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Perverse equivalences and Broué's conjecture

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Abstract

We give a new approach to the construction of derived equivalences between blocks of finite groups, based on perverse equivalences, in the setting of Broué's abelian defect group conjecture. We provide in particular local and global perversity data describing the principal blocks and the derived equivalences for a number of finite simple groups with Sylow subgroups elementary abelian of order 9. We also examine extensions to automorphism groups in a general setting.
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1. Introduction

This paper proposes a new approach to the construction of derived equivalences, based on perverse equivalences. These equivalences, introduced by Joe Chuang and the second author, aim to encode combinatorially the Morita equivalence class of a block within the derived equivalence class of a fixed block [12]. The derived equivalences between blocks of symmetric groups constructed in [11] are compositions of perverse equivalences, and it is expected that the conjectural derived equivalences provided by Deligne–Lusztig varieties for finite groups of Lie type in non-describing characteristic are perverse. This has motivated our search for perverse equivalences in the case of sporadic groups, in the setting of Broué's abelian defect conjecture (cf. [10] for a survey on Broué's conjecture). We also consider certain finite groups of Lie type: in those cases, we provide equivalences which should coincide with the conjectural ones coming from Deligne–Lusztig varieties.

We provide lifts of stable equivalences to perverse equivalences. Our method requires only calculations within the normalizer of a defect group. In the cases we consider here, Broué's conjecture was already known to hold. The equivalences we provide are often different from the known ones, which were usually not perverse. Also, to obtain perverse equivalences, we sometimes need to change the "obvious" stable equivalence, given by Green correspondence: this depends on local data. For those groups of Lie type we consider, this is dictated by the properties of Deligne–Lusztig varieties. We consider finite groups with elementary abelian Sylow 3-subgroups of order 9: for these groups, Green correspondence provides a stable equivalence. In addition, we have a complete description of the local normalized self-derived equivalences, which enables us to parametrize splendid self-stable equivalences. Note that there are stable equivalences between blocks that can be lifted to a derived equivalence but not to a perverse derived equivalence. This occurs for example when all simple modules of one of the blocks can be lifted to characteristic 0 (that happens for the local block in the setting of Broué's conjecture), while the decomposition matrix of the other block cannot be put in a triangular form.

Our approach can also be viewed as an attempt to mimic the extra structure carried by representations of finite groups of Lie type in non-describing characteristic to the case of arbitrary finite groups. Our equivalences depend on the datum of a perversity function π, which is related to Lusztig's A-function for finite groups of Lie type. The precise conjecture for π for groups of Lie type in non-defining characteristic is given in [14]. We actually work in the setting of splendid Rickard equivalences, and we have a collection of perversity functions, associated with p-subgroups. The collection of perversity functions determines the isomorphism type of the source algebra of a block, within the class of blocks splendidly Rickard equivalent to a fixed block.

Extensions of equivalences through ℓ'-groups of automorphisms are easy to carry over for perverse equivalences, and we devote an important part of this paper to the study of extensions of equivalences. Our main point is that checking that a two-sided tilting complex will extend depends only on the underlying (one-sided) tilting complex. We deduce from our results new methods to check that equivalences extend. This enables us to show for instance that Broué's abelian defect conjecture holds for principal blocks in characteristic 2.

Section 3 is devoted to constructing equivalences. In Section 3.1, we explain first the description of the images of simple modules under perverse equivalences associated to increasing perversity. We define the notion of (increasing) perverse splendid equivalences and show that the data of perversity functions associated to the automizer and some of its subgroups determine the isomorphism class of the source algebra of the block. We explain next our method for lifting stable equivalences to perverse equivalences (Section 3.2). In Section 3.3, we explain the construction of stable equivalences for principal blocks with elementary abelian Sylow ℓ-subgroups of order ℓ^2. We describe in detail the images of modules and we describe a family of stable equivalences dependent on local perversity functions when $\ell = 3$. Finally, we recall in Section 3.4 the setting of Broué's conjecture for finite groups of Lie type, where two-sided tilting complexes are expected to arise from Deligne–Lusztig varieties. We study in particular those finite groups of Lie type with a Sylow 3-subgroup elementary abelian of order 9. We also recall Puig's construction of equivalences for the case "$\ell \mid (q-1)$".

Section 4 is devoted to automorphisms and extensions of equivalences. In Section 4.1, we set up a general formalism that allows a reduction to finite simple groups for equivalences of a suitable type between the principal block of a finite group with an abelian Sylow ℓ-subgroup P, and the principal block of $N_G(P)$. This is meant to encompass the various forms of Broué's abelian defect group conjecture. In Section 4.2, we provide extension theorems for equivalences. We give criteria that ensure that a two-sided tilting complex can be made equivariant for the action of a group of automorphisms; we recover results of Rickard and Marcus. We consider in particular (compositions of) perverse equivalences. In Section 4.3, we apply the general results of the previous sections to various forms of Broué's conjecture: derived equivalences, Rickard equivalences, splendid or perverse properties, positivity of gradings and perfect isometries. This provides us in Section 4.4 with a general reduction theorem to simple groups, generalizing a result of Marcus. We apply this to show that Broué's conjecture can be solved using perverse equivalences for certain cases when $\ell = 2$ or 3.

This last result is obtained by a case-by-case study of finite simple groups with elementary abelian Sylow 3-subgroups of order 9 in Section 5. We provide a perverse equivalence with the normalizer of a Sylow 3-subgroup for all such groups except $\mathfrak{A}_6$ and M_{22}, for which we need the composition of two perverse equivalences. Perverse equivalences are encoded in global and local perversity functions. Note that these combinatorial data determine the source algebra of the block up to isomorphism. While Broué's conjecture was known to hold in all cases considered (work of Koshitani, Kunugi, Miyachi, Okuyama, Waki), we have been led to construct a number of new equivalences. In Section 6, we provide an analysis of simple groups with abelian Sylow 2-subgroups.

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2. Notation and basic definitions

2.1. Algebras

2.1.1. Modules

All modules are finitely generated left modules, unless otherwise specified.

Let R be a commutative ring. We write $\otimes$ for $\otimes_R$. Given q a prime power, $\mathbf{F}_q$ denotes a finite field with q elements.

Let A be an R-algebra. We denote by A^{opp} the opposite algebra to A and we put $A^{\mathrm{en}} = A \otimes A^{\mathrm{opp}}$. Given an R-module M, we put $AM = A \otimes M$, an A-module. We denote by $\mathcal{S}_A$ a complete set of representatives of isomorphism classes of simple A-modules.

Let M be a finitely generated module over an artinian algebra. The *head* of M is defined to be its largest semi-simple quotient. We denote by I_M (resp. P_M or $\mathcal{P}(M)$) an injective hull (resp. a projective cover) of M. We denote by $\Omega(M)$ the kernel of a surjective map $P_M \to M$ and by $\Omega^{-1}(M)$ the cokernel of an injective map $M \to I_M$. We define by induction $\Omega^i(M) = \Omega(\Omega^{i-1}(M))$ and $\Omega^{-i}(M) = \Omega^{-1}(\Omega^{-i+1}(M))$ for $i > 1$.

Let $\sigma : B \to A$ be a morphism of algebras and let M be an A-module. We define a B-module ${}_\sigma M$: it is equal to M as an R-module, and the action of $b \in B$ given by the action of $\sigma(b)$ on M.

The algebra A is *symmetric* if it is finitely generated and projective as an R-module, and if $\mathrm{Hom}_R(A, R) \simeq A$ as A^{en}-modules.

2.1.2. Categories

Assume R is noetherian and A is a finitely generated R-module. We denote by A-mod the category of finitely generated A-modules.

Let $\mathcal{C}$ be an additive category and $\mathcal{A}$ an abelian category. We denote by

- $\mathrm{Comp}(\mathcal{C})$ the category of complexes of objects of $\mathcal{C}$,
- $\mathrm{Ho}(\mathcal{C})$ the homotopy category of $\mathrm{Comp}(\mathcal{C})$, and
- $D(\mathcal{A})$ the derived category of $\mathcal{A}$.

A complex in $\mathcal{C}$ is *contractible* if it is 0 in $\mathrm{Ho}(\mathcal{C})$, and a complex in $\mathcal{A}$ is *acyclic* if it is 0 in $D(\mathcal{A})$. We write $\mathrm{Comp}(A)$ for $\mathrm{Comp}(A\text{-mod})$, and so on.

We write $0 \to M \to N \to \cdots \to X \to 0$ (or sometimes $M \to N \to \cdots \to X$) for a complex where $X \neq 0$ is in degree 0.

Given $M, N \in \mathrm{Comp}(\mathcal{C})$, we denote by $\mathrm{Hom}^\bullet(M, N)$ the complex with degree n term $\bigoplus_{j-i=n} \mathrm{Hom}(M^i, N^j)$. When $\mathcal{C} = \mathcal{A}$ is abelian, we denote the derived version by $R\,\mathrm{Hom}^\bullet = \mathrm{Hom}^\bullet(P, N)$, where $P \in \mathrm{Comp}(\mathcal{A})$ is quasi-isomorphic to M and P is homotopically projective, *i.e.*, $\mathrm{Hom}^\bullet(P, C)$ is acyclic for any acyclic complex $C \in \mathrm{Comp}(\mathcal{A})$.

Given two R-algebras A and B, we say that a functor $D^b(A) \to D^b(B)$ is *standard* if it is of the form $C \otimes_A -$, where C is a bounded complex of (B, A)-bimodules, finitely generated and projective as B-modules and as A^{opp}-modules.

A *tilting complex* C for A is a perfect complex of A-modules (*i.e.*, quasi-isomorphic to a bounded complex of finitely generated projective A-modules) such that A is in the thick subcategory of $D(A)$ generated by C and $\mathrm{Hom}_{D(A)}(C, C[i]) = 0$ for $i \neq 0$.

A *two-sided tilting complex* C for (A, B) is a bounded complex of (A, B)-bimodules such that the functor $C \otimes^{\mathbf{L}}_B - : D(B) \to D(A)$ is an equivalence.

A *Rickard complex* C for (A, B) is a bounded complex of (A, B)-bimodules, finitely generated and projective as left A-modules and as right B-modules, such that the functor $C \otimes_B - : \mathrm{Ho}(B) \to \mathrm{Ho}(A)$ is an equivalence. We also say that C induces a *Rickard equivalence*.

Assume that $R = k$ is a field and A is a symmetric k-algebra. We denote by A-stab the stable category, the triangulated quotient of $D^b(A)$ by the thick subcategory of perfect complexes. The canonical functor A-mod $\to A$-stab identifies A-stab with the additive quotient of A-mod by its subcategory of projective modules.

Let B be a symmetric k-algebra. A bounded complex C of (A, B)-bimodules *induces a stable equivalence* if its terms are projective as left A-modules and as right B-modules, and there are isomorphisms of complexes of bimodules $\mathrm{End}^\bullet_A(C) \simeq B \oplus R_1$ and $\mathrm{End}^\bullet_{B^{\mathrm{opp}}}(C) \simeq A \oplus R_2$, where R_1 and R_2 are homotopy equivalent to bounded complexes of projective bimodules. There is an equivalence $C \otimes_B - : B\text{-stab} \xrightarrow{\sim} A\text{-stab}$.

We say that a Rickard complex C *lifts* a complex D inducing a stable equivalence if C and D are isomorphic in the quotient of $\mathrm{Ho}^b(A \otimes B^{\mathrm{opp}})$ by its thick subcategory of complexes of projective modules.

2.2. *Modular setting*

We will denote by $\mathcal{O}$ a complete discrete valuation ring with residue field k of characteristic $\ell > 0$ and field of fractions K of characteristic 0. Let $\mathbf{Z}_\ell$ denote the ring of ℓ-adic integers.

2.3. *Groups*

We denote by Z_n a cyclic group of order n, by D_n a dihedral group of order n, by SD_n a semi-dihedral group of order n, and by $\mathfrak{A}_n$ and $\mathfrak{S}_n$ the alternating and symmetric groups of degree n respectively. If G is a finite group, we denote by G^{opp} the opposite group, we put $G^{\mathrm{en}} = G \times G^{\mathrm{opp}}$ and we set $\Delta G = \{(g, g^{-1}) \mid g \in G\} \leqslant G^{\mathrm{en}}$.

We denote by $b_0(G)$ the principal block idempotent of $\mathbf{Z}_\ell G$ and by $B_0(G) = b_0(G)\mathbf{Z}_\ell G$ the principal block algebra. Given an $\mathcal{O}G$-module M and an ℓ-subgroup P of G, we denote by $\mathrm{Br}_P(M)$ the image of M^P in the coinvariants $M/\{g(m) - m\}_{m \in M, g \in P}$; this is an $\mathcal{O}(N_G(P)/P)$-module.

Let R be either $\mathcal{O}$ or k and let H be a finite group. Assume that H and G have a common Sylow ℓ-subgroup P. We say that a bounded complex C of $RB_0(H \times G^{\mathrm{opp}})$-modules is *splendid* if its terms are direct summands of finite direct sums of modules of the form $\mathrm{Ind}_{\Delta Q}^{H \times G^{\mathrm{opp}}} R$, where $Q \leqslant P$. We say that C induces a *splendid Rickard equivalence* (resp. a *splendid Morita equivalence*) if C is splendid and $C \otimes_B -$ induces a Rickard equivalence (resp. a Morita equivalence) between $RB_0(G)$ and $RB_0(H)$.

3. Constructions of equivalences

3.1. Perverse equivalences

We explain some constructions and results of [12] (see also [53, §2.6]).

3.1.1. Definition

Let A and B be two finite-dimensional algebras over a field k. Fix $r \geqslant 0$ and $q : [0, r] \to \mathbf{Z}$, where $[0, r] = \{0, 1, \ldots, r\}$. Fix filtrations $\emptyset = \mathcal{S}_{-1} \subset \mathcal{S}_0 \subset \cdots \subset \mathcal{S}_r = \mathcal{S}_A$ and $\emptyset = \mathcal{S}'_{-1} \subset \mathcal{S}'_0 \subset \cdots \subset \mathcal{S}'_r = \mathcal{S}_B$.

A functor $F : D^b(B) \to D^b(A)$ is *perverse* relative to $(q, \mathcal{S}_\bullet, \mathcal{S}'_\bullet)$ if, whenever T is in $\mathcal{S}'_i$, the composition factors of $H^{-j}(F(T))$ are in $\mathcal{S}_{i-1}$ for $j \neq q(i)$ and in $\mathcal{S}_i$ for $j = q(i)$.

If F is an equivalence, then given $T \in \mathcal{S}'_i$, the A-module $H^{-q(i)}(F(T))$ is the extension of an object with composition factors in $\mathcal{S}_{i-1}$ by an object $f(T)$ in $\mathcal{S}_i$ by an object with composition factors in $\mathcal{S}_{i-1}$. The map f gives a bijection $\mathcal{S}_B \xrightarrow{\sim} \mathcal{S}_A$ compatible with the filtrations. Note that F^{-1} is perverse relative to $(-q, \mathcal{S}'_\bullet, \mathcal{S}_\bullet)$.

Consider a finite-dimensional k-algebra C, a filtration $\emptyset = \mathcal{S}''_{-1} \subset \mathcal{S}''_0 \subset \cdots \subset \mathcal{S}''_r = \mathcal{S}_C$ and $q' : [0, r] \to \mathbf{Z}$. Given $G : D^b(C) \to D^b(B)$ perverse relative to $(q', \mathcal{S}'_\bullet, \mathcal{S}''_\bullet)$, then the composition $FG : D^b(C) \to D^b(A)$ is perverse relative to $(q + q', \mathcal{S}_\bullet, \mathcal{S}''_\bullet)$.

3.1.2. Increasing perversity

Let A be a symmetric k-algebra. Given $M \in A$-mod and $E \subset \mathcal{S} = \mathcal{S}_A$, we denote by M^E the largest submodule N of I_M containing M and such that all composition factors of N/M are in E.

Consider a map $\pi : \mathcal{S} \to \mathbf{Z}_{\geqslant 0}$, and let $S \in \mathcal{S}$. We define a complex of A-modules $C = C_S = 0 \to C^{-\pi(S)} \to \cdots \to C^0 \to 0$.

- If $\pi(S) = 0$ then we set $C^0 = S$.
- Assume that $\pi(S) > 0$. We put $C^{-\pi(S)} = I_S$. Let $E = \pi^{-1}([0, \pi(S) - 1])$. Let $C^{-\pi(S)+1} = I_{\Omega^{-1}(S^E)}$. We define $d^{-\pi(S)}$ as the composition of canonical maps $C^{-\pi(S)} = I_{S^E} \twoheadrightarrow \Omega^{-1}(S^E) \hookrightarrow C^{-\pi(S)+1}$.

Fix $0 < i < \pi(S)$. Assume that $0 \to C^{-\pi(S)} \xrightarrow{d^{-\pi(S)}} C^{-\pi(S)+1} \to \cdots \xrightarrow{d^{-\pi(S)+i-1}} C^{-\pi(S)+i}$ has been constructed with the property that $C^{-\pi(S)+i} = I_T$, where $T = \operatorname{im} d^{-\pi(S)+i-1}$. Let $E = \pi^{-1}([0, \pi(S) - i - 1])$ and $C^{-\pi(S)+i+1} = I_{\Omega^{-1}(T^E)}$. We define $d^{-\pi(S)+i}$ as the composition of canonical maps $C^{-\pi(S)+i} = I_{T^E} \twoheadrightarrow \Omega^{-1}(T^E) \hookrightarrow C^{-\pi(S)+i+1}$.

Finally, let $T = \operatorname{im} d^{-2}$, $E = \pi^{-1}(0)$ and $C^0 = \Omega^{-1}T^E$. We define d^{-1} as the canonical map $C^{-1} = I_{T^E} \twoheadrightarrow C^0$.

There is a symmetric k-algebra B, well defined up to Morita equivalence, and a standard equivalence $F : D^b(B) \xrightarrow{\sim} D^b(A)$ such that $\{F(T)\}_{T \in \mathcal{S}_B} = \{C_S\}_{S \in \mathcal{S}_A}$. This equivalence is perverse, relative to the filtration $\emptyset \subset \pi^{-1}(0) \subset \pi^{-1}([0, 1]) \subset \pi^{-1}([0, 2]) \subset \cdots \subset \mathcal{S}_A$ and the corresponding filtration on $\mathcal{S}_B$, and relative to the perversity function $i \mapsto i$. Note conversely that given a perversity datum $(q, \mathcal{S}_\bullet, \mathcal{S}'_\bullet)$ where q is increasing, the perverse equivalence arises from a function π where $\pi(S) = \min\{q(n) \mid S \in \mathcal{S}_n\}_{n \geqslant 0}$. We write

$$B \xrightarrow{\pi} A$$

to denote the perverse equivalence.

3.1.3. Elementary equivalences

Assume that $\pi(\mathcal{S}) = \{0, 1\}$. We will describe a tilting complex for A in this case.

Let U be the smallest submodule of the A-module A such that all composition factors of A/U are in $\pi^{-1}(0)$. Let $f : P_U \twoheadrightarrow U$ be a projective cover. Denote by $J(A)$ the Jacobson radical of A. Let P_V be a projective cover of the largest submodule V of $A/J(A)$ all of whose composition factors are in $\pi^{-1}(1)$. Let $X = (0 \to P_V \oplus P_U \xrightarrow{(0,f)} A \to 0)$ be a complex of A-modules with A in degree 0. Let $B = \mathrm{End}_{\mathrm{Ho}(A)}(X)$. Then B is a symmetric algebra and there is a standard perverse equivalence $D^b(B) \xrightarrow{\sim} D^b(A)$, $B \mapsto X$.

Every perverse equivalence is a composition of elementary perverse equivalences or their inverses, *i.e.*, a composition of perverse equivalences associated to two-step filtrations with $q(1) - q(0) = \pm 1$.

3.1.4. Perverse splendid equivalences

Let G and H be two finite groups with a common Sylow ℓ-subgroup P and the same ℓ-local structure. Let R be either $\mathcal{O}$ or k.

Given $Q \leqslant P$, let $\pi_Q : \mathcal{S}_{B_0(C_G(Q)/Z(Q))} \to \mathbf{Z}_{\geqslant 0}$ be a map. We assume that π_Q is invariant under $N_G(Q)$ and independent of Q up to G-conjugacy.

Definition 3.1. An *increasing perverse splendid equivalence* between $RB_0(G)$ and $RB_0(H)$ relative to $\{\pi_Q\}_Q$ is a standard Rickard equivalence of the form $C \otimes_{RB_0(G)} -$, where C is splendid and such that for every $Q \leqslant P$, $\mathrm{Br}_{\Delta Q}(C)$ induces a perverse equivalence relative to π_Q between $kB_0(C_G(Q))$ and $kB_0(C_H(Q))$.

Remark 3.2. One can normalize the equivalence by assuming $\pi_P = 0$ and $\pi_Q(k) = 0$ for all Q. Also, if $\tilde{G}$ is a finite group containing G as a normal subgroup of ℓ'-index, $\tilde{H}$ is a finite group containing H as a normal subgroup of ℓ'-index and $\tilde{G}/G \simeq \tilde{H}/H$, then one can ask for an equivariant form of the definition above by requiring the maps π_Q to be invariant under the action of $N_{\tilde{G}}(Q)$.

Note that given an increasing perverse equivalence as above, we obtain perversity functions $\pi_Q^H : \mathcal{S}_{B_0(C_H(Q)/Z(Q))} \to \mathbf{Z}_{\geqslant 0}$, via the bijections $\mathcal{S}_{B_0(C_G(Q)/Z(Q))} \xrightarrow{\sim} \mathcal{S}_{B_0(C_H(Q)/Z(Q))}$ induced by the perverse equivalence provided by $\mathrm{Br}_{\Delta Q}(C)$.

Remark 3.3. Assume $P \lhd H$. Then we have a canonical isomorphism of algebras $RB_0(H) \xrightarrow{\sim} R(H/O_{\ell'}(H))$ and we have $H/O_{\ell'}(H) \simeq P \rtimes E$, where $E \simeq H/PO_{\ell'}(H)$. The map π_1^H corresponds to a map $\rho_1 : \mathcal{S}_{KE} \to \mathbf{Z}_{\geqslant 0}$. Similarly, π_Q^H corresponds to a map $\rho_Q : \mathcal{S}_{KC_E(Q)} \to \mathbf{Z}_{\geqslant 0}$ coming from $C_H(Q)/C_P(Q)O_{\ell'}C_H(Q) \simeq C_E(Q)$.

Proposition 3.4. *Let G' be a finite group with P as a Sylow ℓ-subgroup and the same local ℓ-structure as G. Consider a perversity data $\{\pi'_Q\}_Q$ and assume there is a splendid complex C' inducing an increasing perverse splendid equivalence between $RB_0(G')$ and $RB_0(H)$, relative to that perversity data.*

Then, $\mathrm{Hom}^\bullet_{RH}(C', C)$ has homology concentrated in degree 0. That homology induces a splendid Morita equivalence between $RB_0(G)$ and $RB_0(G')$. In particular, those blocks have isomorphic source algebras.

Proof. We have an isomorphism of complexes of $(RC_{G'}(Q), RC_G(Q))$-bimodules

$$\mathrm{Br}_{\Delta Q}\big(\mathrm{Hom}^\bullet_{RH}(C', C)\big) \simeq \mathrm{Hom}^\bullet_{RC_H(Q)}\big(\mathrm{Br}_{\Delta Q}(C'), \mathrm{Br}_{\Delta Q}(C)\big)$$

by [48, Lemma 4.2 and proof of Theorem 4.1]. It follows that $X = \mathrm{Hom}^\bullet_{RH}(C', C)$ induces an increasing perverse splendid equivalence associated to the 0 perversity data. So, $\mathrm{Br}_{\Delta Q}(X)$ has homology concentrated in degree 0 for all Q. By [4, Théorème 1.3], it follows that X is homotopy equivalent to a complex Y concentrated in degree 0. So, Y^0 induces a splendid Morita equivalence. It follows that the principal blocks of G and G' have isomorphic source algebras (cf. [44] or [57]). □

Note that Proposition 3.4 generalizes to perverse equivalences which are not increasing. It generalizes also to sequences of perversity data, corresponding to compositions of perverse splendid equivalences.

The definition and the proposition above generalize to the case of non-principal blocks, using the general notion of splendid equivalences [35,55]. An important property of splendid Rickard equivalences is that they lift from k to $\mathcal{O}$ [48, Theorem 5.2].

