# The Representation Theory of Finite Groups: A Guidebook, Errata

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I distinguish four types of corrections, in order of increasing seriousness:

(Extra) Additional information that was not available at the time of writing, or that I did not know about.

(Improve) Typographical issues, where what is written is still correct, but there is a nicer way of phrasing it, or I could choose a better symbol.

(Typo) Typographical errors, where I have spelled a word wrongly, used the wrong symbol, and so on.

(Error) Errors in proofs or statements.

When I give each correction, I will label it with one of these monikers.

To avoid confusion with the bibliography from the book, the bibliography here will follow an author-year format, with other references meaning those in the book.

(i) (Extra) p. 89. In March 2021 An, Hiss and Lübeck posted a paper on the arXiv [AHL21] that proves the inductive BAW for the groups $F_4(q)$.

(ii) (Extra) p. 94. The modular isomorphism problem has been resolved, in the negative. This was a suspicion of mine for a while now, but I didn't think the solution would be as small. A paper of García, Margolis and del Río [GMR21] proves that there are two groups of order $2^n$ whose group algebras are isomorphic over any field of characteristic 2. (They find two such groups of order $2^n$ for all $n \geqslant 9$, in fact, and $n = 9$ is minimal since the modular isomorphism problem has a positive answer for groups of order $2^8$.)

(iii) (Extra) p. 96. There has been progress on the Morita–Frobenius conjecture, over both $\mathcal{O}$ and $k$. In recent work, only available on the arXiv so far [Liv21], Livesey has found examples of blocks with arbitrarily large Morita–Frobenius number over $\mathcal{O}$, so the possibility (i) from the list on this page cannot occur. This fits with my guess on p. 97. As there are, as of yet anyway, no examples of $k$-Morita equivalent blocks that are not $\mathcal{O}$-Morita equivalent, this might well also imply the $k$ version of the Morita–Frobenius number has no global bound.

Recent progress on the interaction between $k$- and $\mathcal{O}$-Morita equivalences has been made by Eisele [Eis19]. He has shown that the Picard groups over $\mathcal{O}$ are always finite. I didn't mention Picard groups in the text, because I was working almost exclusively over $k$, but this is good progress towards the conjecture that two blocks are equivalent over $k$ if and only if they are over $\mathcal{O}$. (That $\mathcal{O}$ implies $k$ is not so hard to see.)

(Note that the negative solution to the modular isomorphism problem—see the note in this list for p. 94—means that two $\mathcal{O}$-blocks that are not Morita equivalent can be Morita equivalent over $k$, just by taking the group algebra of 2-groups with isomorphic $k$-group algebras. So if we want to say that all $k$-Morita equivalences lift to $\mathcal{O}$, we should specify that the two blocks have the same defect group.)

Lastly, Eisele and Livesey collaborated to extend to $k$ the above statement on Morita–Frobenius numbers [EL20].

(iv) (Typo) p.112. The second Brauer tree with labels 4, 6 and 2 should be 6, 2 and 4 respectively. It is correct in the general tree further down the page.

Thanks to John Murray for spotting this.

(v) (Error) p.121. The description of which Green correspondents are 'big' and which are 'small' is wrong, in the sense that what 'adjacent' means was not explained. It means adjacent on a path to the exceptional node. So if $M_1$ and $M_2$ are, modulo 2, the same distance from the exceptional node, then the Green correspondent of $M_1$ has at most $e$ composition factors if and only if the same holds for $M_2$.

That 'adjacent' cannot purely mean on the tree is clearly the case, as it isn't even well defined if a vertex on the tree has odd valency.

Thanks to John Murray for noticing that the description is incorrect.

(vi) (Extra) p.138. The reference [201] at the bottom of that page contains an error, which was corrected in [Kos82].

Thanks to Shigeo Koshitani for this.

