therefore there is a 'global' lower bound to the representation growth of a profinite group. However, Theorem B asserts that there is no function f that tends to infinity such that $R_n(G) > f(n)$ for all sufficiently large n and all infinite, finitely generated profinite groups G.

In [18], Neumann constructed finitely generated profinite groups whose finite images are iterated wreath products of finite simple groups. Kassabov and Nikolov, in [10], constructed profinite groups with finite images consisting of precisely direct products of specified finite simple groups.

More specifically, let $\mathscr{S}$ be any infinite collection of finite simple groups, where each group may appear with a finite multiplicity. In [10] it was proved that, under suitable conditions for the multiplicities, there is a finitely generated profinite group whose finite images are *exactly* the finite direct products of elements of $\mathscr{S}$. (One such suitable condition that we will use later is that all elements of $\mathscr{S}$ have multiplicity 1.)

The Kassabov–Nikolov examples have representation growths that are reasonably easy to compute. Using alternating groups of varying degrees, Kassabov and Nikolov constructed profinite groups G for which $R_n(G)$ is bounded between n^b and $n^{b+\varepsilon}$ for any $b > 0$ and $\varepsilon > 0$ (and n sufficiently large), so that the abcissca of convergence of the zeta function is exactly b. (It could be that the representation growth of G is, for example, $n^b \log n$.)

For functions f that are supermultiplicative (i.e., $f(x)f(y) \geqslant f(xy)$), grow faster than any polynomial, and are below $n!$, it was also proved [10, Theorem 1.8 (a)] that there is a finitely generated profinite group G such that $R_n(G)/f(n) \to 1$ as $n \to \infty$. One can achieve even faster growth by constructing groups $\hat{G}$, such that the finite quotients are extensions of elementary abelian subgroups by simple groups, using the same technique. Thus there can be no upper bound on the rate at which the sequence $(R_n(G))$ may grow.

Let $\mathscr{S} = \{A_{n_1}, A_{n_2}, \ldots\}$ be a collection of alternating groups, with $n_i < n_{i+1}$. Suppose that sequence (n_i) grows very quickly; more precisely, suppose that

$n_i > \prod_{j<i}(n_j!)$. This condition implies that the representations of degree at most $n_i - 2$ are all representations of the product of the first $i-1$ elements of $\mathcal{S}$. Therefore, for $i \geqslant 2$,

$$R_{n_i-2}(G) = \prod_{j<i} k(A_{n_j}) \approx \prod_{j<i} p(n_j)/2,$$

where again $p(m)$ denotes the number of partitions of m. (The number of conjugacy classes of A_m is approximately $p(m)/2$.) The reason for the $n_i - 2$ is that A_{n_i} has no representations of degree less than $n_i - 1$, and exactly one of degree $n_i - 1$, at least if $n_i \geqslant 7$.

Given any non-decreasing function $f : \mathbb{N} \to \mathbb{N}$ that tends to infinity, we may construct an infinite, finitely generated profinite group G such that $R_n(G) < f(n)$ for infinitely many n. To see this, simply define G to be as follows: let $n_1 = 7$, and choose n_2 such that $k(A_{n_1}) > f(n_2)$ (and also $n_2 > n_1!$). This ensures that $R_{n_2-2}(G) = k(A_{n_1})$. We repeat the process, choosing n_3 such that $n_3 > (n_1!)(n_2!)$ and $f(n_3) > k(A_{n_1})k(A_{n_2})$, and so on.

This process produces a finitely generated group G such that $R_{n_i-2}(G) < f(n_i - 2)$ for all i. Thus it is not possible to produce a global lower bound, proving Theorem B. If we are to make claims about 'lower bounds' for representation growth, the most we can say is that the function $R_n(G)$ is greater than a given function $f(n)$ *infinitely many times*. In the example we constructed above, for any non-decreasing, divergent f, we can choose the n_i so that $R_n(G) < f(n)$ for arbitrarily large intervals in $\mathbb{N}$. Therefore we cannot, given a divergent non-decreasing function $f : \mathbb{N} \to \mathbb{N}$, even give lower bounds on 'the proportion of $\mathbb{N}$' (e.g., using density) for which any finitely generated profinite group G satisfies $R_n(G) > f(n)$, for example for $f(n) = c \log n (\log\log n)^{1-\varepsilon}$ as in Proposition 2.2.

Since $R_{n_i-2}(G)$ is the product of partition functions (roughly) for the groups above, we actually have that these groups G satisfy $R_n(G)/f(n) > 1$ for infinitely many n, where $f(n)$ is of the form $\mathrm{e}^{\alpha\sqrt{\log n}}$ for some $\alpha > 0$, and so are a long way from the bound in Proposition 2.2.

Acknowledgments. Some of the work done here was completed during the *Batsheva seminar on representation growth*, which took place between 21 and 26 June, 2009. The author wishes to thank the organizers for the opportunity to enter discussions with the various attendees. He would also like to thank Martin Kassabov for many interesting discussions during the Berlin summer school on finite simple groups and algebraic groups, from 31 August to 11 September, 2009. In particular, he explained more fully the ideas behind [10]. The author would like to thank Andrei Jaikin-Zapirain for suggesting the proof of Proposition 2.2.

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Received 17 November, 2009

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