Theorem 3.5. *Let C be a splendid Rickard complex for $(kB_0(G), kB_0(H))$. There is a splendid Rickard complex $\tilde{C}$ for $(B_0(G), B_0(H))$ such that $k\tilde{C} \simeq C$ in* $\mathrm{Comp}(kB_0(G \times H^{\mathrm{opp}}))$. *Furthermore, $\tilde{C}$ is unique up to isomorphism.*

Note that the "elementary" splendid Rickard equivalences between blocks of symmetric groups constructed in [11] are increasing perverse splendid equivalences [12]. Two blocks of symmetric groups with isomorphic defect groups are connected by compositions of such equivalences, and their inverses.

Remark 3.6. We do not know examples of splendid equivalences that are perverse, but not perverse splendid (*i.e.*, the local equivalences are not perverse).

3.2. Lifts of stable equivalences

3.2.1. Recognition criteria

Let A and A' be two symmetric algebras over a field k, with no simple direct factors, and let $L : D^b(A') \to D^b(A)$ be a standard functor inducing a stable equivalence $\bar{L} : A'\text{-stab} \xrightarrow{\sim} A\text{-stab}$.

Let $\pi : \mathcal{S} \to \mathbf{Z}_{\geqslant 0}$. There is a symmetric algebra B and a standard perverse equivalence $F : D^b(B) \xrightarrow{\sim} D^b(A)$ [12]. Assume that $\{F(T)\}_{T \in \mathcal{S}_B}$ coincides, up to isomorphism in A-stab, with $\{L(S')\}_{S' \in \mathcal{S}_{A'}}$.

The composition $F^{-1}L : D^b(A') \to D^b(B)$ is given by tensoring by a complex X of (B, A')-bimodules. There is a (B, A')-bimodule M with no non-zero projective direct summand, projective as a B-module and as a right A'-module, that is isomorphic to X in $(B \otimes A'^{\mathrm{opp}})$-stab. The functor $M \otimes_{A'} - : A'\text{-stab} \to B\text{-stab}$ is an equivalence and it preserves isomorphism classes of simple modules. Since M has no non-zero projective direct summand, it follows that $M \otimes_{A'} S'$ is indecomposable whenever $S' \in \mathcal{S}_{A'}$ and we deduce that we have an equivalence $M \otimes_{A'} - : A'\text{-mod} \xrightarrow{\sim} B\text{-mod}$ [34, Theorem 2.1]. The composition $G = F \circ (M \otimes_{A'} -) : D^b(A') \xrightarrow{\sim} D^b(A)$ is a standard perverse equivalence lifting $\bar{L}$.

Let k_0 be a subfield of k such that the extension k/k_0 is separable. Let A_0 and A'_0 be two symmetric k_0-algebras such that $A = kA_0$ and $A' = kA'_0$. Assume that

- there is a standard functor $L_0 : D^b(A'_0) \to D^b(A_0)$ with $L = kL_0$, and
- given $S \in \mathcal{S}_{A_0}$ and S_1, S_2 two simple direct summands of kS, then $\pi(S_1) = \pi(S_2)$.

The second assumption gives a function $\pi_0 : \mathcal{S}_{A_0} \to \mathbf{Z}_{\geqslant 0}$, $S \mapsto \pi(S_1)$ where S_1 is a simple direct summand of kS. There is a symmetric k_0-algebra B_0 and a standard perverse equivalence $D^b(B_0) \xrightarrow{\sim} D^b(A_0)$. As above, we obtain a standard stable equivalence A'_0-stab $\xrightarrow{\sim} B_0$-stab that preserves semi-simple modules, and hence simple modules. We deduce that there is a standard perverse equivalence $G_0 : D^b(A'_0) \xrightarrow{\sim} D^b(A_0)$ such that G_0 and L_0 induce isomorphic stable equivalences and such that $kG_0 \simeq G$.

3.2.2. Strategy

Assume that we are given a stable equivalence $\bar{L}$ as above. Our strategy to lift $\bar{L}$ to a derived equivalence is to look for a function π as in Section 3.1.2 such that the set of A-modules $\{C_S^0\}_{S\in\mathcal{S}_A}$, coincides with the set $\{\bar{L}(S')\}_{S'\in\mathcal{S}_{A'}}$. Note that C_S is isomorphic to C_S^0 in A-stab.

In the setting of Broué's conjecture, we take for A a block with a normal abelian defect group (for example, $A = k(P \rtimes E)$ where k is a field of characteristic ℓ, P is an abelian ℓ-group and E an ℓ'-group). The determination of the $L(S')$ requires the determination of the Green correspondents of simple modules: this computation is not directly feasible for larger groups (for example the Monster). Given a perversity function π, the calculation of the C_S is a reasonable computational task. A more tricky matter is the determination of an appropriate function π (in general, there are infinitely many). There are constraints: the filtration on $\mathcal{S}_A$ should make the decomposition matrix of A triangular. Also, the datum π modulo 2 should come from a perfect isometry. As for the specific value of π, we have proceeded by trying systematically all possibilities, increasing progressively the values of π. (For finite groups of Lie type in non-defining characteristic then we use the perversity function given in [14].)

Let us explain this more precisely. Let G be a finite group, $A = \mathcal{O}B_0(G)$ and H another finite group with principal block $B = \mathcal{O}B_0(H)$. Assume that we are given a standard equivalence $F : D^b(A) \xrightarrow{\sim} D^b(B)$ inducing a perverse equivalence $D^b(kA) \xrightarrow{\sim} D^b(kB)$ relative to $\pi : \mathcal{S}_B \to \mathbf{Z}_{\geqslant 0}$. Note that this provides a bijection $\mathcal{S}_A \xrightarrow{\sim} \mathcal{S}_B$. The map $I : K_0(KA) \xrightarrow{\sim} K_0(KB)$ induced by KF is a perfect isometry [6]. There is a map $\varepsilon : \mathcal{S}_{KA} \to \{\pm 1\}$ and a bijection $J : \mathcal{S}_{KA} \xrightarrow{\sim} \mathcal{S}_{KB}$ such that $I(\chi) = \varepsilon(\chi)J(\chi)$ for $\chi \in \mathcal{S}_{KA}$. We have $\chi(1) \equiv \varepsilon(\chi)J(\chi)(1) \pmod{\ell}$, if $I(K) = K$.

Let Z be a subset of $\mathcal{S}_{KB}$ whose image in $K_0(kB)$ by the decomposition map is a basis (Z is a "basic set"). In this case, $J^{-1}(Z)$ is a basic set for A.

Assume now that the image of Z in $K_0(kB)$ is the basis given by $\mathcal{S}_B$; this provides a bijection $Z \xrightarrow{\sim} \mathcal{S}_B$. Given $V \in \mathcal{S}_B$ corresponding to $\psi \in Z$, we have $\pi(V) \equiv \varepsilon(J^{-1}(\psi)) \pmod 2$.

Define a partial order on $\mathcal{S}_B$ by $V > V'$ if $\pi(V) > \pi(V')$. This gives an order on Z and on $\mathcal{S}_A$. Then the decomposition of the irreducible characters in $J^{-1}(Z)$ is given by a unitriangular matrix.

3.3. Stable equivalences for $\ell \times \ell$

3.3.1. Construction of a complex of bimodules

We recall the construction of [51, §6.2]. Let G be a finite group, ℓ a prime and P a Sylow ℓ-subgroup of G. We assume in this subsection that K contains all $|G|$th roots of unity. Let $H = N_G(P)$. We assume that P is elementary abelian of order ℓ^2 and G is not ℓ-nilpotent.

Let Q be a subgroup of P of order ℓ. Let $\bar{N}_H(Q)$ be a complement to Q in $N_H(Q)$, so that $N_H(Q) = Q \rtimes \bar{N}_H(Q)$. Let $\bar{C}_H(Q) = C_H(Q) \cap \bar{N}_H(Q) \lhd N_H(Q)$. Let $\bar{N}_G(Q)$ be a complement to Q in $N_G(Q)$ containing $\bar{N}_H(Q)$. Let $\bar{C}_G(Q) = C_G(Q) \cap \bar{N}_G(Q) \lhd N_G(Q)$. We have $\bar{N}_G(Q) = \bar{C}_G(Q)\bar{N}_H(Q)$.

Let d be the distance from the edge corresponding to k to the exceptional vertex in the Brauer tree of $kB_0(\bar{C}_G(Q))$. Let $\mathcal{E}$ be the set of simple modules (up to isomorphism) of $kB_0(\bar{C}_G(Q))$ whose distance to the exceptional vertex is $d+1$ (mod 2); hence, $k \notin \mathcal{E}$. Define an injection $\gamma : \mathcal{E} \hookrightarrow \mathcal{S}_{kB_0(\bar{C}_H(Q))}$: $\gamma(S)$ is the unique simple $k\bar{C}_H(Q)$-module such that $\operatorname{Hom}_{k\bar{C}_H(Q)\text{-stab}}(\operatorname{Res}^{\bar{C}_G(Q)}_{\bar{C}_H(Q)} S, \gamma(S)) \neq 0$. The set $\mathcal{E}$ and the map γ are $\bar{N}_H(Q)$-stable.

Let $N_\Delta = (\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}})\Delta\bar{N}_H(Q)$. We have a decomposition of $\mathbf{F}_\ell N_\Delta$-modules

$$b_0\bigl(\bar{C}_H(Q)\bigr)\mathbf{F}_\ell\bar{C}_G(Q)b_0\bigl(\bar{C}_G(Q)\bigr) = M_Q \oplus T,$$

where T is projective and M_Q restricts to an indecomposable $(\mathbf{F}_\ell B_0(\bar{C}_H(Q)) \otimes \mathbf{F}_\ell B_0(\bar{C}_G(Q))^{\mathrm{opp}})$-module inducing a stable equivalence. A projective cover of kM_Q is of the form

$$\bigoplus_{S \in \mathcal{S}_{kB_0(\bar{C}_H(Q))}} P_{\gamma(S)} \otimes P_S^* \to kM_Q.$$

The map may be chosen so that its restriction to $\bigoplus_{S\in\mathcal{E}} P_{\gamma(S)} \otimes P_S^*$ is defined over $\mathbf{F}_\ell$ and we obtain a complex of $\mathbf{F}_\ell N_\Delta$-modules

$$X = (0 \to U_Q \xrightarrow{a} M_Q \to 0)$$

with $kU_Q = \bigoplus_{S\in\mathcal{E}} P_{\gamma(S)} \otimes P_S^*$. The restriction of X to $\mathbf{F}_\ell B_0(\bar{C}_H(Q)) \otimes \mathbf{F}_\ell B_0(\bar{C}_G(Q))^{\mathrm{opp}}$ is a Rickard complex. This is the complex $C(M_Q, \mathcal{E})$ defined in a more general setting in Section 4.2.6. We put $T_Q = U_Q \oplus P$, $f = a + \mathrm{id}$, and

$$D = \bigl(0 \to T_Q \xrightarrow{f} b_0\bigl(\bar{C}_H(Q)\bigr)\mathbf{F}_\ell\bar{C}_G(Q)b_0\bigl(\bar{C}_G(Q)\bigr) \to 0\bigr),$$

a complex of $\mathbf{F}_\ell N_\Delta$-modules homotopy equivalent to X.

Define

$$T'_Q = \operatorname{Res}^{N_{H\times G^{\mathrm{opp}}}(\Delta Q)/\Delta Q}_{N_{H\times G^{\mathrm{opp}}}(\Delta Q)} \operatorname{Ind}^{N_{H\times G^{\mathrm{opp}}}(\Delta Q)/\Delta Q}_{N_\Delta} T_Q.$$

We have $T'_Q = \mathbf{F}_\ell Q \otimes T_Q$: the action of Q^{en} is the canonical action on $\mathbf{F}_\ell Q$, the action of $\bar{C}_H(Q) \times \bar{C}_G(Q)^{\mathrm{opp}}$ comes from the action on T_Q and the action of $\Delta N_H(Q)$ comes from the tensor product of the actions on $\mathbf{F}_\ell Q$ and T_Q.

The map f provides by induction a morphism of $\mathbf{F}_\ell N_{H\times G^{\mathrm{opp}}}(\Delta Q)$-modules

$$f' : T'_Q \to b_0\bigl(C_H(Q)\bigr)\mathbf{F}_\ell C_G(Q)b_0\bigl(C_G(Q)\bigr)$$

and the associated complex gives a Rickard complex by restriction to $\mathbf{F}_\ell B_0(C_H(Q)) \otimes \mathbf{F}_\ell B_0(C_G(Q))^{\mathrm{opp}}$.

Consider finally the morphism of $\mathbf{F}_\ell(H \times G^{\mathrm{opp}})$-modules

$$g_Q : b_0(H \times G)\operatorname{Ind}^{H\times G^{\mathrm{opp}}}_{N_{H\times G^{\mathrm{opp}}}(\Delta Q)} T'_Q \to b_0(H)\mathbf{F}_\ell G b_0(G)$$

deduced from f' by adjunction. Then,

$$C = \Bigl(0 \to \bigoplus_Q b_0(H \times G)\operatorname{Ind}^{H\times G^{\mathrm{opp}}}_{N_{H\times G^{\mathrm{opp}}}(\Delta Q)} T'_Q \xrightarrow{\sum g_Q} b_0(H)\mathbf{F}_\ell G b_0(G) \to 0\Bigr)$$

induces a stable equivalence between the principal blocks of $\mathbf{F}_\ell G$ and $\mathbf{F}_\ell H$ [51, Theorem 6.3]. Here, Q runs over subgroups of P of order ℓ up to H-conjugacy.

Fix a decomposition $b_0(H)\mathbf{F}_\ell G b_0(G) = M \oplus R$, where M is an indecomposable $\mathbf{F}_\ell(G \times H^\circ)$-module with vertex ΔP and R is a direct sum of indecomposable modules with vertices strictly contained in ΔP. Then, $\mathrm{Br}_{\Delta Q}(M) \simeq M_Q$ for Q a subgroup of P of order ℓ (see [5, Theorem 3.2] for example). We proceed with the construction above with T_Q replaced by U_Q: define $U'_Q = \mathrm{Res}^{N_{H\times G^{\mathrm{opp}}}(\Delta Q)/\Delta Q}_{N_{H\times G^{\mathrm{opp}}}(\Delta Q)} \mathrm{Ind}^{N_{H\times G^{\mathrm{opp}}}(\Delta Q)/\Delta Q}_{N_\Delta} U_Q$. We obtain a complex homotopy equivalent to C:

$$0 \to \bigoplus_Q b_0(H \times G) \,\mathrm{Ind}^{H\times G^{\mathrm{opp}}}_{N_{H\times G^{\mathrm{opp}}}(\Delta Q)} U'_Q \to M \to 0.$$

Remark 3.7. Note that $U'_Q = 0$ if $N_H(Q) = C_H(Q)$, or if the Brauer tree of $kB_0(\bar{C}_G(Q))$ is a star with exceptional vertex in the centre (this happens for example if $\ell = 3$). In that case, M_Q induces a splendid Morita equivalence. If this holds for all subgroups Q of P of order ℓ, then M induces a splendid stable equivalence.

3.3.2. Images

Let L be a $kB_0(G)$-module. Let Q be a subgroup of P of order ℓ. We keep the notation of Section 3.3.1. Let $\Gamma = N_\Delta \times N_G(Q)$. We have an embedding $\alpha : N_H(Q) \hookrightarrow \Gamma$, $g \mapsto ((\bar{g}, \bar{g}^{-1}), g)$, where $\bar{g} \in \bar{N}_H(Q)$ is the image of g. The action of Γ on $T_Q \otimes \mathrm{Res}^G_{N_G(Q)} L$ restricts via α to an action of $N_H(Q)$ on $T_Q \otimes_{\mathbf{F}_\ell \bar{C}_G(Q)} L$, and f induces a morphism of $kN_H(Q)$-modules $T_Q \otimes_{\mathbf{F}_\ell \bar{C}_G(Q)} L \to \mathrm{Res}^G_{N_H(Q)} L$. By adjunction, this provides a morphism of kH-modules

$$h_Q : b_0(H)\,\mathrm{Ind}^H_{N_H(Q)}(T_Q \otimes_{\mathbf{F}_\ell \bar{C}_G(Q)} L) \to b_0(H)\,\mathrm{Res}^G_H L.$$

Thus

$$C \otimes_{\mathbf{F}_\ell G} L \simeq \left(0 \to \bigoplus_Q b_0(H)\,\mathrm{Ind}^H_{N_H(Q)}(T_Q \otimes_{\mathbf{F}_\ell \bar{C}_G(Q)} L) \xrightarrow{\sum h_Q} b_0(H)\,\mathrm{Res}^G_H L \to 0\right).$$

3.3.3. Self-derived equivalences for $k\mathfrak{S}_3$

Let $G = \mathfrak{S}_3$, $\ell = 3$ and $A = kG$. Let P_1 be the projective cover of the trivial A-module S_1 and P_2 the projective cover of the non-trivial simple A-module S_2. A projective cover of A, viewed as an A^{en}-module, is

$$P_1 \otimes P_1^* \oplus P_2 \otimes P_2^* \xrightarrow{b} A.$$

We set $C = C(A, \{S_2\}) = (0 \to P_2 \otimes P_2^* \xrightarrow{b} A \to 0)$ (cf. Section 4.2.6). This is a Rickard complex. Given $n \geqslant 0$, the equivalence induced by the Rickard complex $C^{\otimes_A n}$ is perverse relative to the function π given by $\pi(1) = 0$ and $\pi(2) = n$.

We have $C \otimes_A P_2 \simeq P_2[1]$ in $\mathrm{Ho}(A)$. Assume that $n > 0$. We deduce that

$$\mathrm{Hom}_{\mathrm{Ho}(A)}\big(P_2, \mathrm{Res}_A C^{\otimes_A n}[i]\big) = \mathrm{Hom}_{\mathrm{Ho}(A^{\mathrm{opp}})}\big(P_2^*, \mathrm{Res}_{A^{\mathrm{opp}}} C^{\otimes_A n}[i]\big) = 0 \quad \text{for } i \neq -n.$$

Thus the composition factors of $H^i(C^{\otimes_A n})$ are isomorphic to $S_1 \otimes S_1^*$ for $i \neq -n$ and we deduce that there is an isomorphism in $\mathrm{Ho}(A^{\mathrm{en}})$:

$$C^{\otimes_A n} \simeq \big(0 \to P_2 \otimes P_2^* \to \cdots \to P_2 \otimes P_2^* \xrightarrow{b} A \to 0\big)$$

where the non-zero terms are in degrees $-n, \dots, 0$ and the complex on the right is the unique indecomposable complex with the given terms and with homology isomorphic to $S_1 \otimes S_1^*$ in degrees $-n+1, \dots, 0$.

Let F be a standard self-equivalence of $D^b(A)$ such that $F(S_1) \simeq S_1$. Then, F is a perverse equivalence for a perversity function π with $\pi(1) = 0$. Consequently, F (or F^{-1}) is given by the Rickard complex $C^{\otimes_A n}$ for some $n \geqslant 0$.

Remark 3.8. The group $\mathrm{TrPic}(A)$ of isomorphism classes of standard self-derived equivalences of A has been determined in [56, §4]. The result above on the subgroup of those self-equivalences that fix the trivial representation can be deduced easily.

3.3.4. Local twists for 3×3

The construction of stable equivalences in Section 3.3.1 builds on the "simplest" possible local derived equivalences. Assume that $\ell = 3$; then we have $U_Q = 0$ for all subgroups Q of P of order ℓ (see Remark 3.7). We have $b_0(H)\mathbf{F}_\ell G b_0(G) = M \oplus R$, where M is an indecomposable $(\mathbf{F}_\ell B_0(G) \otimes \mathbf{F}_\ell B_0(H))$-module inducing a stable equivalence and the indecomposable summands of R have vertex of order at most ℓ (Remark 3.7, see also [28, Lemma 3.8]).

We explain here how to modify the stable equivalence induced by M using a self-stable equivalence of $B_0(H)$. Let $\mathcal{T}$ be a set of representatives of H-conjugacy classes of subgroups Q of P of order ℓ such that $|C_H(Q)/C_H(P)| = 2$. Fix a map $\eta : \mathcal{T} \to \mathbf{Z}_{\geqslant 0}$.

Let $Q \in \mathcal{T}$. There is an ℓ'-subgroup E'_Q of $\bar{N}_H(Q)$ such that $\bar{N}_H(Q) = (P \cap \bar{C}_H(Q)) \rtimes E'_Q$. Let $E_Q = E'_Q \cap \bar{C}_H(Q)$. We have $\bar{C}_H(Q) = (P \cap \bar{C}_H(Q)) \rtimes E_Q$. Let V_Q be a non-trivial simple $\mathbf{F}_\ell B_0(\bar{N}_H(Q))$-module with non-trivial restriction to E_Q. Note that $\mathrm{Res}_{E_Q} V_Q$ is uniquely defined: this is the 1-dimensional non-trivial $\mathbf{F}_\ell(E_Q/O_{\ell'}(\bar{C}_H(Q)))$-module.

Let $N'_\Delta = \bar{C}_H(Q)^{\mathrm{en}} \Delta \bar{N}_H(Q)$. The construction of Section 3.3.3 provides an indecomposable complex of $\mathbf{F}_\ell N'_\Delta$-modules

$$X_Q = \bigl(0 \to P_{V_Q} \otimes P^*_{V_Q} \to \cdots \to P_{V_Q} \otimes P^*_{V_Q} \to \mathbf{F}_\ell B_0\bigl(\bar{C}_H(Q)\bigr) \to 0\bigr),$$

where the non-zero terms are in degrees $-\eta(Q), \dots, 0$ and whose restriction to $\mathbf{F}_\ell B_0(\bar{C}_H(Q))^{\mathrm{en}}$ is a Rickard complex.

We proceed now as in Section 3.3.1 to glue the complexes X_Q. We have $N_{H^{\mathrm{en}}}(\Delta Q) = Q^{\mathrm{en}} \times N'_\Delta$. Let

$$U'_Q = b_0\bigl(H^{\mathrm{en}}\bigr) \operatorname{Ind}_{N_{H^{\mathrm{en}}}(\Delta Q)}^{H^{\mathrm{en}}} \bigl(\mathbf{F}_\ell Q \otimes \bigl(P_{V_Q} \otimes P^*_{V_Q}\bigr)\bigr).$$

We have a bounded complex of $\mathbf{F}_\ell B_0(H^{\mathrm{en}})$-modules

$$C'(\eta) = \Bigl(\cdots \to \bigoplus_{Q \in \mathcal{T}, \eta(Q) \geqslant 3} U'_Q \xrightarrow{\sum h_Q} \bigoplus_{Q \in \mathcal{T}, \eta(Q) \geqslant 2} U'_Q \xrightarrow{\sum h_Q} \bigoplus_{Q \in \mathcal{T}, \eta(Q) \geqslant 1} U'_Q \to \mathbf{F}_\ell B_0(H) \to 0\Bigr)$$

which induces a self-stable equivalence of $\mathbf{F}_\ell B_0(H)$, for certain $h_Q \in \mathrm{End}_{\mathbf{F}_\ell(H^{\mathrm{en}})}(U'_Q)$. We put $C(\eta) = C'(\eta) \otimes_{\mathbf{F}_\ell H} M$: this complex of $(\mathbf{F}_\ell B_0(H) \otimes \mathbf{F}_\ell B_0(G)^{\mathrm{opp}})$-modules induces a stable equivalence.

Let L be a simple $\mathbf{F}_\ell B_0(G)$-module, and let L' be the unique indecomposable direct summand of $b_0(H) \operatorname{Res}_H^G L$ with vertex P. Given $Q \in \mathcal{T}$, let

$$L'_Q = \mathrm{Res}_{\Delta(Q\rtimes E'_Q)}\bigl(\mathrm{Res}^{(Q\rtimes E'_Q)/Q}_{Q\rtimes E'_Q} V_Q \\ \otimes \mathrm{Res}^{(Q\rtimes E'_Q)/E_Q}_{Q\rtimes E'_Q}\bigl(\mathrm{Hom}_{\mathbf{F}_\ell E_Q}\bigl(\mathrm{Res}^{(Q\rtimes E'_Q)/Q}_{Q\rtimes E'_Q} V_Q, L'\bigr)\bigr)\bigr).$$

Thus, $L'_Q = V_Q \otimes \mathrm{Hom}_{\mathbf{F}_\ell E_Q}(V_Q, L')$, the action of $x \in Q$ is given by $v \otimes f \mapsto v \otimes xf$ and the action of $y \in E'_Q$ is $v \otimes f \mapsto yv \otimes yfy^{-1}$, for $v \in V_Q$ and $f \in \mathrm{Hom}_{\mathbf{F}_\ell E_Q}(V_Q, L')$. We have a decomposition $\mathrm{Res}_{Q\rtimes E'_Q}(L') = L^1_Q \oplus L^2_Q$, where L^1_Q is the maximal direct summand such that $\mathrm{Res}_{E_Q}(L^1_Q)$ is a multiple of V_Q. Then $L'_Q \simeq V_Q \otimes L^1_Q$.