(vii) (Extra) pp.139–145. The case (iii) from Theorem 6.2.2, for $n \geqslant 4$, has been shown not to occur by Macgregor [Mac21]. Thus there is a complete list for blocks with dihedral defect group up to Morita equivalence. This also eliminates the case $n \geqslant 5$ from Theorem 6.2.7 and the third decomposition matrix.

In the same paper, Macgregor eliminated some semidihedral cases: from Theorem 6.2.6 (three modules), the last decomposition matrix cannot occur. The third and fourth cannot occur for $n \geqslant 5$. For $n = 4$, the Monster has a semidihedral block, and it has decomposition matrix either the third or the fourth one. Whichever it is, the other one cannot occur as a block of a finite group.

Finally, for semidihedral and two simple modules, Theorem 6.2.5, Macgregor eliminated (v) for $n \geqslant 5$, and (vi) for all $n$.

(viii) (Error) p.144. I tried hard to correct all of the errors that had appeared in various papers about this but this one slipped through. The Morita class for which examples are known is $\mathrm{SD}(3\mathcal{D})$, not $\mathrm{SD}(3\mathcal{B}_1)$. In [193], it was correctly proved that the principal blocks of $\mathrm{PSL}_3(q)$ possess two simple modules with a non-trivial self-extension. Looking at the quivers in [200], this is class $3\mathcal{D}$, whereas class $3\mathcal{B}_1$ has a single simple module with a self-extension. The table in [200] incorrectly claims that $\mathrm{PSL}_3(q)$ lies in $3\mathcal{B}_1$.

This is class III in [199], where it is correct. However, in [199], class IV, which is meant to be $\mathrm{PSU}_3(q)$, is wrong, with $P_2$ supposed to be as in I, not as in III, and the condition for a block is that $t = 1$ and $k = 2^{n-2}$, not $t = 2^{n-2}$ and $k = 1$.

Thanks to Dave Benson for noticing that the blocks have two simple modules with self-extensions, not one.

(ix) (Typo/Error) p.145, Theorem 6.2.8. The groups realizing the decomposition matrices are $\mathrm{SL}_2(q).2$, not $\mathrm{SL}_2(q^2).2$.

Thanks to Norman Macgregor for alerting me to this.

(x) (Error) p.148. This one is really annoying. The definition of a source algebra is wrong. In particular, the idempotent I use is not correct. The correct set of idempotents that should be used are as follows.

Let $B$ be a $p$-block of $kG$ with defect group $D$. A *source idempotent* of $B$ is a primitive idempotent $i$ of $B^D$ satisfying $\mathrm{Br}_D(i) \neq 0$. The source algebra definition with respect to any source idempotent is then as in Definition 6.3.5.

Thanks to Burkhard Külshammer for pointing this out.

(xi) (Typo/Error) p.166. Exercise 7.5. I forgot to mention that $B$ covers $B_0$, but this standard notation for the chapter.

(xii) (Error) p.186. At the end of Section 8.2, I mention the vertices of Specht modules labelled by hook partitions. In fact, whereas [558] deals with Specht modules, it also considers simple modules, since the Specht modules are simple in the cases considered there. The rest of that paragraph, the results of Danz, Giannelli, Müller and Zimmermann (from [149, 151, 444]), all consider the simple modules labelled by hook partitions. Thus replace Specht by simple in this paragraph.

In fact, the case for Specht modules is still unresolved. Giannelli, Lim and Wildon in 2016 [GLW16] found the vertices for the Specht module $S^{(kp-p,1^p)}$ for $k \equiv 1 \bmod p$ and $k \not\equiv 1 \bmod p^2$. If the Specht module is simple, as in the results above from [558], more or less complete information about the modules (vertices, Green correspondents, complexities and so on) was found by Danz and Lim [DL17].

(xiii) (Error) p.193. The smallest known counterexample to James's conjecture is the principal 839-block of weight 561. The product of these two numbers is 470679, not 467874, as stated in the text. This error of the product of two numbers being incorrect has appeared in several other papers and books stretching back at least to 2013. An Internet search for '467874 counterexample' yields a number of results. Of course, I failed to check the product of the numbers.