Consider a decomposition $L'_Q = L''_Q \oplus P$ as $\mathbf{F}_\ell(Q \rtimes E'_Q)$-modules, where P is projective. Let $L_Q = b_0(H)\,\mathrm{Ind}^H_{Q\rtimes E'_Q} L''_Q$. We have an isomorphism

$$C(\eta) \otimes_{\mathbf{F}_\ell G} L \simeq \Bigl(\cdots \to \bigoplus_{Q\in\mathcal{T},\eta(Q)\geqslant 3} L_Q \xrightarrow{\sum \mathrm{Ind}(s_Q)} \bigoplus_{Q\in\mathcal{T},\eta(Q)\geqslant 2} L_Q \\ \xrightarrow{\sum \mathrm{Ind}(s_Q)} \bigoplus_{Q\in\mathcal{T},\eta(Q)\geqslant 1} L_Q \to L' \to 0\Bigr) \\ \oplus \text{ bounded complex of projective modules,}$$

where $s_Q \in \mathrm{End}_{\mathbf{F}_\ell N_H(Q)}(\mathrm{Ind}^{N_H(Q)}_{Q\rtimes E'_Q} L''_Q)$ is non-zero, but not invertible.

Remark 3.9. In the examples studied in Section 5, the maps in the complexes are uniquely determined up to scalars, thanks to the fact that the following conditions hold:

- when $\eta(Q) > 0$, $\dim \mathrm{Hom}_{kH}(L_Q, L') = 1$,
- when $\eta(Q) > 1$, $\dim L''_Q = 1$.

Let $\tilde{G}$ be a finite group containing G as a normal subgroup of ℓ'-index. If the function η is invariant under the action of $\tilde{G}$ on conjugacy classes of subgroups of order 3, then $C(\eta)$ extends to a complex of $k((H \times G^{\mathrm{opp}})\Delta N_{\tilde{G}}(P))$-modules.

Remark 3.10. There are six conjugacy classes of $3'$-subgroups E of $\mathrm{GL}_2(\mathbf{F}_3)$ such that $(\mathbf{F}_3^2)^E = 0$. They are determined by their isomorphism type: Z_2, Z_2^2, Z_4, D_8, Q_8 and SD_{16}. Assume that all non-trivial elements of E act fixed-point freely on $(\mathbf{F}_3)^2 - \{0\}$: this corresponds to the types Z_2, Z_4 and Q_8. Let $A = kP \rtimes E$, where $P = Z_3^2$. Let M' be an A^{en}-module inducing a self-stable equivalence. By [9, Theorem 3.2] there is an integer n such that $\Omega^n_{A^{\mathrm{en}}}(M')$ induces a self-Morita equivalence. Let G be a finite group with Sylow 3-subgroup P and with $N_G(P)/C_G(P) = E$. Let C be a two-sided tilting complex for $(A, kB_0(G))$. Let $D = \mathrm{Hom}^\bullet_{kB_0(G)^{\mathrm{opp}}}(C, M)$. This induces a self-stable equivalence of A, so there is an integer n and an invertible A^{en}-module M'' such that $\Omega^n(M'') \otimes_A D$ is isomorphic to A in A^{en}-stab. Let $C' = \mathrm{Hom}^\bullet_A(M''[-n], C)$: this is a two-sided tilting complex for $(A, kB_0(G))$ and it is isomorphic in the stable category to M. So, given a two-sided tilting complex, we obtain all stable equivalences up to a shift and a self-Morita equivalence.

Note that if G is simple then the automizer E will be of type Z_4, D_8, Q_8 or SD_{16}.

3.4. Lie type

3.4.1. Deligne–Lusztig varieties

For finite groups of Lie type in non-describing characteristic, Broué conjectured that a solution of the abelian defect conjecture will arise from the complex of cohomology of a Deligne–Lusztig variety [6, §6]. That is known in very few cases, and in those cases, defect groups are cyclic [52, 3,18–20]. We recall now the setting and constructions of [7].

Let $\mathbf{G}$ be a reductive connected algebraic group endowed with an endomorphism F such that there is $\delta \in \mathbf{Z}_{>0}$ with the property that F^δ is a Frobenius endomorphism relative to an $\mathbf{F}_{q^\delta}$ structure on $\mathbf{G}$. Here, $q \in \mathbf{R}_{>0}$ and we assume there is a choice $q \in K$. Let $G = \mathbf{G}^F$. Let W be the Weyl group of $\mathbf{G}$ and B^+ be the braid monoid of W. We denote by ϕ the automorphisms of W and B^+ induced by F. Let $w \mapsto \mathbf{w} : W \to B^+$ be the length-preserving lift of the canonical map $B^+ \to W$. Let $\pi = \mathbf{w}_0^2$, where w_0 is the longest element of W.

Let ℓ be a prime number that does not divide q^δ, and let P be a Sylow ℓ-subgroup of G. We assume that P is abelian and $C_{\mathbf{G}}(P)$ is a torus. Let d be the multiplicative order of q in $k^\times$: this is a ϕ-regular number for W. There exists $b_d \in B^+$ such that $(b_d\phi)^d = \pi\phi^d$. Let $B_d^+ = C_{B^+}(b_d\phi)$, and let $Y(b_d)$ be the corresponding Deligne–Lusztig variety. The complex $C = R\Gamma(Y(b_d), \mathcal{O})b_0(G)$ has an action of $C_G(P) \times G^{\mathrm{opp}}$. It is conjectured that

- the action extends (up to homotopy) to an action of $(C_G(P) \rtimes B_d^+) \times G^{\mathrm{opp}}$, and
- the canonical map $\mathcal{O}(C_G(P) \rtimes B_d^+) \to \mathrm{End}^\bullet_{D(\mathcal{O}G^{\mathrm{opp}})}(C)$ is a quasi-isomorphism of algebras, with image isomorphic to $\mathcal{O}B_0(N_G(P))$.

It is conjectured further [12] that these equivalences are perverse. Let us explain how the maps π_Q of Section 3.1.4 are encoded in the geometry.

Given χ a unipotent character of G, let A_χ denote the degree of its generic degree. Conjecturally, if $\mathbf{G}$ has connected centre and ℓ is good, the unipotent characters in $B_0(G)$ form a basic set and the decomposition matrix of $B_0(G)$ is unitriangular with respect to that basic set, for the order given by the function A (cf. [24, Conjecture 3.4] and [23, Conjecture 1.3]). This gives a bijection between $\mathcal{S}_{B_0(G)}$ and the set of unipotent characters in $B_0(G)$. The function π_1 should be given by the unique degree of cohomology of $Y(b_d, K)$ where the corresponding unipotent character occurs.

The complex C has a canonical representative $\tilde{R}\Gamma(Y(b_d), \mathcal{O})$ in $\mathrm{Ho}(\mathcal{O}(C_G(P) \times G^{\mathrm{opp}}))$ that is splendid [47,52] and given Q a subgroup of P, we have $k\,\mathrm{Br}_{\Delta Q}(\tilde{R}\Gamma(Y(b_d))) \simeq \tilde{R}\Gamma(Y(b_d)^{\Delta Q}, k)$. Hence, the local derived equivalences are controlled by Deligne–Lusztig varieties associated with Levi subgroups of $\mathbf{G}$ and this gives a corresponding description for the functions π_Q.

There is a conjecture for the unipotent part of the cohomology of Deligne–Lusztig varieties associated with powers of $\mathbf{w}_0$. For applications to Broué's conjecture, the conjecture below covers the cases $\ell \mid (q \pm 1)$.

Conjecture 3.11. *(See [17, §3.3.23].) Let χ be a unipotent character of G. Given $n, i \geqslant 0$, if $[H^i(Y(\mathbf{w}_0^n), K) : \chi] \neq 0$, then $i = nA_\chi$.*

In [14] the first author has proposed a general conjecture for the multiplicities $[H^i(Y(b_d), K) : \chi]$.

Assume that $\ell = 3$. We consider now all groups $(\mathbf{G}, F)$ such that G is semi-simple and $P \simeq (Z_3)^2$. For each such group, and for each conjugacy class of subgroups Q of order 3, we provide the semi-simple type of $(C_{\mathbf{G}}(Q), F)$ and we give an element b in the braid monoid of $C_{\mathbf{G}}(Q)$ such that $Y_G(b_d)^{\Delta Q} = Y_{C_G(Q)}(b)$. We also provide in some cases another semi-simple group and an element in the braid monoid such that the Deligne–Lusztig variety can be identified equivariantly with $Y_{C_G(Q)}(b)$ [36, §1.18].

- B_2, $d = 1$.
  - A_1, $\mathbf{s}^2$,
  - A_1, $\mathbf{s}^2$.
- ${}^2A_3$, $d = 1$.
  - A_1, $\mathbf{s}^2$,
  - $(A_1 \times A_1, (x, y) \mapsto (y, F(x)))$, $(\mathbf{s}^2, \mathbf{s}^2)$. This is equivalent to A_1, $\mathbf{s}^4$.
- ${}^2A_4$, $d = 1$.
  - ${}^2A_2$, $(\mathbf{st})^3$,
  - $(A_1 \times A_1, (x, y) \mapsto (y, F(x)))$, $(\mathbf{s}^2, \mathbf{s}^2)$. This is equivalent to A_1, $\mathbf{s}^4$.
- A_3, $d = 2$.
  - A_1, $\mathbf{s}$,
  - $(A_1 \times A_1, (x, y) \mapsto (y, F(x)))$, $(\mathbf{s}, \mathbf{s})$. This is equivalent to A_1, $\mathbf{s}^2$.
- A_4, $d = 2$.
  - A_2, $\mathbf{sts}$,
  - $(A_1 \times A_1, (x, y) \mapsto (y, F(x)))$, $(\mathbf{s}, \mathbf{s})$. This is equivalent to A_1, $\mathbf{s}^2$.
- B_2, $d = 2$.
  - A_1, $\mathbf{s}$,
  - A_1, $\mathbf{s}$.

Note that there are finite simple groups of Lie type with elementary abelian Sylow 3-subgroups of order 9 that do not arise as rational points of a reductive connected algebraic group, but as a quotient of such a group. There are two classes of such groups:

- $G = \mathrm{PSL}_3(q)$ with $q \equiv 4, 7 \pmod 9$;
- $G = \mathrm{PSU}_3(q)$ with $q \equiv 2, 5 \pmod 9$.

3.4.2. Morita equivalences

Let G be a finite group and ℓ a prime. Let T be an ℓ-nilpotent subgroup of G with Sylow ℓ-subgroup P. Let $W = N_G(T)/T$.

We assume that

- $C_T(P) = C_G(P)$,
- there is an ℓ'-subgroup U of G such that $T \subset N_G(U)$, $T \cap U = 1$ and $G = UTU$, and
- W is an ℓ'-group.

Let us recall a result of Puig [43, Corollaire 3.6].

Theorem 3.12. *The bimodule $e_U \mathbf{Z}_\ell G b_0(G)$ induces a Morita equivalence between $B_0(G)$ and $B_0(N_G(P))$, where $e_U = \frac{1}{|U|}\sum_{x\in U} x$.*

Let E be a group of automorphisms of G that stabilizes U and P. Then, the $(B_0(N_G(P)) \otimes B_0(G)^{\mathrm{opp}})$-module $e_U \mathbf{Z}_\ell G b_0(G)$ extends to a $((B_0(N_G(P)) \otimes B_0(G)^{\mathrm{opp}}) \rtimes E)$-module.

The main example is the following (cf. [8, Theorem 23.12]). We take $\mathbf{G}$, F, and so on as in Section 3.4.1, and we assume that $\delta = 1$. Let $\mathbf{T} \subset \mathbf{B}$ be an F-stable maximal torus contained in an F-stable Borel subgroup of $\mathbf{G}$ and let $\mathbf{U}$ be the unipotent radical of $\mathbf{B}$. Let $U = \mathbf{U}^F$ and $T = \mathbf{T}^F$. The assumptions above are satisfied when $\ell \mid (q-1)$ and $\ell \nmid |W^F|$. We have $N_G(P) = N_{\mathbf{G}}(\mathbf{T})^F$.

Remark 3.13. Consider the same setting for $\mathbf{G}'$ another reductive group, defined over $\mathbf{F}_{q'}$. If the finite groups $N_{\mathbf{G}}(\mathbf{T})^F / O_{\ell'}(N_{\mathbf{G}}(\mathbf{T})^F)$ and $N_{\mathbf{G}'}(\mathbf{T}')^{F'} / O_{\ell'}(N_{\mathbf{G}'}(\mathbf{T}')^{F'})$ are isomorphic, then Theorem 3.12 provides a splendid Morita equivalence between $B_0(G)$ and $B_0(G')$.

Let us be more specific for our needs. The condition above is satisfied when

- $G = \mathrm{PSU}_n(q)$, $\ell \mid (q-1)$ and $2 \neq \ell > n/2$.
- $G = \mathrm{PSp}_4(q)$ and $\ell \mid (q-1)$, $\ell \neq 2$.

Broué's conjecture predicts the existence of another derived equivalence (not a Morita equivalence), provided by the Deligne–Lusztig variety associated with the element π of the braid group. Note that such an equivalence would arise from an action of $G \times (P \rtimes B^+)^{\mathrm{opp}}$ on a geometric object, while in the Harish-Chandra equivalence above, the action of $N_G(P)$ on $\mathbf{Z}_\ell(G/U)$ doesn't arise from a monoid action on G/U.

4. Automorphisms

4.1. Stability of equivalences

Extensions of equivalences and reductions to finite simple groups have been considered in various particular situations: isotypies [22], (splendid) Rickard and derived equivalences [37]. We introduce here a framework that handles various types of equivalences.

4.1.1. Extensions of equivalences

Let R be a commutative $\mathbf{Z}_\ell$-algebra. We consider the data $\mathcal{C}$ consisting, for every finite group G, of a full subcategory $\mathcal{C}(G)$ of the category of bounded complexes of R-projective finitely generated RG-modules. We assume that $\mathcal{C}(G)$ is closed under taking direct sums and direct summands, and closed under automorphisms of G. We assume also that the following holds:

(S1) given $H \leqslant G$ of ℓ'-index and given $X \in \mathcal{C}(H)$, then $\mathrm{Ind}_H^G(X) \in \mathcal{C}(G)$.

A consequence of the assumptions is that given $X \in \mathrm{Comp}^b(RG)$ and given $H \leqslant G$ of ℓ'-index, if $\mathrm{Res}_H(X) \in \mathcal{C}(H)$, then $X \in \mathcal{C}(G)$, since X is a direct summand of $\mathrm{Ind}^G \mathrm{Res}_H X$.

Definition 4.1. Let G and H be two finite groups. We say that $X \in \mathrm{Comp}^b(RB_0(G) \otimes RB_0(H)^{\mathrm{opp}})$ *induces a $\mathcal{C}$-equivalence* between the principal blocks of G and H if

- the canonical map $RB_0(G) \to \mathrm{End}^\bullet_{RH^{\mathrm{opp}}}(X)$ is a split injection with cokernel in $\mathcal{C}(G^{\mathrm{en}})$, and
- the canonical map $RB_0(H) \to \mathrm{End}^\bullet_{RG}(X)$ is a split injection with cokernel in $\mathcal{C}(H^{\mathrm{en}})$.

Given $G \lhd \tilde{G}$ and $H \lhd \tilde{H}$ with $\tilde{H}/H = \tilde{G}/G = E$ an ℓ'-group, we put $\tilde{\Delta}(G, H) = \{(x, y) \in \tilde{G} \times \tilde{H}^{\mathrm{opp}} \mid (xG, yH) \in \Delta E\}$.

Definition 4.2. We say that $X \in \mathrm{Comp}^b(R\tilde{\Delta}(G, H))$ *induces an E-equivariant $\mathcal{C}$-equivalence* between the principal blocks of G and H if $\mathrm{Res}_{G\times H^{\mathrm{opp}}}(X)$ induces a $\mathcal{C}$-equivalence between the principal blocks of G and H and $b_0(\tilde{G})\,\mathrm{Ind}^{\tilde{G}\times\tilde{H}^{\mathrm{opp}}} X = b_0(\tilde{H})\,\mathrm{Ind}^{\tilde{G}\times\tilde{H}^{\mathrm{opp}}} X$.

Lemma 4.3. *Let $G \lhd \tilde{G}$ be finite groups with $\ell \nmid [\tilde{G} : G]$. Let $H \lhd \tilde{H}$ with $\tilde{H}/H = \tilde{G}/G$. Let $X \in \mathrm{Comp}^b(R\tilde{\Delta}(G, H))$ be a complex inducing an equivariant $\mathcal{C}$-equivalence between the principal blocks of G and H.*

Then $b_0(\tilde{G} \times \tilde{H})\,\mathrm{Ind}^{\tilde{G}\times\tilde{H}^{\mathrm{opp}}} X$ induces a $\mathcal{C}$-equivalence between the principal blocks of $\tilde{G}$ and $\tilde{H}$.

Proof. Let $X_1 = \mathrm{Res}_{G\times H^{\mathrm{opp}}}(X)$ and $X_2 = \mathrm{Ind}^{\tilde{G}\times\tilde{H}^{\mathrm{opp}}}(X)$. We have canonical isomorphisms (Mackey formula)

$$\mathrm{Res}_{G\times\tilde{H}^{\mathrm{opp}}} X_2 \xrightarrow{\sim} \mathrm{Ind}^{G\times\tilde{H}^{\mathrm{opp}}} X_1 \quad \text{and} \quad \mathrm{Res}_{\tilde{G}\times H^{\mathrm{opp}}} X_2 \xrightarrow{\sim} \mathrm{Ind}^{\tilde{G}\times H^{\mathrm{opp}}} X_1.$$

We have canonical isomorphisms in $\mathrm{Comp}^b(R(\tilde{G} \times G^{\mathrm{opp}}))$:

$$\begin{aligned}\mathrm{Res}_{\tilde{G}\times G^{\mathrm{opp}}} \mathrm{End}^\bullet_{R\tilde{H}^{\mathrm{opp}}}(X_2) &\xrightarrow{\sim} \mathrm{Hom}^\bullet_{R\tilde{H}^{\mathrm{opp}}}(\mathrm{Res}_{G\times\tilde{H}^{\mathrm{opp}}} X_2, X_2)\\ &\xrightarrow{\sim} \mathrm{Hom}^\bullet_{R\tilde{H}^{\mathrm{opp}}}\big(\mathrm{Ind}^{G\times\tilde{H}^{\mathrm{opp}}} X_1, X_2\big)\\ &\xrightarrow{\sim} \mathrm{Hom}^\bullet_{RH^{\mathrm{opp}}}(X_1, \mathrm{Res}_{\tilde{G}\times H^{\mathrm{opp}}} X_2)\\ &\xrightarrow{\sim} \mathrm{Hom}^\bullet_{RH^{\mathrm{opp}}}\big(X_1, \mathrm{Ind}^{\tilde{G}\times H^{\mathrm{opp}}} X_1\big)\\ &\xrightarrow{\sim} R\tilde{G} \otimes_{RG} \mathrm{End}^\bullet_{RH^{\mathrm{opp}}}(X_1).\end{aligned}$$

We deduce a commutative diagram in $\mathrm{Comp}^b(RG^{\mathrm{en}})$:

$$\begin{tikzcd} RB_0(\tilde{G}) \otimes_{RB_0(G)} RB_0(G) \arrow[r, "1\otimes\mathrm{can}"] \arrow[d, "\sim"', "\mathrm{mult}"] & RB_0(\tilde{G}) \otimes_{RB_0(G)} \mathrm{End}^\bullet_{RH^{\mathrm{opp}}}(X_1) \arrow[d, "\sim"] \\ RB_0(\tilde{G}) \arrow[r, "\mathrm{can}"'] & b_0(\tilde{G})\,\mathrm{End}^\bullet_{R\tilde{H}^{\mathrm{opp}}}(X_2) \end{tikzcd}$$

It follows that the canonical map $RB_0(\tilde{G}) \to \mathrm{End}^\bullet_{R\tilde{H}^{\mathrm{opp}}}(b_0(\tilde{G})X_2)$ is a split injection with cokernel in $\mathcal{C}(\tilde{G}^{\mathrm{en}})$.

The other condition is checked by swapping the roles of G and H^{opp}. $\square$

Using the notation of the proof of Lemma 4.3, note that we have a commutative diagram

$$\begin{tikzcd} \mathrm{Comp}^b(RB_0(\tilde{H})) \arrow[r, "X_2\otimes_{R\tilde{H}}-"] \arrow[d, "\mathrm{Res}"'] & \mathrm{Comp}^b(RB_0(\tilde{G})) \arrow[d, "\mathrm{Res}"] \\ \mathrm{Comp}^b(RB_0(H)) \arrow[r, "X_1\otimes_{RH}-"'] & \mathrm{Comp}^b(RB_0(G)) \end{tikzcd}$$

Remark 4.4. Consider $G \lhd \tilde{G}$ and $H \lhd \tilde{H}$ with $\tilde{H}/H = \tilde{G}/G = E$ an ℓ'-group. Let P be a Sylow ℓ-subgroup of G and Q a Sylow ℓ-subgroup of H. If $C_{\tilde{G}}(P) \subset G$ and $C_{\tilde{H}}(Q) \subset H$, then given $X \in \mathrm{Comp}^b(R\tilde{\Delta}(G,H))$ whose restriction is in $\mathrm{Comp}^b(RB_0(G \times H^{\mathrm{opp}}))$, we have $X \in \mathrm{Comp}^b(RB_0(\tilde{\Delta}(G,H)))$ [2, Theorem 6.4.1].

We can even do a little better to extend equivalences.

Lemma 4.5. *Consider finite groups $G_1 \lhd G_2 \lhd \tilde{G}_2 \leqslant \tilde{G}_1$ and $H_1 \lhd H_2 \lhd \tilde{H}_2 \leqslant \tilde{H}_1$ with $G_1 \lhd \tilde{G}_1$, $H_1 \lhd \tilde{H}_1$, $\ell \nmid [\tilde{G}_1 : G_1]$, $\tilde{G}_1/G_1 = \tilde{H}_1/H_1$, $\tilde{G}_2/G_1 = \tilde{H}_2/H_1$ and $G_2/G_1 = H_2/H_1$ (compatible with the inclusions $G_2/G_1 \leqslant \tilde{G}_2/G_1 \leqslant \tilde{G}_1/G_1$ and $H_2/H_1 \leqslant \tilde{H}_2/H_1 \leqslant \tilde{H}_1/H_1$). Let $E_i = \tilde{G}_i/G_i$.*

Let $X \in \mathrm{Comp}^b(R\tilde{\Delta}(G_1,H_1))$ be a complex inducing an E_1-equivariant $\mathcal{C}$-equivalence between the principal blocks of G_1 and H_1.

Then $b_0(G_2 \times H_2^{\mathrm{opp}}) \operatorname{Ind}^{\tilde{\Delta}(G_2,H_2)} \operatorname{Res}_{\tilde{\Delta}(G_1,H_1)\cap(\tilde{G}_2\times\tilde{H}_2^{\mathrm{opp}})}(X)$ induces an E_2-equivariant $\mathcal{C}$-equivalence between the principal blocks of G_2 and H_2.

Proof. If $G_1 = G_2$, then in this case the result is clear. If $\tilde{G}_1 = \tilde{G}_2$, then

$$\operatorname{Res}_{G_2\times H_2^{\mathrm{opp}}} \operatorname{Ind}^{\tilde{\Delta}(G_2,H_2)}(X) \simeq \operatorname{Ind}^{G_2\times H_2^{\mathrm{opp}}} \operatorname{Res}_{\tilde{\Delta}(G_1,H_1)\cap(G_2\times H_2^{\mathrm{opp}})} X,$$

and the result follows from Lemma 4.3. The general case follows from the two cases studied above. □

Lemma 4.6. *Let $G \lhd \tilde{G} \leqslant \hat{G}$ with $G \lhd \hat{G}$ and $\ell \nmid [\hat{G} : G]$. Let P be a Sylow ℓ-subgroup of G. Let $\tilde{H} = GC_{\tilde{G}}(P)$ and $\hat{H} = GC_{\hat{G}}(P)$. Assume that $\hat{G} = \tilde{G}C_{\hat{G}}(P)$.*

The $B_0(\tilde{H} \times \hat{H}^{\mathrm{opp}})$-module $B_0(\tilde{H}) \otimes_{\mathbf{Z}_\ell G} B_0(\hat{H})$ extends to a $B_0((\tilde{H} \times \hat{H}^{\mathrm{opp}})\Delta(\tilde{G}))$-module M, where $h \in \tilde{G}$ sends $x \otimes y$ to $hxh^{-1} \otimes hyh^{-1}$. The module M induces a splendid $(\tilde{G}/\tilde{H})$-equivariant Morita equivalence between $B_0(\tilde{H})$ and $B_0(\hat{H})$, and the module $\operatorname{Ind}^{\tilde{G}\times\hat{G}^{\mathrm{opp}}}(M)$ provides an isomorphism of algebras

$$B_0(\tilde{G}) \xrightarrow{\sim} B_0(\hat{G}), \quad x \mapsto b_0(\hat{G})x.$$

Proof. The Alperin–Dade theorem ([1, Theorem 2], [15]) shows that there are isomorphisms

$$B_0(G) \xrightarrow{\sim} B_0(\tilde{H}), \quad a \mapsto b_0(\tilde{H})a \quad \text{and} \quad B_0(G) \xrightarrow{\sim} B_0(\hat{H}), \quad a \mapsto b_0(\hat{H})a.$$

We obtain an isomorphism

$$B_0(G) \xrightarrow{\sim} B_0(\tilde{H}) \otimes_{\mathbf{Z}_\ell G} B_0(\hat{H}), \quad a \mapsto b_0(\tilde{H})a \otimes b_0(\hat{H})$$

compatible with the $\Delta(\tilde{G})$-action described in the lemma, and this provides $M \simeq B_0(G)$ with a structure of a $B_0((\tilde{H} \times \hat{H}^{\mathrm{opp}})\Delta(\tilde{G}))$-module. Note that $\operatorname{Res}_{\tilde{H}\times\hat{H}^{\mathrm{opp}}}(M)$ induces a Morita equivalence that sends $B_0(\tilde{H})$ to $B_0(\hat{H})$. We have $\operatorname{Ind}_{\tilde{H}}^{\tilde{G}}(B_0(\tilde{H})) = B_0(\tilde{G})$ and $\operatorname{Ind}_{\hat{H}}^{\hat{G}}(B_0(\hat{H})) = B_0(\hat{G})$ [2, Theorem 6.4.1(v)], hence the Morita equivalence induced by $\operatorname{Ind}^{\tilde{G}\times\hat{G}^{\mathrm{opp}}}(M)$ sends $B_0(\tilde{G})$ to $B_0(\hat{G})$ (cf. the proof of Lemma 4.3) and it gives rise to the isomorphism of algebras described in the lemma. □

We assume now that the data $\mathcal{C}$ satisfy the following additional assumption:

(S2) given G, G' two finite groups, given $X \in \mathcal{C}(G)$ and $Y \in \mathrm{Comp}^b(RG')$ with Y^i projective over R for all i, then $X \otimes_R Y \in \mathcal{C}(G \times G')$.

Lemma 4.7. *Let $G_i \lhd \tilde{G}_i$ and $H_i \lhd \tilde{H}_i$ for $i = 1, 2$. Assume that $\tilde{G}_i/G_i = \tilde{H}_i/H_i$ and $\ell \nmid [\tilde{G}_i : G_i]$. Let X_i be a complex inducing a $\tilde{G}_i/G_i$-equivariant $\mathcal{C}$-equivalence between the principal blocks of G_i and H_i for $i = 1, 2$. Then, $X_1 \otimes_R X_2$ induces a $(\tilde{G}_1 \times \tilde{G}_2)/(G_1 \times G_2)$-equivariant $\mathcal{C}$-equivalence between the principal blocks of $G_1 \times G_2$ and $H_1 \times H_2$.*

Proof. The equivariance part is clear, so we can assume $\tilde{G}_i = G_i$ and $\tilde{H}_i = H_i$. We have a canonical isomorphism $\mathrm{End}^\bullet_{RH_1^{\mathrm{opp}}}(X_1) \otimes_R \mathrm{End}^\bullet_{RH_2^{\mathrm{opp}}}(X_2) \xrightarrow{\sim} \mathrm{End}^\bullet_{R(H_1 \times H_2)^{\mathrm{opp}}}(X_1 \otimes_R X_2)$. The canonical map $R(G_1 \times G_2) \to \mathrm{End}^\bullet_{R(H_1 \times H_2)^{\mathrm{opp}}}(X_1 \otimes_R X_2)$ is a split injection, with cokernel L isomorphic to $(R_1 \otimes_R RG_2) \oplus (RG_1 \otimes_R R_2) \oplus (R_1 \otimes_R R_2)$, where R_i is the cokernel of the canonical map $RG_i \to \mathrm{End}^\bullet_{RH_i^{\mathrm{opp}}}(X_i)$. It follows from (S2) that $L \in \mathcal{C}(G_1 \times G_2)$. The other property is obtained by swapping the roles of G_i and H_i^{opp}. □

If A is an R-algebra, $n \geqslant 0$ is an integer, and X is a complex of A-modules, then there is a canonical extension of $X^{\otimes n}$ from a complex of $A^{\otimes n}$-modules to a complex of $A \wr \mathfrak{S}_n$-modules: it is obtained as the total complex associated with an n-fold complex [16, §1.1] (see also [37, Lemma 4.1] for an explicit description). The following lemma is a consequence of Lemma 4.7.

Lemma 4.8. *Let $G \lhd \tilde{G}$, $H \lhd \tilde{H}$ with $\tilde{G}/G = \tilde{H}/H$ and $\ell \nmid [\tilde{G} : G]$. Let X be a complex inducing a $\mathcal{C}$-equivariant equivalence between the principal blocks of G and H.*

Let $n \geqslant 1$ and let L be an ℓ'-subgroup of $\mathfrak{S}_n$. Then, $X^{\otimes n}$ induces a $\mathcal{C}$-equivariant equivalence between the principal blocks of $G \wr L$ and $H \wr L$.

4.1.2. Stability of properties of finite groups with abelian Sylow p-subgroups

Let $\mathcal{E}_1$ be the set of finite groups with abelian Sylow ℓ-subgroups and let $\mathcal{E}$ be the set of pairs $(G, \tilde{G})$ with $G \in \mathcal{E}_1$, $G \lhd \tilde{G}$ and $\ell \nmid [\tilde{G} : G]$.

Recall that if P is a Sylow ℓ-subgroup of G, then $\tilde{G} = GN_{\tilde{G}}(P)$ (Frattini argument), and hence $N_{\tilde{G}}(P)/N_G(P) = \tilde{G}/G$. There is an ℓ'-subgroup E of $N_{\tilde{G}}(P)$ such that $N_{\tilde{G}}(P) = P \rtimes E$. We have $\tilde{G} = GE$ and $\tilde{G}$ is a quotient of $G \rtimes E$ by an ℓ'-subgroup. Let $N_\Delta(G, \tilde{G}) = \tilde{\Delta}(H, G)$, where $H = N_G(P)$ and $\tilde{H} = N_{\tilde{G}}(P)$. We have $N_\Delta(G, \tilde{G}) = (H \times G^{\mathrm{opp}})\Delta\tilde{H}$.

Definition 4.9. We say that a subset $\mathcal{P}$ of $\mathcal{E}$ satisfies $(*)$ if

(i) $\mathcal{P}$ is closed under direct products;
(ii) given $(H, \tilde{H}) \in \mathcal{P}$ and $(G, \tilde{G}) \in \mathcal{E}$ such that $H \lhd G \lhd \tilde{G} \leqslant \tilde{H}$, we have $(G, \tilde{G}) \in \mathcal{P}$;
(iii) if $(G, \tilde{G}) \in \mathcal{P}$, $n \geqslant 0$ and L is an ℓ'-subgroup of $\mathfrak{S}_n$, then $(G^n, \tilde{G} \wr L) \in \mathcal{P}$;
(iv) if G is an abelian ℓ-group, then $(G, \tilde{G}) \in \mathcal{P}$;
(v) $(G/O_{\ell'}(G), \tilde{G}/O_{\ell'}(\tilde{G})) \in \mathcal{P}$ if and only if $(G, \tilde{G}) \in \mathcal{P}$.

We say that a subset $\mathcal{P}$ of $\mathcal{E}$ satisfies $(*')$ if, in addition, we have

(vi) given $(G, \tilde{G}) \in \mathcal{P}$ and $(G, \hat{G}) \in \mathcal{E}$ with $\tilde{G} \leqslant \hat{G}$, and given a Sylow ℓ-subgroup P of G, if $\hat{G} = \tilde{G}C_{\hat{G}}(P)$ then $(G, \hat{G}) \in \mathcal{P}$.

Proposition 4.10. *Let $\mathcal{P}$ be a subset of $\mathcal{E}$ satisfying $(*)$ (resp. $(*')$). Let $\mathcal{F}$ be a set of non-cyclic finite simple groups with non-trivial abelian Sylow ℓ-subgroups. Given $G \in \mathcal{F}$, let $\hat{G} \leqslant \mathrm{Aut}(G)$ be such that the image of $\hat{G}$ in $\mathrm{Out}(G)$ is a Hall ℓ'-subgroup of $\mathrm{Out}(G)$. Assume that there is a* *pair $(G,\tilde{G})\in\mathcal{P}$ such that $\tilde{G}/GC_{\tilde{G}}(G)=\hat{G}/G$ (resp. such that $\tilde{G}/GC_{\tilde{G}}(G)\leqslant\hat{G}/G$ and given a Sylow ℓ-subgroup P of G, we have $N_{\tilde{G}}(P)/C_{\tilde{G}}(P)=N_{\hat{G}}(P)/C_{\hat{G}}(P)$).*

If $(G,\tilde{G})\in\mathcal{E}$ is such that all non-cyclic composition factors of G of order divisible by ℓ are in $\mathcal{F}$, then $(G,\tilde{G})\in\mathcal{P}$.

Proof. Let us show first that the assumptions for $(*')$ imply those for $(*)$. Let $G\in\mathcal{F}$ and $\tilde{G}$ be as in the "resp." case of the proposition. Because of $(*)$(v), we may assume that $O_{\ell'}(\tilde{G})=1$, hence we may assume that $\tilde{G}\leqslant\hat{G}$. We have $\hat{G}=\tilde{G}C_{\hat{G}}(P)$, and so $(G,\hat{G})\in\mathcal{P}$ by $(*')$(vi).

Let us now prove the proposition in the case $(*)$. One may assume that $O_{\ell'}(G)=1$. It follows from the classification of finite simple groups [22, §5] that there is a collection

- $(H_0,\tilde{H}_0)\in\mathcal{E}$ where H_0 is an ℓ-group,
- $H_1,\dots,H_n\in\mathcal{F}$,
- $d_1,\dots,d_n\in\mathbf{Z}_{>0}$, and
- $L_1,\dots,L_n$ a family of ℓ'-subgroups of $\mathfrak{S}_{d_1},\dots,\mathfrak{S}_{d_n}$,

and there are embeddings

$$H_0\times H_1^{d_1}\times\cdots\times H_n^{d_n}\lhd G\lhd\tilde{G}\leqslant\tilde{H}_0\times\hat{H}_1\wr L_1\times\cdots\times\hat{H}_n\wr L_n,$$

where $H_i\lhd\hat{H}_i$ and $\hat{H}_i/H_i$ is a Hall ℓ'-subgroup of $\mathrm{Out}(H_i)$.

Property $(*)$(iv) ensures that $(H_0,\tilde{H}_0)\in\mathcal{P}$. Assume $i>0$. Consider a pair $(H_i,\tilde{H}_i)\in\mathcal{P}$ as in the proposition. We have $\tilde{H}_i/O_{\ell'}(\tilde{H}_i)\simeq\hat{H}_i$ and we deduce from $(*)$(v) that $(H_i,\hat{H}_i)\in\mathcal{P}$. Hence, $(H_i^{d_i},\hat{H}_i\wr L_i)\in\mathcal{P}$ by $(*)$(iii). We deduce that $(H_0\times H_1^{d_1}\times\cdots\times H_n^{d_n},\tilde{H}_0\times\hat{H}_1\wr L_1\times\cdots\times\hat{H}_n\wr L_n)\in\mathcal{P}$ by $(*)$(i), hence $(G,\tilde{G})\in\mathcal{P}$ by $(*)$(ii). □

4.1.3. Equivalences and Broué's conjecture

Proposition 4.11. *Let $\mathcal{C}$ be data satisfying* (S1) *and* (S2). *Let $\mathcal{P}$ be the set of pairs $(G,\tilde{G})\in\mathcal{E}$ such that there is a $\tilde{G}/G$-equivariant $\mathcal{C}$-equivalence between the principal blocks of G and $N_G(P)$. The set $\mathcal{P}$ satisfies property $(*')$.*

Proof. Conditions (i), (ii) and (iii) follow from Lemmas 4.7, 4.5 and 4.8, respectively. Since the principal blocks of G and $G/O_{\ell'}(G)$ are isomorphic, we have $(G,\tilde{G})\in\mathcal{P}$ if and only if $(G/O_{\ell'}(G),\tilde{G}/O_{\ell'}(G))\in\mathcal{P}$. Assume that $O_{\ell'}(G)=1$. Then $O_{\ell'}(\tilde{G})$ centralizes G, and hence $(G,\tilde{G})\in\mathcal{P}$ if and only if $(G,\tilde{G}/O_{\ell'}(\tilde{G}))\in\mathcal{P}$. So, condition (v) holds. Condition (iv) holds as well: take $X=RG$.

Consider now $G\lhd\tilde{G}\leqslant\hat{G}$ with $G\lhd\hat{G}$ and $\ell\nmid[\hat{G}:G]$. Let P be a Sylow ℓ-subgroup of G and assume that $\hat{G}=\tilde{G}C_{\hat{G}}(P)$. Let $X\in\mathrm{Comp}^b(RN_\Delta(G,\tilde{G}))$ inducing an equivariant $\mathcal{C}$-equivalence between the principal blocks of G and $N_G(P)$. We have

$$N_\Delta(G,\hat{G})=\bigl(N_G(P)\times G^{\mathrm{opp}}\bigr)\Delta N_{\tilde{G}}(P)\Delta C_{\hat{G}}(P)=N_\Delta(G,\tilde{G})C_{N_\Delta(G,\hat{G})}\bigl(P^{\mathrm{en}}\bigr).$$

Let $Y=b_0(N_\Delta(G,\tilde{G}))\,\mathrm{Ind}_{N_\Delta(G,\tilde{G})}^{N_\Delta(G,\hat{G})}(X)$. We have $\mathrm{Res}_{N_\Delta(G,\tilde{G})}(Y)\simeq X$ by Lemma 4.6, and this shows condition (vi). □

The next proposition is clear.

Proposition 4.12. *The following data $\mathcal{C}$ satisfy properties* (S1) *and* (S2):

- *acyclic complexes with R-projective components* ($\mathcal{C}$*-equivalences are standard derived equivalences*);
- *contractible complexes with R-projective components* ($\mathcal{C}$*-equivalences are Rickard equivalences*).

4.2. Automorphisms

We provide extension results for derived equivalences in the presence of automorphism groups. The main results are due to Marcus [37,39,40] (except for Section 4.2.7).

4.2.1. Extensions of modules

Let A be a k-algebra and G a finite group endowed with a homomorphism $\phi : G \to \mathrm{Aut}(A)$. Let M be an $(A \rtimes G)$-module. The structure of $(A^{\mathrm{en}} \rtimes G^{\mathrm{en}})$-module on $\mathrm{End}_k(M)$ restricts to a structure of $k\Delta G$-module on $\mathrm{End}_A(M)$. The corresponding morphism $G \to \mathrm{Aut}(\mathrm{End}_A(M))$ is the following: given $g \in G$ and $f \in \mathrm{End}_A(M)$, we set $g(f)(m) = g(f(g^{-1}(m)))$ for $m \in M$. We have a canonical isomorphism $kG \otimes M \xrightarrow{\sim} \mathrm{Ind}^{A\rtimes G} \mathrm{Res}_A M$, and an isomorphism of algebras

$$\mathrm{End}_A(M) \rtimes G \xrightarrow{\sim} \mathrm{End}_{A\rtimes G}(kG \otimes M), \quad f \otimes g \mapsto \bigl(h \otimes m \mapsto hg \otimes hf\bigl(h^{-1}m\bigr)\bigr).$$

Recall also that $J(A)G \subset J(A \rtimes G)$, hence $\mathrm{Res}_A^{A\rtimes G}$ preserves semi-simplicity.

4.2.2. Two-sided tilting complexes

Let k be a commutative ring. Let A and B be two flat k-algebras and G a finite group with homomorphisms $G \to \mathrm{Aut}(A)$ and $G \to \mathrm{Aut}(B)$.

We start with a classical result [37].

Lemma 4.13. *Let X be a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules. Then $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} X$ is a two-sided tilting complex for (A, B) if and only if $\mathrm{Ind}^{(A\rtimes G)\otimes(B\rtimes G)^{\mathrm{opp}}} X$ is a two-sided tilting complex for $(A \rtimes G, B \rtimes G)$.*

Proof. Let $X_1 = \mathrm{Res}_{A\otimes B^{\mathrm{opp}}} X$ and $X_2 = \mathrm{Ind}^{(A\rtimes G)\otimes(B\rtimes G)^{\mathrm{opp}}} X$. Following the proof of Lemma 4.3, we obtain a commutative diagram

$$\begin{tikzcd}
B \rtimes G \arrow[r, "\mathrm{can}"] \arrow[d, equal] & R\,\mathrm{End}^\bullet_A(X_1) \rtimes G \arrow[d, "\sim"] \\
B \rtimes G \arrow[r, "\mathrm{can}"] & R\,\mathrm{End}^\bullet_{A\rtimes G}(X_2)
\end{tikzcd}$$

There is a similar commutative diagram with the roles of A and B reversed. The lemma follows. $\square$

4.2.3. Rickard complexes

The following lemma is an equivariant version of a result of Rickard [48, p. 336].

Lemma 4.14. *Assume that k is a regular noetherian ring and A and B are symmetric k-algebras. Let C be a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules which restricts to a two-sided tilting complex* *for (A, B). There is a complex D of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules that is quasi-isomorphic to C and which restricts to a Rickard complex for (A, B).*

Proof. Let C' be a right-bounded complex quasi-isomorphic to C, all of whose terms are finitely generated and projective. Let $n \in \mathbf{Z}$ be such that $H^i(C) = 0$ for $i < n$. Then $D = \tau_{\geqslant n-m}(C')$ satisfies the required property for m the Krull dimension of k (see [48, pp. 135–136]). □

A proof similar to that of Lemma 4.13 shows the following classical result. Note that the assumption on $|G|$ is necessary to ensure that a complex of $(A^{\mathrm{en}} \rtimes G)$-modules is contractible if its restriction to A^{en} is contractible.

Lemma 4.15. *Assume that $|G| \in k^\times$. Let X be a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules. Then $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} X$ is a Rickard complex for (A, B) if and only if $\mathrm{Ind}^{(A\rtimes G)\otimes(B\rtimes G)^{\mathrm{opp}}} X$ is a Rickard complex for $(A \rtimes G, B \rtimes G)$.*

4.2.4. Tilting complexes

Let A be a flat k-algebra and G a finite group endowed with a homomorphism $G \to \mathrm{Aut}(A)$.

Lemma 4.16. *Let $T \in \mathrm{Comp}(A \rtimes G)$. Then $\mathrm{Res}_A T$ is a tilting complex for A if and only if $kG \otimes T$ is a tilting complex for $A \rtimes G$. There is a canonical isomorphism $\mathrm{End}_{D(A)}(T) \rtimes G \xrightarrow{\sim} \mathrm{End}_{D(A\rtimes G)}(kG \otimes T)$.*

Proof. By Section 4.2.1, we have

$$R\,\mathrm{End}^\bullet_A(T) \rtimes G \xrightarrow{\sim} R\,\mathrm{End}^\bullet_{D(A\rtimes G)}(kG \otimes T).$$

Thus, $\mathrm{Hom}_{D(A)}(T, T[i]) = 0$ for $i \neq 0$ if and only if $\mathrm{Hom}_{D(A\rtimes G)}(kG \otimes T, kG \otimes T[i]) = 0$ for $i \neq 0$.

Assume that T is a tilting complex for A. Then, $\mathrm{Ind}^{A\rtimes G}(T)$ is perfect. Also, A is in the thick subcategory of $D(A)$ generated by T, hence $A \rtimes G = \mathrm{Ind}^{A\rtimes G} A$ is in the thick subcategory of $D(A)$ generated by $\mathrm{Ind}^{A\rtimes G} T$. So, $\mathrm{Ind}^{A\rtimes G} T$ is a tilting complex.

Conversely, assume that $\mathrm{Ind}^{A\rtimes G} T$ is a tilting complex. We have $\mathrm{Res}_A(kG \otimes T) \simeq T^{|G|}$, and hence T is a perfect complex for A. Since A is in the thick subcategory of $D(A \rtimes G)$ generated by $\mathrm{Ind}^{A\rtimes G} T$, it follows that A is in the thick subcategory of $D(A \rtimes G)$ generated by $\mathrm{Res}_A \mathrm{Ind}^{A\rtimes G} T \simeq T^{|G|}$. It follows that T is a tilting complex for A. □

The following lemma is an equivariant version of a result of Keller [26, §8.3.1].

Lemma 4.17. *Let $C \in \mathrm{Comp}(A \rtimes G)$ and let $B = \mathrm{End}_{D(A)}(C)$. Assume that $\mathrm{Hom}_{D(A)}(C, C[n]) = 0$ for $n < 0$. There is a complex X of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules and an isomorphism $\phi : C \xrightarrow{\sim} X$ in $D(A \rtimes G)$ such that the composition of canonical maps $\mathrm{End}_{D(A)}(C) = B \to \mathrm{End}_{\mathrm{Comp}(A)}(X) \to \mathrm{End}_{D(A)}(X)$ is given by ϕ.*

Proof. Up to isomorphism in $D(A \rtimes G)$, we may assume that C is homotopically projective. Let $U = (A \otimes \mathrm{End}^\bullet_A(C)^{\mathrm{opp}}) \rtimes G$, a dg algebra. The complex C extends to a dg U-module. Let $U_- = \tau_{\leqslant 0}(U)$, a dg subalgebra of U. We have $H^0(U_-) \xrightarrow{\sim} H^0(U) = (A \otimes B) \rtimes G$. Let $X = H^0(U) \otimes^{\mathbf{L}}_{U_-} C$. The canonical quasi-isomorphism $U_- \xrightarrow{\sim} H^0(U_-)$ induces an isomorphism $\phi : C \xrightarrow{\sim} X$ in $D(A \rtimes G)$. It satisfies the stated property. □

Lemma 4.17 gives the following useful criterion to extend equivalences.

Lemma 4.18. *Let* $C \in \mathrm{Comp}(A \rtimes G)$ *be such that* $\mathrm{Res}_A C$ *is a tilting complex. Let* $B = \mathrm{End}_{D(A)}(C)$. *There is a complex* X *of* $((A \otimes B^{\mathrm{opp}}) \rtimes G)$*-modules and an isomorphism* $\phi : C \xrightarrow{\sim} X$ *in* $D(A \rtimes G)$ *such that* $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(X)$ *is a two-sided tilting complex and the composition of canonical maps* $\mathrm{End}_{D(A)}(C) = B \to \mathrm{End}_{\mathrm{Comp}(A)}(X) \to \mathrm{End}_{D(A)}(X)$ *is given by* ϕ.

Lemma 4.19. *Let* T *be a two-sided tilting complex for* (A, B). *Assume that there is a complex* C *of* $(A \rtimes G)$*-modules such that* $T \simeq C$ *in* $D(A)$. *There is an action of* G *on* B *and a complex* X *of* $((A \otimes B^{\mathrm{opp}}) \rtimes G)$*-modules such that* $X \simeq T$ *in* $D(A \otimes B^{\mathrm{opp}})$.

Proof. Consider X as in Lemma 4.18. Then $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(X)$ and T are two-sided tilting complexes and there is an isomorphism $X \xrightarrow{\sim} T$ in $D(A)$ compatible with the canonical maps $B \to \mathrm{End}_{D(A)}(X)$ and $B \to \mathrm{End}_{D(A)}(T)$. This forces X and T to be isomorphic in $D(A \otimes B^{\mathrm{opp}})$ (Lemma 4.20). □

Lemma 4.20. *Let* $C, D \in \mathrm{Comp}(A \otimes B^{\mathrm{opp}})$ *be such that* C *is a two-sided tilting complex. Assume that there is an isomorphism* $\phi : C \xrightarrow{\sim} D$ *in* $D(A)$ *such that the composition of canonical maps* $\mathrm{End}_{D(A)}(C) = B \to \mathrm{End}_{\mathrm{Comp}(A)}(X) \to \mathrm{End}_{D(A)}(X)$ *is given by* ϕ. *There is an isomorphism* $C \xrightarrow{\sim} D$ *in* $D(A \otimes B^{\mathrm{opp}})$ *restricting to* ϕ.