Thanks to Diego Millan Berdasco for noticing this.

(xiv) (Error) pp.215–216. When discussing highest weight modules, there should be an assumption that $\mathbf{G}$ is simply connected. Alternatively, one obtains only a *projective* representation of $\mathbf{G}$ for some dominant weights. Without a simple connectedness assumption for Steinberg's tensor product theorem, some representations (like the 4-dimensional representation $L(p+1)$ of $\mathrm{PGL}_2$) are only the tensor product of projective representations (i.e., representations of $\mathrm{SL}_2$).

Finally, Steinberg's restriction really needs this assumption. Without it, some modules will be missing. There isn't even a bijection in general. For example, there are five simple modules for $\mathrm{SL}_2(5)$ (in characteristic 5), of dimensions 1 to 5, but six for $\mathrm{PGL}_2(5)$, two each of dimensions 1, 3 and 5. One cannot see the non-trivial 1-dimensional modules for adjoint type groups using highest weight theory, as they are not restrictions of modules for the algebraic group.

(xv) (Error) p.219. The definition of the affine Weyl group is correct, but not if you want its action on weights to be correct. If $W$ is a Weyl group, write $W_a$ for the semidirect product

$$W_a = \mathbb{Z}\Phi^\vee \rtimes W,$$

and this is the *affine Weyl group* as it is usually defined. There is also the group $W_p$, defined by

$$W_p = p \cdot \mathbb{Z}\Phi \rtimes W.$$

This replaces $\mathbb{Z}\Phi^\vee$ by $p$ times it, but *also* replaces the coroot lattice by the root lattice. All of the groups $W_a$, $W_p$ as defined here, and $W_p$ as defined in the book, are isomorphic. But in order for the linkage principle to be correct, you need to act by $p$ times roots, not $p$ times coroots.

Thanks to Chris Parker for noticing this.

(xvi) (Error) p.220. Helpfully, Example 9.1.6 is wrong, in my calculation of the orbit of the affine Weyl group $W_5$. Notice that this error is not due to the error above on p.219, because in type $A$ roots are coroots. The problem is that I added *weights*, not *roots*, when I calculated the orbit under the affine Weyl group. The roots have labels $(2,-1)$ and $(-1,2)$, not $(1,0)$ and $(0,1)$. So instead of obtaining 00, 04, 40, 33, 34 and 43 in the orbit, one obtains only 00 and 33.

I give an example of what went wrong. We start at 00, add the Steinberg weight to obtain 11. The images of this under the Weyl group are $(1,1)$, $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ and $(-1,-1)$. Now we subtract $\rho$ to obtain $(0,0)$, $(0,-1)$, $(-1,0)$, $(-2,-1)$, $(-1,-2)$, $(-2,-2)$. If we add five times a weight onto $(-1,0)$ we obtain $(4,0)$, but adding five times a *root* onto $(-1,0)$ can obtain $(4,5)$, or $(9,-4)$, or something like this. None of these is 5-restricted; in fact, only $(0,0)$ and $(-2,-2)$—which maps to $(3,3)$—can yield 5-restricted weights.

Thanks to Chris Parker for noticing this.

(xvii) (Typo/Extra) p.270. Reference 371 is wrong, because it should have been Math. Z., to appear. (A similar-titled paper of those authors did appear in J. Algebra.) It has since come out. The full reference is [KL20] given below.

Thanks to Shigeo Koshitani for spotting this error.

## References


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[GLW16] Eugenio Giannelli, Kay Jin Lim, and Mark Wildon. Sylow subgroups of symmetric and alternating groups and the vertex of $S^{(kp-p,1^p)}$ in characteristic $p$. *J Algebra*, 455:358–385, 2016.

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[Kos82] Shigeo Koshitani. A remark on blocks with dihedral defect groups in solvable groups. *Math. Z.*, 179:401–406, 1982.

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[Mac21] Norman Macgregor. Morita equivalence classes of tame blocks of finite groups. Preprint, 2021.