Proof. The map ϕ induces isomorphisms of B^{en}-modules $\mathrm{Hom}_{D(A)}(C, C[i]) \xrightarrow{\sim} \mathrm{Hom}_{D(A)}(C, D[i])$ for $i \in \mathbf{Z}$. The canonical map $B \to R\,\mathrm{End}^\bullet_A(C)$ is an isomorphism in $D(B^{\mathrm{en}})$ and we obtain an isomorphism $B \xrightarrow{\sim} R\,\mathrm{Hom}^\bullet_A(C, D)$ in $D(B^{\mathrm{en}})$. Applying $C \otimes^{\mathbf{L}}_B -$ gives the required isomorphism. □

4.2.5. Equivariant lifts of stable equivalences

We assume that k is a field for the rest of Section 4.2.

Lemma 4.21. *Let* A *be a symmetric* k*-algebra and* M, N *two complexes of* A*-modules. Assume* M *and* N *are isomorphic in* A-stab. *Then there exists* U, V *two bounded complexes of projective* A*-modules and* $\alpha : M \to U$, $\beta : V \to C[-1]$ *two morphisms of complexes, where* C *is the cone of* α, *such that the cone of* β *is quasi-isomorphic to* N.

Proof. By definition, there is a complex of A-modules L and there are morphisms of complexes $f : L \to M$ and $g : L \to N$ whose cones U' and V' are perfect complexes. Let U be a bounded complex of (finitely generated) projective modules that is quasi-isomorphic to U'. Since A is self-injective, U is homotopically injective, hence there is a morphism of complexes $\alpha'' : U' \to U$ that is a quasi-isomorphism. We put $\alpha = \alpha'' \circ \alpha'$, where $\alpha' : M \to U$ is the canonical map. We denote by C the cone of α, a complex quasi-isomorphic to $L[1]$. There is a distinguished triangle $V' \xrightarrow{\beta'} C[-1] \to N \rightsquigarrow$ in $D(A)$. Let V be a bounded complex of projective modules quasi-isomorphic to V'. Since such a complex is homotopically projective, the composition $V \xrightarrow{\sim} V' \underset{\sim}{\xrightarrow{\beta'}} C[-1]$ in $D^b(A)$ comes from a morphism of complexes $\beta : V \to C[-1]$. This proves the lemma.

$$\begin{tikzcd}
V \arrow[d, "\beta"'] & & & \\
C[-1] \arrow[r] \arrow[d] & M \arrow[r, "\alpha"] & U \arrow[r, squiggly] & {} \\
N \arrow[d, squiggly] & & & \\
{} & & &
\end{tikzcd}$$

□

Proposition 4.22. *Let A and B be two symmetric k-algebras endowed with an action of a finite group G. Let M be a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} M$ induces a stable equivalence. Assume that there is a complex N of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules isomorphic to M in $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-stab and such that $N \otimes^{\mathbf{L}}_B -$ induces an equivalence $D^b(B) \xrightarrow{\sim} D^b(A)$.*

Then there exists a complex X of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules with the following properties:

- *X is quasi-isomorphic to N,*
- *$\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} X$ induces a Rickard equivalence that lifts the stable equivalence induced by $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} M$,*
- *there are distinguished triangles in $\mathrm{Ho}((A \otimes B^{\mathrm{opp}}) \rtimes G)$*

  $$L \to M \to U \rightsquigarrow \quad \text{and} \quad V \to L \to X \rightsquigarrow$$

  where U, V are bounded complexes of projective $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules.

Proof. Lemma 4.21 shows the existence of X with all the properties except the fact $Y = \mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(X)$ induces a Rickard equivalence. We will deduce the finer property as in [46, Corollary 5.5]. We know that Y induces a derived equivalence, *i.e.*, there is an isomorphism of complexes $\mathrm{End}^\bullet_A(Y) \simeq B \oplus T$, where T is acyclic. The distinguished triangles of the proposition show that Y is isomorphic to $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(M)$ in $\mathrm{Ho}((A \otimes B^{\mathrm{opp}})\text{-stab})$, hence T is homotopy equivalent to a bounded complex of projective modules. As it is acyclic, it is homotopy equivalent to 0. Similarly, $\mathrm{End}^\bullet_{B^{\mathrm{opp}}}(Y) \simeq A$ in $\mathrm{Ho}(A^{\mathrm{en}})$, hence Y is a Rickard complex. □

Remark 4.23. The previous proposition can be applied to the case where k is a field of characteristic $\ell > 0$, G and H are two finite groups with a common Sylow ℓ-subgroup, and $A = B_0(kG)$, $B = B_0(kH)$. The proposition shows that if $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(M)$ is splendid, then $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(X)$ is splendid.

Proposition 4.24. *Let A and B be two symmetric k-algebras with no simple direct factors and endowed with an action of a finite group G. Let M be an $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-module such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} M$ induces a stable equivalence. Assume that there is a two-sided tilting complex T for $A \otimes B^{\mathrm{opp}}$ that is isomorphic to $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(M)$ in $(A \otimes B^{\mathrm{opp}})$-stab, and such that $\mathrm{Res}_A(T)$ extends to a complex of $(A \rtimes G)$-modules.*

There is a bounded complex C of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} C \simeq T$ in $D(A \otimes B^{\mathrm{opp}})$.

Proof. Let $\rho : G \to \mathrm{Aut}(A)$ and $\psi : G \to \mathrm{Aut}(B)$ be the canonical homomorphisms. Lemma 4.19 shows that there is a homomorphism $\psi' : G \to \mathrm{Aut}(B)$ and a complex X of $((A \otimes$

$B^{\mathrm{opp}}) \rtimes_{\rho,\psi'} G)$-modules such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(X)$ is a two-sided tilting complex and $X \simeq T$ in $D(A \otimes B^{\mathrm{opp}})$.

Let N be an $((A \otimes B^{\mathrm{opp}}) \rtimes_{\rho,\psi'} G)$-module that is isomorphic to X in $((A \otimes B^{\mathrm{opp}}) \rtimes_{\rho,\psi'} G)$-stab. Then $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} N \simeq \mathrm{Res}_{A\otimes B^{\mathrm{opp}}} M$ in $(A \otimes B^{\mathrm{opp}})$-stab. We have the structure of an $((A \otimes B^{\mathrm{opp}}) \rtimes_{\rho,\psi} G, (A \otimes B^{\mathrm{opp}}) \rtimes_{\rho,\psi'} G)$-bimodule on $\mathrm{Hom}_k(N, M)$, and this restricts to a structure of $(B^{\mathrm{en}} \rtimes_{\psi,\psi'} G)$-module on $\mathrm{Hom}_A(N, M)$. We have $\mathrm{Hom}_A(N, M) \simeq B \oplus U$ as B^{en}-modules, where U is a projective B^{en}-module. We deduce that the B^{en}-module B extends to a $(B^{\mathrm{en}} \rtimes_{\psi,\psi'} G)$-module L. Let $C = \mathrm{Hom}^{\bullet}_{B^{\mathrm{opp}}}(L, X)$: this is a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes_{\rho,\psi} G)$-modules satisfying the required property. □

We have a descent counterpart.

Proposition 4.25. *Let A and B be two symmetric k-algebras with no simple direct factors and endowed with an action of a finite group G such that $|G| \in k^{\times}$. Let M be a bounded complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} M$ induces a stable equivalence. Assume that there is a two-sided tilting complex T for $((A \rtimes G), (B \rtimes G))$ that is isomorphic to $\mathrm{Ind}^{(A\rtimes G)\otimes(B\rtimes G)^{\mathrm{opp}}}(M)$ in $((A \rtimes G) \otimes (B \rtimes G)^{\mathrm{opp}})$-stab, and such that there is an isomorphism $\mathrm{Res}_{A\rtimes G} T \simeq C \otimes kG$ in $D(A \rtimes G)$, where $C \in \mathrm{Comp}(A \rtimes G)$.*

There is a bounded complex Y of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules such that $\mathrm{Ind}^{(A\rtimes G)\otimes(B\rtimes G)^{\mathrm{opp}}}(Y) \simeq T$ in $D((A \rtimes G) \otimes (B \rtimes G)^{\mathrm{opp}})$.

Proof. Lemma 4.16 shows that C is a tilting complex for A. Let $B' = \mathrm{End}_{D(A)}(C)$. Let $X \in \mathrm{Comp}((A \otimes B'^{\mathrm{opp}}) \rtimes G)$ as in Lemma 4.18. We have an isomorphism $C \xrightarrow{\sim} X$ in $D(A \rtimes G)$, whose restriction to $D(A)$ is compatible with the action of B' up to homotopy. Let N be an $((A \otimes B') \rtimes G)$-module isomorphic to X in $((A \otimes B') \rtimes G)$-stab. We have $\mathrm{Hom}_A(M, N) = L \oplus R$ as (B, B')-bimodules, where L has no projective direct summand. Then L extends to a $((B \otimes B'^{\mathrm{opp}}) \rtimes G)$-module. The $(B \rtimes G, B' \rtimes G)$-bimodule $L' = \mathrm{Ind}^{(B\rtimes G)\otimes(B'\rtimes G)^{\mathrm{opp}}}_{(B\otimes B'^{\mathrm{opp}})\rtimes G}(L)$ induces a stable equivalence that sends simple modules to simple modules. Hence, $L \otimes_B -$ sends simple modules to semi-simple modules. Since it induces a stable equivalence, it sends simple modules to simple modules, and we deduce that $L \otimes_{B'} - : B'\text{-mod} \to B\text{-mod}$ is an equivalence [34, Theorem 2.1] and the equivalence $L' \otimes_{B'\rtimes G} : (B' \times G)\text{-mod} \xrightarrow{\sim} (B \rtimes G)\text{-mod}$ comes from an isomorphism $B \rtimes G \xrightarrow{\sim} B' \rtimes G$. We deduce that there is a G-invariant isomorphism $\sigma : B \xrightarrow{\sim} B'$ such that $B_{\sigma} \xrightarrow{\sim} L$ as $(B \otimes B') \rtimes G$-modules. The complex $Y = \sigma^* X$ satisfies the required property. □

4.2.6. Okuyama's sequential lifts

Let A and B be two symmetric k-algebras with no simple direct factors and acted on by a finite group G. Assume that the simple A and B-modules are absolutely simple. Let M be an $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-module, projective as an A-module and as a B^{opp}-module. We assume that M induces a stable equivalence between A and B and it has no non-zero projective direct summand.

Let $\phi : U \to M$ be a projective cover of M. We have $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} U \simeq \bigoplus_{S\in\mathcal{S}_B} P_{M\otimes_B S} \otimes P_S^*$ (cf. [50, Lemma 2.12] and Lemma 4.27 below).

Let I be a G-invariant subset of $\mathcal{S}_B$. Then, there is a direct summand Q of U such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} Q \simeq \bigoplus_{S\in I} P_{M\otimes_B S} \otimes P_S^*$. Let $C = C(M, I) = 0 \to Q \xrightarrow{\phi_{|Q}} M \to 0$, a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules with M in degree 0.

Assume that $\mathrm{Res}_A(C)$ is a tilting complex and let $B' = \mathrm{End}_{D(A)}(C)$. Lemma 4.18 provides a complex X of $((A \otimes B'^{\mathrm{opp}}) \rtimes G)$-modules such that $\mathrm{Res}_{A\otimes B'^{\mathrm{opp}}}(X)$ is a two-sided tilting complex.

We consider a situation studied in the non-equivariant setting by Okuyama [42] (this is to be used in Section 6.3). Let $I_0, \ldots, I_l$ be G-invariant subsets of $\mathcal{S}_B$. We assume that the sequence satisfies the following conditions. Let $A_0 = A$ and $M_0 = M$. Assume that a G-algebra A_i and an $((A_i \otimes B^{\mathrm{opp}}) \rtimes G)$-module M_i have been defined. We assume that $\mathrm{Res}_{A_i} C(M_i, I_i)$ is a tilting complex. We set $A_{i+1} = \mathrm{End}_{D(A_i)}(C(M_i, I_i))$. We view A_{i+1} as an $((A_{i+1} \otimes B) \rtimes G)$-module by restricting the canonical $(A_{i+1}^{\mathrm{en}} \rtimes G)$-module structure via the canonical map $B \to \mathrm{End}_{D(A_i)}(C(M_i, I_i))$, and we denote by M_{i+1} a maximal direct summand of A_{i+1} with no non-zero projective direct summand. Note that $\mathrm{Res}_{A_{i+1} \otimes B^{\mathrm{opp}}}(M_{i+1})$ induces a stable equivalence.

Let us assume finally that $B = A_{l+1}$ (as algebras, without G-action). We have a sequence of derived equivalences

$$D^b(A) \xrightarrow{\sim} D^b(A_1) \xrightarrow{\sim} \cdots \xrightarrow{\sim} D^b(A_l) \xrightarrow{\sim} D^b(B),$$

whose composition lifts the stable equivalence induced by $\mathrm{Hom}_A(M, -)$. It follows from Proposition 4.24 that there is a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules whose restriction to $A \otimes B^{\mathrm{opp}}$ is a two-sided tilting complex.

4.2.7. Perverse equivalences

Let A be a symmetric k-algebra and G a finite group of automorphisms of A with $|G| \in k^\times$.

Define an equivalence relation on $\mathcal{S}_{A \rtimes G}$ by $M \sim N$ if $\mathrm{Hom}_A(\mathrm{Res}_A M, \mathrm{Res}_A N) \neq 0$. Induction and restriction define a bijection $\mathcal{S}_A/G \xrightarrow{\sim} \mathcal{S}_{A \rtimes G}/\sim$.

The perversity datum $(q, \mathcal{S}_\bullet)$ for $A \rtimes G$ is said to be *G-compatible* if it is compatible with $\sim$.

Proposition 4.26. *Let $(q_1, \mathcal{S}_\bullet^1), \ldots, (q_d, \mathcal{S}_\bullet^d)$ be a family of G-invariant perversity data for A and let $(q'_1, \mathcal{S}'^1_\bullet), \ldots, (q'_d, \mathcal{S}'^d_\bullet)$ be the corresponding G-compatible perversity data for $A \rtimes G$. There are algebras $B_0 = A, \ldots, B_d$ endowed with a G-action and complexes X_i of $((B_{i-1} \otimes B_i^{\mathrm{opp}}) \rtimes G)$-modules such that*

- $\mathrm{Res}_{B_{i-1} \otimes B_i^{\mathrm{opp}}}(X_i)$ *induces a perverse equivalence $D^b(B_i) \xrightarrow{\sim} D^b(B_{i-1})$ relative to $(q_i, \mathcal{S}_\bullet^i)$, and*
- $\mathrm{Ind}^{(B_{i-1} \rtimes G) \otimes (B_i \rtimes G)^{\mathrm{opp}}}(X_i)$ *induces a perverse equivalence relative to $(q'_i, \mathcal{S}'^i_\bullet)$.*

Proof. Since every perverse equivalence is a composition of perverse equivalences associated to two-step filtrations with $q(1) - q(0) = \pm 1$, it is enough to prove the proposition when the $(q_i, \mathcal{S}_\bullet^i)$ satisfy that requirement. Next, it is enough to deal with an individual $(q, \mathcal{S}_\bullet^i)$. Shifting if necessary, it is enough to deal with the case $q(0) = 0$.

Consider an elementary equivalence as in Section 3.1.3. The submodule U of A is G-stable and so is the submodule V of $A/J(A)$. It follows from Lemma 4.27 below that the complex X extends to a complex of $(A \rtimes G)$-modules. Lemma 4.18 shows that there is a complex C of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules inducing a perverse equivalence.

The case of a perverse equivalence associated to a two-step filtration and q given by $q(0) = 0$, $q(1) = -1$ is handled similarly.

The last part of the lemma follows from Lemma 4.29 below. $\square$

Lemma 4.27. *Let M be an $(A \rtimes G)$-module with projective cover P. Then $\mathrm{Res}_A P$ is a projective cover of $\mathrm{Res}_A M$.*

Proof. Set $A' = A \rtimes G$. The A-module $A'/(A'J(A)A')$ is semi-simple, hence it is semi-simple as an A'-module, so $J(A') \subset A'J(A)A' = A'J(A)$ and $\mathrm{soc}(A) \subset \mathrm{soc}(A')$. Let N be the kernel of a projective cover $P \to M$. Since N has no non-zero projective direct summand, we have $\mathrm{soc}(A')N = 0$, hence $\mathrm{soc}(A)N = 0$: this shows that $\mathrm{Res}_A N$ has no non-zero projective direct summand. □

Remark 4.28. Note that if G acts trivially on $\mathcal{S}_A$, then sequences of perversity data are automatically G-invariant.

Lemma 4.29. *Let B be a symmetric k-algebra endowed with an action of G. Let $X \in \mathrm{Comp}^b((A \otimes B^{\mathrm{opp}}) \rtimes G)$. Let $(q, \mathcal{S}_{A,\bullet}, \mathcal{S}_{B,\bullet})$ be a G-invariant perversity datum and $(q', \mathcal{S}_{A\rtimes G,\bullet}, \mathcal{S}_{B\rtimes G,\bullet})$ the corresponding G-compatible datum.*

Then $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} X$ induces a perverse equivalence relative to $(q, \mathcal{S}_{A,\bullet}, \mathcal{S}_{B,\bullet})$ if and only if $\mathrm{Ind}^{(A\rtimes G)\otimes(B\rtimes G)^{\mathrm{opp}}}(X)$ induces a perverse equivalence relative to $(q', \mathcal{S}_{A\rtimes G,\bullet}, \mathcal{S}_{B\rtimes G,\bullet})$.

Proof. Let $Y = \mathrm{Ind}^{(A\rtimes G)\otimes(B\rtimes G)^{\mathrm{opp}}}(X)$. The equivalence part is Lemma 4.13. Let $S' \in \mathcal{S}_{B\rtimes G}$ and $L = Y \otimes^{\mathbf{L}}_{B\rtimes G} S'$. We have $\mathrm{Res}_A L \simeq X \otimes^{\mathbf{L}}_B \mathrm{Res}_B(S')$.

Assume that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} X$ induces a perverse equivalence, and let $S' \in \mathcal{S}_{B\rtimes G,i}$. Since $\mathrm{Res}_B(S')$ is a direct sum of simple modules in $\mathcal{S}_{B,i}$, we deduce that $\mathrm{Res}_A H^j(L)$ has composition factors in $\mathcal{S}_{A,i-1}$ for $j \neq -q(i)$ and composition factors in $\mathcal{S}_{A,i}$ for $j = -q(i)$. We deduce that the composition factors of $H^j(L)$ have the required property and Y induces a perverse equivalence. The converse statement has a similar proof. □

The following result shows that the lifting strategy is well behaved with respect to outer automorphisms.

Corollary 4.30. *Let A and B be two symmetric k-algebras with no simple direct factors and endowed with the action of a finite group G with $|G| \in k^\times$. Let M be a bounded complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}} M$ induces a stable equivalence. Let $(q_1, \mathcal{S}^1_\bullet), \dots, (q_d, \mathcal{S}^d_\bullet)$ be a family of G-invariant perversity data for A and let $(q'_1, \mathcal{S}'^1_\bullet), \dots, (q'_d, \mathcal{S}'^d_\bullet)$ be the corresponding G-compatible perversity data for $A \rtimes G$. Let $B_0 = A, B_1, \dots, B_d$ be algebras endowed with a G-action and let X_i be complexes of $((B_{i-1} \otimes B_i^{\mathrm{opp}}) \rtimes G)$-modules for $i = 1, \dots, d$, such that $\mathrm{Res}_{B_{i-1}\otimes B_i^{\mathrm{opp}}}(X_i)$ induces a perverse equivalence $F_i : D^b(B_i) \xrightarrow{\sim} D^b(B_{i-1})$ relative to $(q_i, \mathcal{S}^i_\bullet)$. Assume that the sets $\{M \otimes_B S\}_{S\in\mathcal{S}_B}$ and $\{F_1 \cdots F_d(T)\}_{T\in\mathcal{S}_{B_d}}$ coincide up to isomorphism in A-stab.*

Then there is a $((B_d \otimes B^{\mathrm{opp}}) \rtimes G)$-module N such that $\mathrm{Res}_{B_d\otimes B^{\mathrm{opp}}} N$ induces a Morita equivalence and such that $X_1 \otimes_{B_1} \cdots \otimes_{B_{d-1}} X_d \otimes_{B_d} N \simeq M$ in $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-stab. In particular, the composition of perverse equivalences

$$D^b(B) \xrightarrow[\sim]{N\otimes_B -} D^b(B_d) \xrightarrow[\sim]{F_d} D^b(B_{d-1}) \to \cdots \to D^b(B_1) \xrightarrow[\sim]{F_1} D^b(A)$$

lifts the stable equivalence induced by M.

Proof. The existence of the algebras B_i and of the complexes X_i is provided by Proposition 4.26.

Let $Y = \mathrm{Hom}^\bullet_A(X_1 \otimes_{B_1} \cdots \otimes_{B_{d-1}} X_d, M)$, a complex of $((B_d \otimes B^{\mathrm{opp}}) \rtimes G)$-modules. Note that $\mathrm{Res}_{B_d\otimes B^{\mathrm{opp}}}(Y)$ induces a stable equivalence that sends simple modules to simple modules.

There is a $((B \otimes B_d^{\mathrm{opp}}) \rtimes G)$-module N with no projective direct summands that is isomorphic to Y in $((B_d \otimes B^{\mathrm{opp}}) \rtimes G)$-stab. Since N has no projective summand, we deduce as in the proof of Lemma 4.27 that $\mathrm{Res}_{B_d \otimes B^{\mathrm{opp}}}(N)$ has no projective summand. The functor $N \otimes_B -$ induces a stable equivalence between B and B_d that sends simple modules to simple modules. It follows from [34, Theorem 2.1] that $N \otimes_B -$ induces a Morita equivalence.

Consider the evaluation map $\rho : X_1 \otimes_{B_1} \otimes \cdots \otimes_{B_{d-1}} X_d \otimes_{B_d} Y \to M$, a morphism of complexes of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules. Since the X_i's induce Rickard equivalences, it follows that ρ is an isomorphism in $\mathrm{Ho}(A \otimes B^{\mathrm{opp}})$, hence ρ is an isomorphism in $\mathrm{Ho}((A \otimes B^{\mathrm{opp}}) \rtimes G)$ (cf. [50, Lemma 10.2.7]). □

4.3. *Equivalences with extra structure*

4.3.1. *Particular equivalences*

We consider data $\mathcal{B}$ consisting, for every $(G, \tilde{G}) \in \mathcal{E}$, of a family $\mathcal{B}(G, \tilde{G})$ of objects of $\mathrm{Comp}(RB_0(N_\Delta(G, \tilde{G})))$.

We say that $(G, \tilde{G}) \in \mathcal{E}$ satisfies Broué's $(\mathcal{B}, \mathcal{C})$-conjecture (for principal blocks) if there exists $X \in \mathcal{B}(G, \tilde{G})$ such that $\mathrm{Res}_{N_G(P) \times G^{\mathrm{opp}}}(X)$ induces a $\mathcal{C}$-equivalence between the principal blocks of G and $N_G(P)$.

We say that $\mathcal{B}$ satisfies (S3) if:

(i) whenever $(G_i, \tilde{G}_i) \in \mathcal{E}$ and $X_i \in \mathcal{B}(G_i, \tilde{G}_i)$ for $i = 1, 2$, we have $X_1 \otimes X_2 \in \mathcal{B}(G_1 \times G_2, \tilde{G}_1 \times \tilde{G}_2)$;

(ii) whenever $G_1 \lhd G_2 \lhd \tilde{G}_2 \leqslant \tilde{G}_1$ with $G_1 \lhd \tilde{G}_1$, if $X \in \mathcal{B}(G_1, \tilde{G}_1)$, we have
$$b_0(G_2)\,\mathrm{Ind}^{N_\Delta(G_2, \tilde{G}_2)} \mathrm{Res}_{N_\Delta(G_1, \tilde{G}_1) \cap (N_{\tilde{G}_2}(P_2) \times \tilde{G}_2^{\mathrm{opp}})}(X) \in \mathcal{B}(G_2, \tilde{G}_2);$$

(iii) given $(G, \tilde{G}) \in \mathcal{E}$, $X \in \mathcal{B}(G, \tilde{G})$, $n \geqslant 0$ and L an ℓ'-subgroup of $\mathfrak{S}_n$, we have $X^{\otimes n} \in \mathcal{B}(G^n, \tilde{G} \wr L)$;

(iv) whenever G is an abelian ℓ-group and $(G, \tilde{G}) \in \mathcal{E}$, we have $RG \in \mathcal{B}(G, \tilde{G})$;

(v) given $(G, \tilde{G}) \in \mathcal{E}$ and $(G, \hat{G}) \in \mathcal{E}$ with $\tilde{G} \leqslant \hat{G}$ and $\hat{G} = \tilde{G} C_{\hat{G}}(P)$ where P is a Sylow ℓ-subgroup of G, if $X \in \mathcal{B}(G, \tilde{G})$, then $b_0(N_\Delta(G, \hat{G}))\,\mathrm{Ind}^{N_\Delta(G, \hat{G})}(X) \in \mathcal{B}(G, \hat{G})$.

Proposition 4.31. *Let $\mathcal{P}$ be the set of pairs $(G, \tilde{G}) \in \mathcal{E}$ satisfying Broué's $(\mathcal{B}, \mathcal{C})$-conjecture. If $\mathcal{B}$ satisfies* (S3) *and $\mathcal{C}$ satisfies* (S1) *and* (S2)*, then $\mathcal{P}$ satisfies property* $(*')$.

Proof. The proposition follows from Proposition 4.11 and its proof. □

4.3.2. *Examples of stable data*

Let us define various data $\mathcal{B}$. Given $(G, \tilde{G}) \in \mathcal{E}$, we describe the condition for X to be in $\mathcal{B}(G, \tilde{G})$. We set $A = RB_0(G)$, $B = RB_0(N_G(P))$ and $Y = \mathrm{Res}_{A \otimes B^{\mathrm{opp}}}(X)$. We denote by E an ℓ'-subgroup of $N_{\tilde{G}}(P)$ such that $N_{\tilde{G}}(P) = PE$.

- *Splendid complexes*: Y^i is a direct summand of a direct sum of modules of the form $\mathrm{Ind}_{\Delta Q}^{N_G(P) \times G^{\mathrm{opp}}}(R)$, where $Q \leqslant P$, for $i \in \mathbf{Z}$.
- *(Increasing) perverse complexes* ($R = k$): $Y \otimes_A -$ is perverse relative to some datum $(q, \mathcal{S}_{A,\bullet}, \mathcal{S}_{B,\bullet})$ (resp. and q is increasing).
- *Iterated perverse complexes* ($R = k$ or $\mathcal{O}$): there is a sequence of algebras $A_1 = B, A_2, \ldots, A_l = A$ with actions of E and complexes $X_i \in \mathrm{Comp}^b((A_i \otimes A_{i+1}^{\mathrm{opp}}) \rtimes E)$ for $i = 1, \ldots, l-1$ such that

  - the actions of E on A_1 and A_l are the canonical actions,
  - $kX_i \otimes^{\mathbf{L}}_{kA_{i+1}} -$ is perverse, and
  - $X \simeq X_1 \otimes^{\mathbf{L}}_{A_2} \cdots \otimes^{\mathbf{L}}_{A_{l-1}} X_{l-1}$ in $D((B \otimes A^{\mathrm{opp}}) \rtimes E)$.
- *Positively gradable complexes* ($R = k$ or $\mathcal{O}$ and k is assumed to be algebraically closed). There is a non-negative grading on $kB_0(G)$ and a structure of graded $(kB \otimes kA^{\mathrm{opp}})$-module on kY. Here, we take a tight grading on kB, *i.e.*, one for which $J^i(kB) = (kB)_i \oplus J^{i+1}(kB)$ [54].
- *Character maps* ($R = K$). $H^i(Y) = 0$ for $i \neq 0, 1$ and $\mathrm{Hom}_{A\otimes B^{\mathrm{opp}}}(H^0(Y), H^1(Y)) = 0$.
- *Perfect character maps*: Y defines a character map (cf. above) and denoting by μ the character of X, the following holds for $g \in \tilde{G}$ and $h \in N_{\tilde{G}}(P)$ such that $(h, g) \in N_\Delta(G, \tilde{G})$:
  - $\mu(h, g) \in \gcd(|C_{\tilde{G}}(g)|, |C_{N_{\tilde{G}}(P)}(h)|)\mathcal{O}$;
  - if one of g, h is an ℓ'-element and the other is not, then $\mu(h, g) = 0$.

Remark 4.32. Note that the data $\mathcal{B}$ defined above prescribe conditions only on $\mathrm{Res}_{N_G(P)\times G^{\mathrm{opp}}}(X)$, except in the cases of iterated perverse complexes and perfect character maps.

Let us explain how the definitions above relate to classical notions.

Let $R = K$ and $\mathcal{B}$ be the data of character maps. Assume that $\tilde{G} = G$. Then $[X \otimes_{KG} -] : K_0(KB_0(G)) \to K_0(KB_0(N_G(P)))$ is a morphism of abelian groups. This gives a bijection

$$\mathcal{B}(G, G)/\text{iso.} \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{Z}\text{-mod}}\big(K_0\big(KB_0(G)\big), K_0\big(KB_0\big(N_G(P)\big)\big)\big).$$

Let $\mathcal{C}$ be the class of acyclic complexes. Then, X induces a $\mathcal{C}$-equivalence if and only if $[X \otimes_{KG} -]$ is an isometry (we require isometries to be bijective). The isometry is perfect [6] if $\mathcal{B}$ is taken to be the data of perfect character maps.

Let R be either k or $\mathcal{O}$, and let $\mathcal{C}$ be given by complexes homotopy equivalent to 0 and $\mathcal{B}$ by splendid complexes. An object $X \in \mathcal{B}(G, G)$ induces a $\mathcal{C}$-equivalence if and only if it induces a splendid Rickard equivalence [48].

Proposition 4.33. *The data $\mathcal{B}$ defined above satisfy property* (S3).

Proof. The case of character maps is immediate, while the case of perfect character maps follows from [22, Theorem 1E, Theorem 2B, §6].

The properties (S3)(i), (iii), (iv), (v) are easy in all other cases.

Property (S3)(ii) for (iterated, increasing) perverse complexes follows from Lemma 4.29.

Let us consider the case of graded complexes. Take $R = k$. Let $(G, \tilde{G}) \in \mathcal{E}$ and X be a complex of $((B \otimes A^{\mathrm{opp}}) \rtimes E)$-modules whose restriction to $B \otimes A^{\mathrm{opp}}$ is a two-sided tilting complex. There is an E-invariant tight grading on B [54, §6.2.1]. Proposition 4.34 below shows that there is an E-invariant grading on A and a compatible grading on X. Furthermore, if there is a non-negative grading on A compatible with that on B via $\mathrm{Res}_{B\otimes A^{\mathrm{opp}}}(X)$, then we can choose an E-invariant compatible grading on A that is non-negative. We deduce that property (S3)(ii) holds.

The property is clear for the other types of data. $\square$

The following proposition examines the behaviour of automorphisms and gradings under actions of finite groups.

Proposition 4.34. *Assume that k is algebraically closed. Let A and B be two finite-dimensional k-algebras endowed with the action of a finite group G. Let $\mathrm{Out}^G(A)$ be the image of $C_{\mathrm{Aut}(A)}(G)$ in $\mathrm{Out}(A)$.*

Let C be a complex of $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-modules such that $\mathrm{Res}_{A\otimes B^{\mathrm{opp}}}(C)$ is a two-sided tilting complex. Then the canonical isomorphism $\mathrm{Out}(A)^0 \xrightarrow{\sim} \mathrm{Out}(B)^0$ induced by C restricts to an isomorphism $\mathrm{Out}^G(A)^0 \xrightarrow{\sim} \mathrm{Out}^G(B)^0$.

Assume that $|G| \in k^\times$, and fix a grading on B invariant under G. Then there is a grading on A invariant under G and a structure of graded $((A \otimes B^{\mathrm{opp}}) \rtimes G)$-module on C, where G is in degree 0.

Assume that the morphism $\mathbf{G}_m \to \mathrm{Out}^G(B)$ induced by the grading on B can be lifted to a morphism $\mathbf{G}_m \to \mathrm{Aut}(B)$ such that the corresponding grading is non-negative. Then, it can also be lifted to a morphism $\mathbf{G}_m \to C_{\mathrm{Aut}(B)}(G)$ such that the corresponding G-invariant grading is non-negative.

Proof. We use the results of [54, §4.2]. Let $D\,\mathrm{Pic}(A)$ be the locally algebraic group whose S points are the group of quasi-isomorphism classes of invertible complexes of $(A^{\mathrm{en}} \otimes \mathcal{O}_S)$-modules [58]. Let $D\,\mathrm{Pic}^G(A)$ be its subgroup of complexes that extend to a structure of complexes of $((A^{\mathrm{en}} \rtimes G) \otimes \mathcal{O}_S)$-modules. Let $\mathrm{Pic}^f(A)$ be the subgroup of $D\,\mathrm{Pic}(A)$ whose S-points consist of isomorphism classes of $(A^{\mathrm{en}} \otimes \mathcal{O}_S)$-modules that are locally free of rank 1 as $(A \otimes \mathcal{O}_S)$-modules and as $(A^{\mathrm{opp}} \otimes \mathcal{O}_S)$-modules. Let $\mathrm{Pic}^{f,G}(A) = \mathrm{Pic}^f(A) \cap D\,\mathrm{Pic}^G(A)$. The canonical map $\mathrm{Pic}^f(A) \to \mathrm{Out}(A)$ restricts to a map $\mathrm{Pic}^{f,G}(A) \to \mathrm{Out}^G(A)$. The canonical isomorphism $R\,\mathrm{Hom}^\bullet_A(C, - \otimes^{\mathbf{L}}_A C) : D\,\mathrm{Pic}(A) \xrightarrow{\sim} D\,\mathrm{Pic}(B)$ restricts to isomorphisms $\mathrm{Pic}^f(A) \xrightarrow{\sim} \mathrm{Pic}^f(B)$ and $D\,\mathrm{Pic}^G(A) \xrightarrow{\sim} D\,\mathrm{Pic}^G(B)$, and hence to $\mathrm{Pic}^{f,G}(A) \xrightarrow{\sim} \mathrm{Pic}^{f,G}(B)$. Passing to quotients, we obtain an isomorphism between connected components $\mathrm{Out}^G(A)^0 \xrightarrow{\sim} \mathrm{Out}^G(B)^0$.

Let us assume now that there is a G-invariant grading on B, *i.e.*, a morphism $\phi : \mathbf{G}_m \to C_{\mathrm{Aut}(B)}(G)$. This induces a morphism $\mathbf{G}_m \to \mathrm{Out}^G(B)$, and hence a morphism $\psi : \mathbf{G}_m \to \mathrm{Out}^G(A)$. In order to show that this morphism lifts to a morphism $\mathbf{G}_m \to C_{\mathrm{Aut}(A)}(G)$, it is enough to prove that there is a lift as a morphism of varieties, in a neighbourhood of the identity of $\mathbf{G}_m$.

We have the canonical structure of a $((B^{\mathrm{en}} \rtimes G) \otimes \mathcal{O}_{C_{\mathrm{Aut}(B)}(G)})$-module on $B \otimes \mathcal{O}_{C_{\mathrm{Aut}(B)}(G)}$: the action of $(B \rtimes G) \otimes \mathcal{O}_{C_{\mathrm{Aut}(B)}(G)}$ is the canonical one, while the right action of $b \in B$ is given by right multiplication by $\rho(b)$, where $\rho : B \to B \otimes \mathcal{O}_{C_{\mathrm{Aut}(B)}(G)}$ is the universal algebra map: at a closed point of $C_{\mathrm{Aut}(B)}(G)$, it is the corresponding automorphism of B.

Let $L = R\,\mathrm{Hom}^\bullet_{B^{\mathrm{opp}}}(C, C \otimes^{\mathbf{L}}_B (B \otimes \mathcal{O}_{C_{\mathrm{Aut}(B)}(G)}))$, a complex of $((A^{\mathrm{en}} \rtimes G) \otimes \mathcal{O}_{C_{\mathrm{Aut}(B)}(G)})$-modules. We have $L(1) \simeq A$ in $D(A^{\mathrm{en}} \rtimes G)$, hence there is a neighbourhood U of the identity in $C_{\mathrm{Aut}(A)}(G)$ such that $H^i(L) = 0$ for $i \neq 0$ and $H^0(L)$ is an $((A^{\mathrm{en}} \rtimes G) \otimes \mathcal{O}_U)$-module that is free of rank 1 as a left $(A \otimes \mathcal{O}_U)$-module and as a right $(A^{\mathrm{opp}} \otimes \mathcal{O}_U)$-module. Shrinking U, we can assume that $H^0(L)$ is isomorphic to $A \otimes \mathcal{O}_U$ as an $((A \rtimes G) \otimes \mathcal{O}_U)$-module, since A is projective as an $(A \rtimes G)$-module, hence it is rigid. Fixing an isomorphism of $((A \rtimes G) \otimes \mathcal{O}_U)$-modules $H^0(L) \xrightarrow{\sim} A \otimes \mathcal{O}_U$ provides a morphism $U \to C_{\mathrm{Aut}(A)}(G)$. We obtain a morphism $\phi^{-1}(U) \to C_{\mathrm{Aut}(A)}(G)$ lifting ψ locally and we are done.

Assume that there is a lift $\mathbf{G}_m \to \mathrm{Aut}(B)$ such that the grading is non-negative. Consider the original grading on B. Given two simple B-modules S and T in degree 0, let $f(S,T)$ be the smallest integer d such that there are simple B-modules $S_1, \ldots, S_n$ in degree 0 and integers $d_0, \ldots, d_n$ with $\sum d_i = d$ and

$$\begin{gathered}\mathrm{Ext}^1_B\big(S, S_1\langle -d_0\rangle\big) \neq 0, \quad \mathrm{Ext}^1_B\big(S_1, S_2\langle -d_1\rangle\big) \neq 0, \quad \ldots, \\ \mathrm{Ext}^1_B\big(S_{n-1}, S_n\langle -d_{n-1}\rangle\big) \neq 0, \quad \mathrm{Ext}^1_B\big(S_n, T\langle -d_n\rangle\big) \neq 0.\end{gathered}$$

[54, Proposition 5.14 and Lemma 5.15] shows the existence of a function $d : \mathcal{S}_B \to \mathbf{Z}$ such that given two simple B-modules S and T in degree 0, then $f(S,T) + d(S) - d(T) \geqslant 0$. Furthermore, the proof of [54, Lemma 5.15] provides a function that is G-invariant, since f is G-invariant. Given $l \in \mathbf{Z}$, there is a decomposition as $(B \rtimes G)$-modules $B = \bigoplus_l P_l$, where $P_l \simeq \bigoplus_{d(S)=l} P_S^{\dim S}$. The graded algebra of endomorphisms of $\bigoplus_l P_l\langle l\rangle$ has a non-negative grading, it is isomorphic to B as a G-algebra, and the induced morphism $\mathbf{G}_m \to \operatorname{Out}(B)$ is the same as the one coming from the original grading on B [54, proof of Proposition 5.14]. □

We deduce from Propositions 4.10, 4.12, 4.31 and 4.33 our main reduction theorem.

Theorem 4.35. *Let $\mathcal{C}$ be given by contractible or acyclic complexes, with R-projective components. Let $\mathcal{B}$ be one of the data defined above.*

Let $\mathcal{F}$ be a set of non-cyclic finite simple groups with non-trivial abelian Sylow ℓ-subgroups. Assume that, given $G \in \mathcal{F}$, there is a pair $(G, \tilde{G})$ satisfying Broué's $(\mathcal{B}, \mathcal{C})$-conjecture with the following property. Let $\hat{G}$ be a finite group containing $\tilde{G}/O_{\ell'}(\tilde{G})$ with $G \lhd \hat{G}$ and $\hat{G}/G$ a Hall ℓ'-subgroup of $\operatorname{Out}(G)$. *Then, we require that $N_{\hat{G}}(P)/C_{\hat{G}}(P) = N_{\tilde{G}}(P)/C_{\tilde{G}}(P)$, where P is a Sylow ℓ-subgroup of G.*

Let G be a finite group with an abelian Sylow ℓ-subgroup. Assume that all non-cyclic composition factors of G with order divisible by ℓ are in $\mathcal{F}$. Let $\tilde{G}$ be a finite group containing G as a normal subgroup of ℓ'-index. Then $(G, \tilde{G})$ satisfies Broué's $(\mathcal{B}, \mathcal{C})$-conjecture.

4.4. Broué's conjecture for principal blocks

Theorem 4.36. *Let G be a finite group with an abelian Sylow ℓ-subgroup P, where $\ell = 2$ or $\ell = 3$.*

(a) *If $\ell = 2$ or ($\ell = 3$ and $|P| \leqslant 9$), then there is a splendid Rickard equivalence between $B_0(G)$ and $B_0(N_G(P))$.*
(b) *If $|P| \leqslant 9$, such an equivalence can be chosen to be a composition of perverse equivalences.*
(c) *If $|P| \leqslant \ell^2$ and P has no simple factor $\mathfrak{A}_6$ or M_{22} (when $\ell = 3$), then the equivalence can be chosen to be a single increasing perverse equivalence.*

The case $\ell = 3$ of (a) is due to Koshitani and Kunugi [28], while the case $\ell = 2$ is due to Marcus [38].

The existence of a splendid increasing perverse Rickard equivalence when P is cyclic is already known [12]. In the next two sections, we show that the theorem holds by reduction to simple groups, using Theorem 4.35.

5. Defect 3 × 3

This section gives a combinatorial description of all principal blocks of finite groups with Sylow 3-subgroup Z_3^2, up to splendid Morita equivalence, *i.e.*, the source algebra is determined by the combinatorics. The description is done in terms of perversity functions (both global and local). Some blocks Morita equivalent to the local block are given non-zero perversity function: in doing so, we try to follow the precise form of the abelian defect group conjecture for finite groups of Lie type.

5.1. *Local structure*

In this section we will collate the local information that we need to prove that the maps we obtain from the algorithm are really derived equivalences. This includes information on the centralizers of elements of order 3, and on automizers of Sylow 3-subgroups.

5.1.1. *Decompositions*

The structure of finite groups with elementary abelian Sylow 3-subgroups of order 9 is described in the following proposition (cf. the proof of Proposition 4.10).

Proposition 5.1. *Let G be a finite group with a Sylow* 3*-subgroup P isomorphic to Z_3^2. Assume that $O_{3'}(G) = 1$. Then $O^{3'}(G)$ is simple or $O^{3'}(G) = G_1 \times G_2$ where G_1 and G_2 are simple groups with Sylow* 3*-subgroups of order* 3.

When $O^{3'}(G) = G_1 \times G_2$ in the proposition above, then $B_0(G_i)$ is splendidly Morita equivalent to the principal block of the normalizer of a Sylow 3-subgroup (cf. Remark 3.7) and we deduce that the same holds for G. We describe below perversity functions for other equivalences.

Let G be a finite simple group with Sylow 3-subgroup P of order 3. Then, G is of one of the following types

- J_1;
- $\mathrm{PSL}_2(q)$, $\mathrm{PSL}_2(r)$;
- $\mathrm{PSL}_3(q)$;
- $\mathrm{PSU}_3(r)$.

Here, $q \equiv 2, 5 \pmod 9$ ($q > 2$ for $\mathrm{PSL}_2(q)$) and $r \equiv 4, 7 \pmod 9$.

In all of those cases, $|N_G(P)/C_G(P)| = 2$. Denote by k (resp. ε) the trivial (resp. non-trivial) simple $kB_0(N_G(P))$-module. There is a perverse equivalence between the principal blocks of G and $N_G(P)$ corresponding to the following perversity functions:

	k	ε
J_1	0	0
$\mathrm{PSL}_2(q)$	0	1
$\mathrm{PSL}_2(r)$	0	2
$\mathrm{PSL}_3(q)$	0	3
$\mathrm{PSU}_3(r)$	0	6

Remark 5.2. Note that $\mathrm{PSL}_2(4) \simeq \mathrm{PSL}_2(5)$ and the Deligne–Lusztig theory provides two different perversity functions.

5.1.2. *Automizers*

We begin by describing the automizers for almost simple groups with Sylow 3-subgroups isomorphic to Z_3^2.

Let $\tilde{G}$ be an almost simple group, *i.e.*, a finite group whose derived subgroup G is simple and is the unique minimal non-trivial normal subgroup. Let P be a Sylow 3-subgroup of G. Assume that $P \simeq Z_3^2$ and $3 \nmid [\tilde{G} : G]$. We list the almost simple groups $\tilde{G}$ modulo the equivalence relation generated by $\tilde{G} \sim \tilde{H}$ if $[G, G] = [H, H]$, $\tilde{H} \leqslant \tilde{G}$ and $\tilde{G} = \tilde{H} C_{\tilde{G}}(P)$.

The classification is the following. We indicate first the simple groups and then the almost simple ones, modulo equivalence.

- Z_4: $\mathfrak{A}_6, \mathfrak{A}_7$;
- Z_8: $\mathfrak{A}_6.2_2 = \mathrm{PGL}_2(9)$;
- Q_8: M_{22}, $\mathrm{PSU}_3(q)$, $\mathrm{PSL}_3(r)$; $\mathfrak{A}_6.2_3 = M_{10}$;
- D_8: $\mathfrak{A}_8$, $\mathrm{PSp}_4(q)$, $\mathrm{PSp}_4(r)$, $\mathrm{PSL}_4(q)$, $\mathrm{PSL}_5(q)$, $\mathrm{PSU}_4(r)$, $\mathrm{PSU}_5(r)$; $\mathfrak{S}_6, \mathfrak{S}_7$;
- SD_{16}: M_{11}, M_{23}, HS; $\mathfrak{A}_6.2^2 = \mathrm{Aut}(\mathfrak{A}_6)$, $M_{22}.2$,
  - $\mathrm{PSp}_4(q).Z_{2n}$ if $q = 2^n$ and $\mathrm{PSp}_4(r).Z_{2n}$ if $r = 2^n$ (extension by the extraordinary graph automorphism),
  - $\mathrm{PSL}_3(r).Z_2$ (extension by the graph automorphism),
  - $\mathrm{PSU}_3(q).Z_n$ where $q^2 = p^n$ and p is prime (extension by the Frobenius automorphism over $\mathbf{F}_p$).

Here, $q \equiv 2, 5 \pmod 9$ ($q > 2$ for $\mathrm{PSU}_3(q)$ and $\mathrm{PSp}_4(q)$) and $r \equiv 4, 7 \pmod 9$, so that $\ell \mid \Phi_1(r)$ and $\ell \mid \Phi_2(q)$. Note that n is even in the extended group $\mathrm{PSU}_3(q).Z_n$. Note finally that the square of the extraordinary graph automorphism is the Frobenius over $\mathbf{F}_2$.

5.1.3. *Centralizers of* 3*-elements*

The structure of the centralizers of 3-elements influences the local perversity functions, for finite groups of Lie type. We provide here a description of centralizers for those finite groups considered in Section 5.1.2.

Proposition 5.3. *Let $\tilde{G}$ be one of the groups listed in Section 5.1.2 and assume that G is an alternating group or is sporadic. Let x be an element of order* 3 *in $\tilde{G}$. We have $\mathrm{C}_{\tilde{G}}(x) = \langle x\rangle \times A$, where A is given in the table below. When there is more than one conjugacy class of elements of order* 3*, we list all possibilities.*

$\tilde{G}$	$\mathfrak{A}_6$	$\mathfrak{A}_7$	$\mathrm{PGL}_2(9)$	M_{10}	M_{22}	$\mathfrak{S}_6$	$\mathfrak{S}_7$	$\mathfrak{A}_8$
A	Z_3, Z_3	$\mathfrak{A}_4, Z_3$	Z_3	Z_3	$\mathfrak{A}_4$	$\mathfrak{S}_3, \mathfrak{S}_3$	$\mathfrak{S}_3, \mathfrak{S}_4$	$\mathfrak{S}_3, \mathfrak{A}_5$

$\tilde{G}$	$\mathrm{Aut}(\mathfrak{A}_6), M_{11}$	$M_{22}.2$	M_{23}	HS
A	$\mathfrak{S}_3$	$\mathfrak{S}_4$	$\mathfrak{A}_5$	$\mathfrak{S}_5$

Proof. The proof of this is trivial for the alternating groups and follows from the information in the Atlas for the sporadic groups [13]. $\square$

We now move on to the groups of Lie type, where we choose convenient representatives for $\tilde{G}$ up to equivalence.

Proposition 5.4. *Let $\tilde{G}$ be one of the groups of Lie type in Section 5.1.2. Let x be an element of order* 3 *in $\tilde{G}$. We have $\mathrm{C}_{\tilde{G}}(x) = \langle x\rangle \times A$, where A is given in the table below. When there is more than one conjugacy class of elements of order* 3*, we list all possibilities.*

$\tilde{G}$	A
$\mathrm{PSL}_3(r)$	$(Z_{(r-1)/3})^2 \rtimes Z_3$
$\mathrm{PSL}_3(r).Z_2$	$((Z_{(r-1)/3})^2 \rtimes Z_3) \rtimes Z_2$
$\mathrm{PSU}_3(q)$	$(Z_{(q+1)/3})^2 \rtimes Z_3$
$\mathrm{PSU}_3(q).Z_n$	$((Z_{(q+1)/3})^2 \rtimes Z_3) \rtimes Z_n$
$\mathrm{Sp}_4(q)$	$\mathrm{SL}_2(q) \times Z_{(q+1)/3}$, $\mathrm{GU}_2(q)/Z_3$
$\mathrm{Sp}_4(q).Z_{2n}$	$(\mathrm{SL}_2(q) \times Z_{(q+1)/3}) \rtimes Z_n$, $(\mathrm{GU}_2(q)/Z_3) \rtimes Z_n$
$\mathrm{Sp}_4(r)$	$\mathrm{SL}_2(r) \times Z_{(r-1)/3}$, $\mathrm{GL}_2(r)/Z_3$
$\mathrm{Sp}_4(r).Z_{2n}$	$(\mathrm{SL}_2(r) \times Z_{(r-1)/3}) \rtimes Z_n$, $(\mathrm{GL}_2(r)/Z_3) \rtimes Z_n$
$\mathrm{GL}_4(q)$	$\mathrm{GL}_2(q) \times Z_{(q^2-1)/3}$, $\mathrm{GL}_2(q^2)/Z_3$
$\mathrm{GU}_4(r)$	$\mathrm{GU}_2(r) \times Z_{(r^2-1)/3}$, $\mathrm{GL}_2(r^2)/Z_3$
$\mathrm{GL}_5(q)$	$\mathrm{GL}_3(q) \times Z_{(q^2-1)/3}$, $Z_{q-1} \times \mathrm{GL}_2(q^2)/Z_3$
$\mathrm{GU}_5(r)$	$\mathrm{GU}_3(r) \times Z_{(r^2-1)/3}$, $Z_{r+1} \times \mathrm{GL}_2(r^2)/Z_3$

Proof. These descriptions are well known. Here are some references

(1) $\mathrm{PSL}_3(r)$: see [32, Lemma 3.1].
(2) $\mathrm{PSU}_3(q)$: see [27, Lemma 2.5].
(3) $\mathrm{Sp}_4(q)$ ($\mathrm{Sp}_4(r)$ is similar): see [41, Example 3.6].
(4) $\mathrm{GL}_4(q)$: see [30, Lemma 2.2].
(5) $\mathrm{GU}_4(r)$: see the proof of [31, Lemma 2.2].
(6) $\mathrm{GL}_5(q)$: see [30, Lemma 2.6].
(7) $\mathrm{GU}_5(r)$: see the proof of [31, Lemma 2.2]. □

5.1.4. Reductions for Lie type

We now reduce the number of groups that need to be checked to finitely many by using splendid Morita equivalences between groups of a given Lie type.

- For $G = \mathrm{PSL}_3(r)$, in [32, Theorem 1.2] it is shown that the principal block of G is splendidly Morita equivalent to that of $\mathrm{PSL}_3(4)$.
- For $G = \mathrm{PSU}_3(q)$, in [27, Theorem 0.2] it is shown that the principal block of G is splendidly Morita equivalent to that of $\mathrm{PSU}_3(2) = Z_3^2 \rtimes Q_8$.
- For $G = \mathrm{PSL}_4(q)$, in [30, Theorem 0.3] it is shown that the principal block of G is splendidly Morita equivalent to that of $\mathrm{PSL}_4(2)$.
- For $G = \mathrm{PSL}_5(q)$, in [30, Theorem 0.2] it is shown that the principal block of G is splendidly Morita equivalent to that of $\mathrm{PSL}_5(2)$ (and also to that of N).
- For $G = \mathrm{PSp}_4(q)$, in [41, Example 3.6] it is shown that the principal block of G is splendidly Morita equivalent to that of $\mathrm{PSp}_4(2)$.
- The principal 3-blocks of $\mathrm{PSp}_4(r)$, $\mathrm{PSU}_4(r)$, and $\mathrm{PSU}_5(r)$ are splendidly Morita to those corresponding to $r = 4$ (cf. Remark 3.13).

Hence the simple groups that we have to analyze are $\mathrm{PSL}_3(4)$, $\mathrm{PSU}_3(2)$, $\mathrm{PSL}_4(2)$, $\mathrm{PSL}_5(2)$, $\mathrm{PSU}_4(4)$, $\mathrm{PSU}_5(4)$, $\mathrm{PSp}_4(2)$ and $\mathrm{PSp}_4(4)$ of Lie type, and $\mathfrak{A}_6$, $\mathfrak{A}_7$, M_{11}, M_{22}, M_{23} and HS.

5.2. Results of the algorithm

In the next four subsections we will describe the result of the algorithm (see Section 3.2.2) on various simple (and in two cases almost simple) groups G with a Sylow 3-subgroup P

isomorphic to Z_3^2. These are divided according to the automizer $E = N_G(P)/C_G(P)$, in the order Z_4, Q_8, D_8 and finally SD_{16}.

We assume that K is big enough for the finite groups considered. Each section will follow the same template for automizer E: we start by giving information on the group $P \rtimes E$, in particular its simple modules and the radical series for trivial source modules (including projectives). We write $N = N_G(P)$. Note that the canonical isomorphism $N/O_{3'}(N) \xrightarrow{\sim} P \rtimes E$ gives an isomorphism of algebras $kB_0(N) \xrightarrow{\sim} k(P \rtimes E)$ and we will freely identify modules in the principal blocks of N with $k(P \rtimes E)$-modules. We list in a table the perversity functions and the local twists (local perversity functions) as a summary of the results to be described in the subsections. Note that a row of the table determines the block up to a splendid Morita equivalence. This applies as well for composite perverse equivalences.

In the subsections we examine each (almost) simple group in turn, describing first the simple modules and Green correspondents, then giving the perversity function π together with the decomposition matrix. The Green correspondents are known in Lie type when "$\ell \mid (q-1)$" (Theorem 3.12). In the other cases, they can be determined by a computer by constructing the simple modules and decomposing their restriction. The identification of N with $Z_3^2 \rtimes Z_4, \dots, Z_3^2 \rtimes SD_{16}$ is not canonical. The choice we make affects the description of the Green correspondents C_i. When $G = G(q_0)$ is a finite group of Lie type, we provide the generic degree of the irreducible characters, a polynomial in q that specializes to the actual degree for $q = q_0$.

We give the decomposition matrix of $B_0(G)$ in an upper triangular form (in some cases, we only provide the upper square part). This gives rise to a basic set of "unipotent characters" $\{\chi_i\}_i$ in bijection with simple modules (we always choose $\chi_1 = 1$). They agree with the unipotent characters for Lie type, except for $\mathrm{PSU}_3(r)$ and $\mathrm{PSL}_3(q)$, where we need a different (and larger) set. Our numbering of simple modules gives an implicit bijection between simple $B_0(G)$-modules S_i and simple $B_0(N)$-modules T_i. We construct the images X_i in $B_0(N)$ of the simple modules under the perverse equivalence determined by π. We give explicit descriptions of the complexes X_i in all cases where feasible, and when they are not simple, *i.e.*, when $\pi(i) \neq 0$. We describe the cohomology of the complexes X_i in table form. Write $[X_i] = \sum_j a_{ij}[T_j]$ in $K_0(kN)$. We have $[S_i] = \sum_j (-1)^{\pi(j)} a_{ij} d(\chi_j)$, where $d : K_0(KG) \to K_0(kG)$ is the decomposition map. We indicate $\sum_j (-1)^{\pi(j)} a_{ij} \chi_j = \sum_{j,r} (-1)^{\pi(j)+r} [H^r(X_i) : T_j] \chi_j$ in the table (column "$[S_i]$"). This explains how the classes $[X_i]$ determine the decomposition matrix of $B_0(G)$.

The last ingredient in the construction is a twist of the stable equivalence between $B_0(G)$ and $B_0(N)$. This is determined by functions η_R, where R runs over N-conjugacy classes of subgroups of order 3 of P. This twist is only needed in some cases where $E = D_8$ or $E = SD_{16}$. We determine the images Y_i of the modules S_i under the twisted stable equivalence, following Section 3.3.4. The maps in the complexes are uniquely determined, as we are in the setting described in Remark 3.9. The problem of actually finding appropriate twist functions is discussed in Remark 5.8. Note that in some cases we have been unable to find perverse equivalences without introducing local perverse twists.

We show that X_i is isomorphic to Y_i in the stable category, as needed to obtain a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ lifting the twisted stable equivalence (cf. Section 3.2.2).

In all cases it is immediate to check that the perversity function π is invariant under the action of field automorphisms and under the action of the outer automorphism group of G. Whenever this automorphism group is non-trivial, this follows from the fact that the function π takes the same value on non-trivial "unipotent" characters of the same degree. Also, the twisted stable equivalences are invariant under the outer automorphism group of G. This enables us to use Corollary 4.30 and the descent method of Section 3.2.1.

In the last paragraph of Section 5.6.4.3, we give an example of a non-principal block.

All the equivalences we construct are actually defined over $\mathbf{F}_3$, except the one for $\mathfrak{A}_7$ in Section 5.3.1. Note that when $E \not\cong Z_4$ then the simple $\mathbf{F}_3 E$-modules are absolutely simple.

Let us introduce some more notation. The trivial modules are labelled S_1 and T_1. Often when describing the structure of a kN-module we will abbreviate T_i to i, and to save space we separate the radical layers by /, so a module with T_1 in the head and T_2 in the second radical layer would be described as 1/2. The projective cover of T_i will be denoted by $\mathcal{P}(i)$.

Remark 5.5. Okuyama has constructed derived equivalences for all blocks of simple groups with Sylow 3-subgroup Z_3^2 [41]. Note that the equivalences in [41] are all compositions of perverse equivalences. If the subsets $I_0, \ldots, I_r$ used by Okuyama are nested (*i.e.*, $I_l \subset I_m$ or $I_m \subset I_l$ for all l, m), then the composition itself is perverse.

5.3. Automizer Z_4

Over an algebraically closed field of characteristic 3, the group $Z_3^2 \rtimes Z_4$ has four simple, 1-dimensional modules, but over $\mathbf{F}_3$ (or any other field without a fourth root of unity) it possesses only three. We denote the two 1-dimensional modules over $\mathbf{F}_3$ by T_1 and T_2, and the 2-dimensional simple (but not absolutely simple) module by T_3, which over $\mathbf{F}_9$ splits into $T_{3,1}$ and $T_{3,2}$.

$$\mathcal{P}(1) = \begin{matrix} 1 \\ 3 \\ 122 \\ 3 \\ 1 \end{matrix}, \qquad \mathcal{P}(2) = \begin{matrix} 2 \\ 3 \\ 112 \\ 3 \\ 2 \end{matrix}, \qquad \mathcal{P}(3) = \begin{matrix} 3 \\ 1122 \\ 333 \\ 1122 \\ 3 \end{matrix}.$$

In this section we will prove that the perverse form of Broué's conjecture holds for $\mathfrak{A}_7$, but not for $\mathfrak{A}_6$. Instead we set up a perverse equivalence from $\mathfrak{A}_6$ to $\mathfrak{A}_7$. (Note that if $\tilde{G}$ is an extension of $\mathfrak{S}_6$, rather that just $\mathfrak{A}_6$, then the perverse form of Broué's conjecture *does* hold for $\tilde{G}$, as we shall see in the next section.)

The perversity function is given in the following table.

	T_1	T_2	T_3
$\mathfrak{A}_7$	0	1	0

In addition, there is a composition of two perverse equivalences for $\mathfrak{A}_6$

$$kB_0(\mathfrak{A}_6) \xrightarrow{(0,0,1)} kB_0(\mathfrak{A}_7) \xrightarrow{(0,1,0)} kB_0(N).$$

5.3.1. *The alternating group $\mathfrak{A}_7$*

Let $G = \mathfrak{A}_7$.

5.3.1.1. Simple modules. There are four simple modules in $kB_0(G)$, of dimensions 1, 10, 10 and 13. Over $\mathbf{F}_3$, the two 10-dimensional modules $S_{3,1}$ and $S_{3,2}$ amalgamate into a 20-dimensional module S_3. Write S_2 for the 13-dimensional simple module. The Green correspondents are

$$C_1 = 1, \qquad C_2 = \begin{matrix}2\\3,\\2\end{matrix} \qquad C_3 = 3.$$

5.3.1.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with the π-values on the left, which makes the decomposition matrix look as follows. Note that this equivalence has been constructed already by Okuyama [41, Example 4.1].

π	Ord. Char.	S_1	$S_{3,1}$	$S_{3,2}$	S_2
0	1	1			
0	10		1		
0	10			1	
1	35	2	1	1	1
	14	1			1
	14	1			1

We have

$$X_2 : 0 \to \mathcal{P}(2) \to C_2 \to 0,$$

and $H^{-1}(X_2) = \Omega^{-1}(C_2) = 11/3/2$.

5.3.2. The alternating group $\mathfrak{A}_6$

Let $G = \mathfrak{A}_6$.

5.3.2.1. Simple modules. There are four simple modules in $kB_0(G)$, of dimensions 1, 3, 3 and 4. Over $\mathbf{F}_3$, the two 3-dimensional modules amalgamate into a 6-dimensional module S_3. We label the 4-dimensional simple module S_2.

We construct a perverse equivalence between the principal blocks of G and $H = \mathfrak{A}_7$ lifting the stable equivalence given by induction and restriction. We label the three simple $\mathbf{F}_3H$-modules in the principal block by U_1, U_2 and U_3, with the ordering taken from Section 5.3.1. The images C_i of S_i in H under the stable equivalence are

$$C_1 = U_1, \qquad C_2 = U_2, \qquad C_3 = \begin{matrix}U_3\\U_1 \oplus U_1.\\U_3\end{matrix}$$

5.3.2.2. The perverse equivalence. We set $\pi(U_1) = \pi(U_2) = 0$ and $\pi(U_3) = 1$. Let X_i be the complex for the image of the ith simple module for the corresponding perverse equivalence. We have $X_1 = C_1$ and $X_2 = C_2$. The structure of $\mathcal{P}(U_3)$ is

$$\mathcal{P}(U_3) = \begin{matrix}U_3\\U_1 \oplus U_1\\U_2 \oplus U_3 \oplus U_2.\\U_1 \oplus U_1\\U_3\end{matrix}$$

We deduce that $X_3 = (0 \to \mathcal{P}(U_3) \to C_3 \to 0)$, with cohomology

$$H^{-1}(X_3) = (U_2 \oplus U_2)/(U_1 \oplus U_1)/U_3.$$

It satisfies the conditions of the algorithm, so produces a perverse equivalence. This equivalence has been constructed by Okuyama [41, Example 4.2].

5.3.2.3. Outer automorphisms. The group $\mathrm{Out}(\mathfrak{A}_6)$ has order 4, with three order-2 extensions, yielding the groups $\mathfrak{S}_6$, $\mathrm{PGL}_2(9)$ and M_{10} (the one-point stabilizer of the Mathieu group M_{11}). In Section 5.5.1 we will provide a perverse equivalence between the principal block of $\mathfrak{S}_6$ and the principal block of its normalizer, and this will be compatible with the outer automorphism of $\mathfrak{S}_6$. For the other two extensions $\mathrm{PGL}_2(9)$ and M_{10}, the decomposition matrices are not triangular, and so there can be no perverse equivalence for their principal blocks. However, since both of the equivalences $D^b(kB_0(G)) \xrightarrow{\sim} D^b(kB_0(H))$ (from above) and $D^b(kB_0(H)) \xrightarrow{\sim} D^b(kB_0(N))$ (from Section 5.3.1) are compatible with exchanging the two simple modules defined over $\mathbf{F}_9$, the derived equivalence obtained by composing these two perverse equivalences will extend to both $\mathrm{PGL}_2(9)$ and M_{10}.

Remark 5.6. While the decomposition matrix of $B_0(G)$ is triangular, the fact that the principal block of $\mathrm{PGL}_2(9)$ has a non-triangular decomposition matrix means that there can be no perverse equivalence between $kB_0(G)$ and $kB_0(N)$. Indeed, all standard stable equivalences are compatible with the action of $\mathrm{PGL}_2(9)/G$ by Remark 3.10, hence all perverse equivalences extend to $\mathrm{PGL}_2(9)$ since the two modules of dimension 3 in $B_0(G)$ are fixed by $\mathrm{PGL}_2(9)$ (cf. Corollary 4.30).

5.4. Automizer Q_8

For the group $Z_3^2 \rtimes Q_8$, there are five simple $\mathbf{F}_3$-modules, all absolutely simple. The first four are 1-dimensional, and the last is 2-dimensional. The three non-trivial 1-dimensional module are permuted transitively by the $\mathfrak{S}_3$-group of outer automorphisms of Q_8. The projective indecomposable modules are as follows.

$$\mathcal{P}(1)=\begin{matrix}1\\5\\234\\5\\1\end{matrix},\quad \mathcal{P}(2)=\begin{matrix}2\\5\\134\\5\\2\end{matrix},\quad \mathcal{P}(3)=\begin{matrix}3\\5\\124\\5\\3\end{matrix},\quad \mathcal{P}(4)=\begin{matrix}4\\5\\123\\5\\4\end{matrix},\quad \mathcal{P}(5)=\begin{matrix}5\\1234\\555\\1234\\5\end{matrix}.$$

The perversity functions are given in the following table.

	T_1	T_2	T_3	T_4	T_5
$\mathrm{PSL}_3(r)$	0	3	3	3	2
$\mathrm{PSU}_3(q)$	0	6	6	6	4

In addition, there is a composition of two perverse equivalences for principal blocks of M_{22} and for M_{10} (these blocks are splendidly Morita equivalent) given by the composite π-values $(0,1,1,0,0)$, $(0,0,0,0,1)$.

$$kB_0(M_{22})\text{-mod} \simeq kB_0(M_{10})\text{-mod} \xrightarrow{(0,0,0,0,1)} \bullet \xrightarrow{(0,1,1,0,0)} kB_0(N)\text{-mod}.$$

5.4.1. The group $\mathrm{PSL}_3(4)$

Let $G = \mathrm{PSL}_3(4)$. Okuyama has shown that the principal blocks of G and N are derived equivalent [41, Example 4.6]. In this section, we produce a perverse equivalence.

5.4.1.1. Simple modules. There are five simple modules in the principal 3-block, of dimensions 1, 15, 15, 15, and 19. We label the 19-dimensional module S_5, and the three 15-dimensional simple modules S_2 to S_4. There is an $\mathfrak{S}_3$-group of outer automorphisms that permutes transitively S_2, S_3 and S_4. We choose the S_i so that the Green correspondents C_i are

$$C_1 = 1, \qquad C_2 = \begin{matrix}5\\12\\5\end{matrix}, \qquad C_3 = \begin{matrix}5\\13\\5\end{matrix}, \qquad C_4 = \begin{matrix}5\\14\\5\end{matrix}, \qquad C_5 = \begin{matrix}234\\55\\234\end{matrix}.$$

5.4.1.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with the π-values on the left, which makes the decomposition matrix look as follows. We provide the generic degree for the corresponding irreducible characters of $\mathrm{PSL}_3(q)$.

π	Ord. Char.	S_1	S_5	S_2	S_3	S_4
0	1	1				
2	$q(q+1)$	1	1			
3	$(q+1)(q^2+q+1)/3$	1	1	1		
3	$(q+1)(q^2+q+1)/3$	1	1		1	
3	$(q+1)(q^2+q+1)/3$	1	1			1
	q^3		1	1	1	1

The explicit complexes are as follows.

$$X_5: \qquad 0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to C_5 \to 0,$$
$$X_2: \ 0 \to \mathcal{P}(2) \to \mathcal{P}(34) \to \mathcal{P}(5) \to C_2 \to 0,$$
$$X_3: \ 0 \to \mathcal{P}(3) \to \mathcal{P}(24) \to \mathcal{P}(5) \to C_3 \to 0,$$
$$X_4: \ 0 \to \mathcal{P}(4) \to \mathcal{P}(23) \to \mathcal{P}(5) \to C_4 \to 0.$$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
5		$1/5$	11	$5-1$
2	$1/5/2$	1		$2-5$
3	$1/5/3$	1		$3-5$
4	$1/5/4$	1		$4-5$

5.4.2. The group $\mathrm{PSU}_3(2)$

Let $G = \mathrm{PSU}_3(2) = Z_3^2 \rtimes Q_8$.

We construct a self-perverse equivalence of kG with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_5	S_2	S_3	S_4
0	1	1				
4	$q(q-1)$		1			
6	$(q-1)(q^2-q+1)/3$			1		
6	$(q-1)(q^2-q+1)/3$				1	
6	$(q-1)(q^2-q+1)/3$					1
	q^3	1	2	1	1	1

The explicit complexes are as follows.

$$X_5:\quad 0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to \mathcal{P}(234) \to \mathcal{P}(5) \to C_5 \to 0,$$
$$X_2:\quad 0 \to \mathcal{P}(2) \to \mathcal{P}(34) \to \mathcal{P}(234) \to \mathcal{P}(25) \to \mathcal{P}(2) \to C_2 \to 0,$$
$$X_3:\quad 0 \to \mathcal{P}(3) \to \mathcal{P}(24) \to \mathcal{P}(234) \to \mathcal{P}(35) \to \mathcal{P}(3) \to C_3 \to 0,$$
$$X_4:\quad 0 \to \mathcal{P}(4) \to \mathcal{P}(23) \to \mathcal{P}(234) \to \mathcal{P}(45) \to \mathcal{P}(4) \to C_4 \to 0.$$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
5			$1/5$	11	11	1	5
2	$1/5/2$	$1 \oplus 1/5$	11	1			2
3	$1/5/3$	$1 \oplus 1/5$	11	1			3
4	$1/5/4$	$1 \oplus 1/5$	11	1			4

Remark 5.7. Note that the perversity function for $\mathrm{PSU}_3(r)$ is twice that of $\mathrm{PSL}_3(q)$, after identification of the "unipotent characters". This is our reason for providing this perverse equivalence, instead of the identity.

5.4.3. The Mathieu group M_{22}

Let $G = M_{22}$. By [41, Example 4.5], there is a splendid Morita equivalence between $kB_0(G)$ and $kB_0(H)$, where $H = M_{10}$. There is an embedding of H inside G so that $N_H(P) = N$. The composition of the splendid Morita equivalence from $kB_0(G)$ to $kB_0(H)$ and the derived equivalence $kB_0(H) \to kB_0(N)$ from Section 5.3.2, yields a derived equivalence $kB_0(G) \to kB_0(N)$. There can be no perverse equivalence between $kB_0(G)$ and $kB_0(N)$ since there is none between $kB_0(H)$ and $kB_0(N)$.

5.4.3.1. Outer automorphisms. There is an outer automorphism of M_{22}, and this will be discussed in Section 5.6.3. Note that the splendid Morita equivalence from $kB_0(G)$ to $kB_0(H)$ extends to one between $kB_0(M_{22}.2)$ and $kB_0(\mathfrak{A}_6.2^2)$.

5.5. Automizer D_8

For the group $N = Z_3^2 \rtimes D_8$, there are five simple $\mathbf{F}_3$-modules, all absolutely simple. The first four are 1-dimensional, and the last is 2-dimensional. We denote by T_4 the exterior square of the

2-dimensional module. The modules T_2 and T_3 are permuted by the outer automorphism of D_8. There are two Klein four subgroups lying in D_8. One acts trivially on T_2 and not on T_3, while the other acts trivially on T_3 and not on T_2. The projective indecomposable modules are as follows.

$$\mathcal{P}(1)=\begin{array}{c}1\\5\\123\\5\\1\end{array},\quad \mathcal{P}(2)=\begin{array}{c}2\\5\\124\\5\\2\end{array},\quad \mathcal{P}(3)=\begin{array}{c}3\\5\\134\\5\\3\end{array},\quad \mathcal{P}(4)=\begin{array}{c}4\\5\\234\\5\\4\end{array},\quad \mathcal{P}(5)=\begin{array}{c}5\\1234\\555\\1234\\5\end{array}.$$

We also need relative projective modules. There are two conjugacy classes of subgroups of order 3 in N, with representatives Q_1 and Q_2. We denote by $M_{1,j},\ldots,M_{4,j}$ the indecomposable summands of the permutation module $\mathrm{Ind}_{Q_i}^{N} k$.

$$M_{1,1}=\begin{array}{c}5\\12\\5\end{array},\quad M_{2,1}=\begin{array}{c}5\\34\\5\end{array},\quad M_{3,1}=\begin{array}{c}12\\5\\12\end{array},\quad M_{4,1}=\begin{array}{c}34\\5\\34\end{array},$$

$$M_{1,2}=\begin{array}{c}5\\13\\5\end{array},\quad M_{2,2}=\begin{array}{c}5\\24\\5\end{array},\quad M_{3,2}=\begin{array}{c}13\\5\\13\end{array},\quad M_{4,2}=\begin{array}{c}24\\5\\24\end{array}.$$

The perversity and local twist functions are given in the following table.

	T_1	T_2	T_3	T_4	T_5	$\eta(Q_1)$	$\eta(Q_2)$
$\mathfrak{S}_7$	0	1	1	0	0	0	0
$\mathrm{PSp}_4(q), \mathfrak{S}_6$	0	3	3	4	3	1	1
$\mathrm{PSp}_4(r)$	0	6	6	8	6	2	2
$\mathrm{PSL}_4(q), \mathfrak{A}_8$	0	3	5	6	4	2	1
$\mathrm{PSU}_4(r)$	0	6	10	12	8	4	2
$\mathrm{PSL}_5(q)$	0	6	8	10	7	3	2
$\mathrm{PSU}_5(r)$	0	12	16	20	14	6	4

In addition, there is a composition of two perverse equivalences for $\mathrm{PGL}_2(9)=\mathfrak{A}_6.2_2$ given by the composite π-values $(0,1,1,0,0)$, $(0,0,0,0,1)$ (and $\eta=0$).

Remark 5.8. Let us explain how we find the non-trivial local twists η. They are needed when the local equivalences are not Morita. If one merely runs the algorithm to find a perverse equivalence, one finds that the $C_{S_i}^0$'s (constructed as in Section 3.1.2) are not isomorphic to the C_i's in the stable category. In the examples, there is a module N_i and a complex $N_i \to C_i$ that is isomorphic to $C_{S_i}^0$ in the stable category. More precisely, the complex C_{S_i} is quasi-isomorphic to a complex $0 \to C_{S_i}^{\prime -\pi(i)} \to \cdots \to C_{S_i}^{\prime -2} \to C_{S_i}^{\prime -1} \oplus N_i \to C_i \to 0$ where the $C_{S_i}^{\prime r}$ are projective. If the functions η take value in $\{0,1\}$, then N_i is a sum of a module projective relatively to Q_1 and a module projective relatively to Q_2.

For $\mathrm{PSp}_4(4)$, $\mathrm{PSL}_4(2)$, $\mathrm{PSU}_4(4)$, $\mathrm{PSL}_5(2)$ and $\mathrm{PSU}_5(4)$ below, the modules N_i do not have these relative projectivity properties, and the local twists will need to take values >1. Note nevertheless that N_i is *filtered* by relative projective modules.

5.5.1. The group $\mathrm{PSp}_4(2)=\mathfrak{S}_6$

Let $G=\mathrm{PSp}_4(2)=\mathfrak{S}_6$.

5.5.1.1. Simple modules. There are five simple modules in the principal 3-block, of dimensions 1, 1, 4, 4, and 6. We label the non-trivial 1-dimensional module S_5, the 6-dimensional simple module S_4, and the two 4-dimensional simple modules S_2 and S_3. There is an outer automorphism that swaps S_2 and S_3 and we choose the labelling so that the Green correspondents C_i are

$$C_1 = 1, \qquad C_2 = \begin{matrix} 3 \\ 5, \\ 3 \end{matrix} \qquad C_3 = \begin{matrix} 2 \\ 5, \\ 2 \end{matrix} \qquad C_4 = \begin{matrix} 5 \\ 14, \\ 5 \end{matrix} \qquad C_5 = 4.$$

5.5.1.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_{Q_1} = \eta_{Q_2} = 1$ and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_5	S_2	S_3	S_4
0	1	1				
3	$q(q-1)^2/2$		1			
3	$q(q^2+1)/2$	1		1		
3	$q(q^2+1)/2$	1			1	
4	q^4	1	1	1	1	1
	$(q-1)(q^2+1)$		1		1	
	$(q-1)(q^2+1)$		1	1		
	$q(q-1)(q^2+1)$			1		1
	$q(q-1)(q^2+1)$				1	1

The explicit complexes are as follows.

X_5: $0 \to \mathcal{P}(5) \to \mathcal{P}(234) \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0,$

X_2: $0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0,$

X_3: $0 \to \mathcal{P}(3) \to \mathcal{P}(5) \to \mathcal{P}(2) \oplus M_{1,1} \to C_3 \to 0,$

X_4: $0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(23) \to \mathcal{P}(5) \to C_4 \to 0.$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
5		1/5	1		5
2		2		1	$2-1$
3		3		1	$3-1$
4	23/5/4		1		$4+1-2-3-5$

5.5.2. The group $\mathrm{PSp}_4(4)$

Let $G = \mathrm{PSp}_4(4)$.

5.5.2.1. Simple modules. There are five simple modules in the principal 3-block, of dimensions 1, 34, 34, 50, and 256. We label the 50-dimensional module S_4 and the 256-dimensional module S_5. The two 34-dimensional modules S_2 and S_3 are permuted by an outer automorphism and we choose the S_i so that the Green correspondents are $C_i = T_i$.

5.5.2.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_{Q_1} = \eta_{Q_2} = 2$ and the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_5	S_2	S_3	S_4
0	1	1				
6	$q(q+1)^2/2$		1			
6	$q(q^2+1)/2$			1		
6	$q(q^2+1)/2$				1	
8	q^4					1
	$(q+1)(q^2+1)$	1	1	1		
	$(q+1)(q^2+1)$	1	1		1	
	$q(q+1)(q^2+1)$		1		1	1
	$q(q+1)(q^2+1)$		1	1		1

The explicit complexes are as follows.

$$X_2:\quad \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(35) \to \mathcal{P}(234) \to \mathcal{P}(4) \oplus M_{4,2} \to M_{4,2} \to C_2,$$

$$X_3:\quad \mathcal{P}(3) \to \mathcal{P}(5) \to \mathcal{P}(25) \to \mathcal{P}(234) \to \mathcal{P}(4) \oplus M_{4,1} \to M_{4,1} \to C_3,$$

$$X_5:\ \mathcal{P}(234) \to \mathcal{P}(2344) \to \mathcal{P}(455) \to \mathcal{P}(5) \oplus M_{1,1} \oplus M_{1,2} \to M_{1,1} \oplus M_{1,2} \to C_5,$$

$$\begin{aligned} X_4:\ & \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(5) \\ & \to \mathcal{P}(55) \to \mathcal{P}(23445) \to \mathcal{P}(234) \oplus M_{4,1} \oplus M_{4,2} \to M_{4,1} \oplus M_{4,2} \to C_4. \end{aligned}$$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
2			2		1	1			2
3			3		1	1			3
5			1/5	1			1	1	5
4	23/5/4	23/5		1	1	1	1		4

5.5.3. The group $\mathrm{PSL}_4(2) = \mathfrak{A}_8$

Let $G = \mathrm{PSL}_4(2) = \mathfrak{A}_8$. There is an easy perverse equivalence constructed by Okuyama [41, Example 4.3]: the perversity function vanishes on all simple modules except one, where the π-value is 1. However, this is not compatible with the Deligne–Lusztig theory for $\mathrm{PSL}_4(q)$, and we provide a different perverse equivalence.

5.5.3.1. Simple modules. There are five simple modules in the principal 3-block, of dimensions 1, 7, 13, 28 and 35. We label the modules S_1 to S_5 so that they have dimensions 1, 13, 35, 28 and 7 respectively. The Green correspondents C_i are

$$C_1 = 1, \qquad C_2 = \begin{matrix} 3 \\ 5, \\ 3 \end{matrix} \qquad C_3 = 5, \qquad C_4 = 2, \qquad C_5 = 4.$$

5.5.3.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_{Q_1} = 2$ and $\eta_{Q_2} = 1$ and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
3	$q(q^2+q+1)$	1	1			
4	$q^2(q^2+1)$		1	1		
5	$q^3(q^2+q+1)$	1	1	1	1	
6	q^6	1			1	1
	$(q-1)^2(q^2+q+1)$			1		
	$q^2(q-1)^2(q^2+q+1)$					1
	$(q-1)(q^2+1)(q^2+q+1)$				1	
	$q(q-1)(q^2+1)(q^2+q+1)$			1	1	1

The explicit complexes are as follows.

$$X_2: \quad 0 \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(3) \oplus M_{1,2} \to C_2 \to 0,$$

$$X_5: \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(234) \oplus M_{4,1} \to M_{4,1} \oplus M_{4,2} \to C_5 \to 0,$$

$$X_3: \quad 0 \to \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(5) \oplus M_{1,1} \to M_{1,1} \oplus M_{1,2} \to C_3 \to 0,$$

$$X_4: \quad 0 \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(4) \to M_{4,2} \to C_4 \to 0.$$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
2				2		1	$2-1$
5			12/5	1	1		$5-2+1$
3		1/5/3			1	1	$3-5-1$
4	23/5/4	2	1	1			$4+5-3$

5.5.4. The group $\mathrm{PSU}_4(4)$

Let $G = \mathrm{PSU}_4(4)$. In [31, Corollary 2.7], it is shown that the principal block of G is splendidly Morita equivalent to that of N. We provide a different equivalence.

5.5.4.1. Simple modules. There are five simple modules in the principal 3-block, of dimensions 1, 52, 272, 832 and 4096. We label the modules S_1 to S_5 so that they have dimensions 1, 52, 832, 4096 and 272 respectively. The Green correspondents are $C_i = T_i$.

5.5.4.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_{Q_1} = 4$ and $\eta_{Q_2} = 2$ and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
6	$q(q^2-q+1)$		1			
8	$q^2(q^2+1)$			1		
10	$q^3(q^2-q+1)$				1	
12	q^6					1
	$(q+1)^2(q^2-q+1)$	1	1	1		
	$(q+1)(q^2+1)(q^2-q+1)$	1		1	1	
	$q^2(q+1)^2(q^2-q+1)$			1	1	1
	$q(q+1)(q^2+1)(q^2-q+1)$		1	1		1

The explicit complexes are too long to write down here, but we make a record of the relatively projective modules involved.

X_i	-4	-3	-2	-1
X_2			$M_{4,2}$	$M_{4,2}$
X_5	$M_{1,1}$	$M_{1,1}$	$M_{1,1} \oplus M_{1,2}$	$M_{1,1} \oplus M_{1,2}$
X_3	$M_{4,1}$	$M_{4,1}$	$M_{4,1}$	$M_{4,1}$
X_4	$M_{4,1}$	$M_{4,1}$	$M_{4,1} \oplus M_{4,2}$	$M_{4,1} \oplus M_{4,2}$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-12}	H^{-11}	H^{-10}	H^{-9}	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
X_2							2		1	1			2
X_5					12/5	12	1	1			1	1	5
X_3			1/5/3	12/5	2				1	1			3
X_4	23/5/4	5/23	2/5	12/5	1			1	1	1	1		4

5.5.5. The group $\mathrm{PSL}_5(2)$

Let $G = \mathrm{PSL}_5(2)$. In [30, Theorem 0.2] it is shown that the principal block of G is splendidly Morita equivalent to that of N. We provide here a different equivalence.

5.5.5.1. Simple modules. There are five simple modules in the principal 3-block, of dimensions 1, 124, 155, 217 and 868. We label the modules S_1 to S_5 so that they have dimensions 1, 124, 217, 868 and 155 respectively. The Green correspondents C_i are

$$C_1 = 1, \qquad C_2 = 2, \qquad C_3 = 4, \qquad C_4 = 3, \qquad C_5 = 5.$$

5.5.5.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twists $\eta_{Q_1} = 3$ and $\eta_{Q_2} = 2$ and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
6	$q^2(q^4+q^3+q^2+q+1)$		1			
7	$q^3(q^2+1)(q^2+q+1)$	1	1	1		
8	$q^4(q^4+q^3+q^2+q+1)$		1	1	1	
10	q^{10}	1		1		1
	$(q-1)(q^2+1)(q^4+q^3+q^2+q+1)$			1		
	$(q-1)^2(q^2+q+1)(q^4+q^3+q^2+q+1)$				1	
	$q^2(q-1)^2(q^2+q+1)(q^4+q^3+q^2+q+1)$					1
	$q^3(q-1)(q^2+1)(q^4+q^3+q^2+q+1)$			1	1	1

The explicit complexes are as follows.

$$X_2\colon \quad \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(35) \to \mathcal{P}(234) \to \mathcal{P}(4) \oplus M_{4,2} \to M_{4,2} \to C_2,$$

$$\begin{aligned} X_5\colon \quad & \mathcal{P}(5) \to \mathcal{P}(345) \to \mathcal{P}(23344) \to \mathcal{P}(23445) \\ & \to \mathcal{P}(455) \oplus M_{1,1} \to \mathcal{P}(5) \oplus M_{1,1} \oplus M_{1,2} \to M_{1,1} \oplus M_{1,2} \to C_5, \end{aligned}$$

$$\begin{aligned} X_3\colon \quad & \mathcal{P}(3) \to \mathcal{P}(34) \to \mathcal{P}(45) \to \mathcal{P}(55) \to \mathcal{P}(3455) \\ & \to \mathcal{P}(23445) \oplus M_{4,1} \to \mathcal{P}(234) \oplus M_{4,1} \oplus M_{4,2} \to M_{4,1} \oplus M_{4,2} \to C_4, \end{aligned}$$

$$\begin{aligned} X_4\colon \quad & \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(4) \to \mathcal{P}(34) \to \mathcal{P}(35) \\ & \to \mathcal{P}(55) \to \mathcal{P}(2345) \to \mathcal{P}(234) \oplus M_{4,1} \to \mathcal{P}(4) \oplus M_{4,1} \to M_{4,1} \to C_3. \end{aligned}$$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-10}	H^{-9}	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
X_2					2		1	1			2
X_5				12/5	1	1			1	1	$5-1-2$
X_3			1/5/3			1	1	1	1		$3-5+1$
X_4	23/5/4	5/23	2/5				1	1			$4+2-5$

5.5.6. The group $\mathrm{PSU}_5(4)$

Let $G = \mathrm{PSU}_5(4)$.

5.5.6.1. Simple modules. There are five simple modules in the principal 3-block, of dimensions 1, 3280, 14 144, 52 840 and 1 048 576. We label the modules S_1 to S_5 so that they have dimensions 1, 52 840, 3280, 1 048 576 and 14 144. The Green correspondents are $C_i = T_i$.

5.5.6.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_{Q_1} = 6$ and $\eta_{Q_2} = 4$ and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_2	S_5	S_3	S_4
0	1	1				
12	$q^2(q^4-q^3+q^2-q+1)$		1			
14	$q^3(q^2+1)(q^2-q+1)$			1		
16	$q^4(q^4-q^3+q^2-q+1)$				1	
20	q^{10}					1
	$(q+1)(q^2+1)(q^4-q^3+q^2-q+1)$	1	1	1		
	$(q+1)^2(q^2-q+1)(q^4-q^3+q^2-q+1)$	1		1	1	
	$q^3(q+1)(q^2+1)(q^4-q^3+q^2-q+1)$			1	1	1
	$q^2(q+1)^2(q^2-q+1)(q^4-q^3+q^2-q+1)$		1	1		1

The explicit complexes are too long to write down here, but we make a record of the relatively projective modules involved.

X_i	-6	-5	-4	-3	-2	-1
X_2			$M_{4,2}$	$M_{4,2}$	$M_{4,2}$	$M_{4,2}$
X_5	$M_{1,1}$	$M_{1,1}$	$M_{1,1} \oplus M_{1,2}$	$M_{1,1} \oplus M_{1,2}$	$M_{1,1} \oplus M_{1,2}$	$M_{1,1} \oplus M_{1,2}$
X_3	$M_{4,1}$	$M_{4,1}$	$M_{4,1}$	$M_{4,1}$	$M_{4,1}$	$M_{4,1}$
X_4	$M_{4,1}$	$M_{4,1}$	$M_{4,1} \oplus M_{4,2}$	$M_{4,1} \oplus M_{4,2}$	$M_{4,1} \oplus M_{4,2}$	$M_{4,1} \oplus M_{4,2}$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-20}	H^{-19}	H^{-18}	H^{-17}	H^{-16}	H^{-15}	H^{-14}	H^{-13}	H^{-12}	H^{-11}
X_2									2	
X_5							12/5	12	1	1
X_3					1/5/3	12/5	2			
X_4	23/5/4	5/23	23/5	5/23	2/5	12/5	1			1

X_i	H^{-10}	H^{-9}	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
X_2	1	1					1	1			2
X_5			1	1	1	1			1	1	5
X_3	1	1					1	1			3
X_4	1	1	1			1	1	1	1		4

5.6. Automizer SD_{16}

The group $Z_3^2 \rtimes SD_{16}$ has seven simple $\mathbf{F}_3$-modules, all absolutely simple. There are four of dimension 1 – denoted T_1 to T_4 – and three of dimension 2 – denoted T_5 to T_7. They are chosen so that $\ker T_3 = D_8$ and $\ker T_4 = Q_8$. There is a unique labelling so that the projective indecomposable modules are given below:

$$\mathcal{P}(1) = \begin{array}{c} 1 \\ 7 \\ 35 \\ 6 \\ 1 \end{array}, \qquad \mathcal{P}(2) = \begin{array}{c} 2 \\ 7 \\ 45 \\ 6 \\ 2 \end{array}, \qquad \mathcal{P}(3) = \begin{array}{c} 3 \\ 6 \\ 15 \\ 7 \\ 3 \end{array}, \qquad \mathcal{P}(4) = \begin{array}{c} 4 \\ 6 \\ 25 \\ 7 \\ 4 \end{array},$$

$$\mathcal{P}(5)=\begin{array}{c}5\\67\\12345\\67\\5\end{array},\qquad \mathcal{P}(6)=\begin{array}{c}6\\125\\677\\345\\6\end{array},\qquad \mathcal{P}(7)=\begin{array}{c}7\\345\\667\\125\\7\end{array}.$$

Let Q be a representative of the unique conjugacy class of subgroups of P of order 3. There are four modules with vertex Q and trivial source, labelled as below.

$$M_{1,1}=\begin{array}{c}135\\67\\135\end{array},\qquad M_{2,1}=\begin{array}{c}245\\67\\245\end{array},\qquad M_{3,1}=\begin{array}{c}67\\135\\67\end{array},\qquad M_{4,1}=\begin{array}{c}67\\245\\67\end{array}.$$

There are also four modules with vertex C_3 and 2-dimensional source, labelled as below.

$$M_{1,2}=\begin{array}{c}135\\6677\\123455\\67\end{array},\qquad M_{2,2}=\begin{array}{c}245\\6677\\123455\\67\end{array},\qquad M_{3,2}=\begin{array}{c}67\\123455\\6677\\135\end{array},\qquad M_{4,2}=\begin{array}{c}67\\123455\\6677\\245\end{array}.$$

The perversity and local twist functions are given in the following table.

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	$\eta(Q)$
M_{11}	0	4	2	5	7	6	3	1
M_{23}	0	1	0	0	2	0	0	0
$M_{22}.2, \mathfrak{A}_6.2^2, \mathrm{PSp}_4(q).Z_{2n}$	0	4	0	4	3	3	3	1
$\mathrm{PSp}_4(r).Z_{2n}$	0	8	0	8	6	6	6	2
$\mathrm{PSL}_3(r).Z_2$	0	2	0	2	3	3	3	0
$\mathrm{PSU}_3(q).Z_n$	0	4	0	4	6	6	6	0
HS	0	7	0	4	3	10	3	1

5.6.1. The Mathieu group M_{11}

Let $G=M_{11}$. By [41, Example 4.9], there is a derived equivalence between $kB_0(G)$ and $kB_0(N)$. In this section we will produce a perverse equivalence between the two blocks.

5.6.1.1. Simple modules. There are seven simple modules in the principal 3-block, of dimensions 1, 5, 5, 10, 10, 10 and 24. The ordering on the S_i is the chosen such that S_2 and S_4 are 5-dimensional (dual) modules, S_3 and S_5 are 10-dimensional (dual) modules, S_6 is the 24-dimensional module, and S_7 is the self-dual 10-dimensional module. The choice of S_2 through to S_7 is such that the Green correspondents are given below.

$$C_1=1,\qquad C_2=\begin{array}{c}3\\6\\5\end{array},\qquad C_3=\begin{array}{c}7\\45\\6\\27\end{array},\qquad C_4=\begin{array}{c}5\\7\\3\end{array},$$

$$C_5=\begin{array}{c}2\\67\\45\\6\end{array},\qquad C_6=\begin{array}{c}6\\12\\7\end{array},\qquad C_7=4.$$

5.6.1.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_Q = 1$ and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_3	S_7	S_2	S_4	S_6	S_5
0	1	1						
2	10		1					
3	10			1				
4	16	1	1		1			
5	11	1			1	1		
6	44			1	1	1	1	
7	55	1	1		1	1	1	1
	10							1
	16	1				1		1

The explicit complexes are as follows.

$$\begin{aligned}
&X_3: && 0 \to \mathcal{P}(3) \to \mathcal{P}(7) \to C_3 \to 0,\\
&X_7: && 0 \to \mathcal{P}(7) \to \mathcal{P}(25) \to M_{2,1} \to C_7 \to 0,\\
&X_2: && 0 \to \mathcal{P}(2) \to \mathcal{P}(6) \to \mathcal{P}(67) \to \mathcal{P}(3) \oplus M_{4,2} \to C_2 \to 0,\\
&X_4: && 0 \to \mathcal{P}(4) \to \mathcal{P}(5) \to \mathcal{P}(25) \to \mathcal{P}(246) \to M_{2,2} \to C_4 \to 0,\\
&X_6: && 0 \to \mathcal{P}(6) \to \mathcal{P}(5) \to \mathcal{P}(4) \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(6) \to C_6 \to 0,\\
&X_5: && 0 \to \mathcal{P}(5) \to \mathcal{P}(5) \to \mathcal{P}(6) \to \mathcal{P}(46) \to \mathcal{P}(45) \to \mathcal{P}(25) \to \mathcal{P}(26) \to C_5 \to 0.
\end{aligned}$$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
3						3		3
7					$1/7$	1		7
2				2	3		1	$2-3-1$
4			$2/7/4$	$1/7, 3$	1			$4-2+3$
6		$7/34/6$	3			1		$6-4-7+1$
5	$1234/67/5$	$7/34, 1, 2$	$1/7/3$	3	3	1	1	$5-6+7-3-1$

Here, “$1/7, 3$” means a direct sum of $1/7$ and 3 and “$7/34, 1, 2$” a direct sum of $7/34$, 1 and 2.

5.6.2. The Mathieu group M_{23}

Let $G = M_{23}$.

5.6.2.1. Simple modules. There are seven simple modules in the principal 3-block, of dimensions 1, 22, 104, 104, 253, 770 and 770. The ordering on the S_i is such that S_2 and S_5 are 104-dimensional (dual) modules, S_3 is 253-dimensional, S_4 is 22-dimensional, and S_6 and S_7 are (dual) 770-dimensional modules. The choice of S_2 and S_5 through to S_7 is such that the Green correspondents are given below.

$$C_1 = 1, \qquad C_2 = \begin{matrix}2\\7\\5\end{matrix}, \qquad C_3 = 3, \qquad C_4 = 4, \qquad C_5 = \begin{matrix}5\\6\\2\end{matrix}, \qquad C_6 = 6, \qquad C_7 = 7.$$

5.6.2.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_Q = 0$ and with the π-values on the left, which makes the decomposition matrix look as follows. This equivalence has been constructed previously by Okuyama [41, Example 4.7].

π	Ord. Char.	S_1	S_3	S_4	S_6	S_7	S_2	S_5
0	1	1						
0	253		1					
0	22			1				
0	770				1			
0	770					1		
1	896			1	1		1	
2	230			1			1	1
	896			1		1		1
	2024	1	1	1	1	1	1	1

The explicit complexes are as follows.

$$X_2\colon \quad 0 \to \mathcal{P}(2) \to C_2 \to 0,$$
$$X_5\colon \quad 0 \to \mathcal{P}(5) \to \mathcal{P}(5) \to C_5 \to 0.$$

The cohomology of the complexes above is as follows: the first complex X_2 has $H^{-1}(X_2) = 4/6/2$, and the second complex X_5 has $H^{-2}(X_5) = 1234/67/5$ and $H^{-1}(X_5) = 1 \oplus 7/34$.

5.6.3. The extended Mathieu group $M_{22}.2$

Let $G = M_{22}.2$. In this section we will produce a perverse equivalence between $kB_0(G)$ and $kB_0(N)$. As we mentioned in Section 5.4.3, there is a splendid Morita equivalence between the principal blocks of M_{22} and $M_{10} = \mathfrak{A}_6.2_3$, and this extends to a splendid Morita equivalence between the principal blocks of $M_{22}.2$ and $\mathfrak{A}_6.2^2$. As we saw in Section 5.5.1, there is a perverse equivalence between the principal blocks of $\mathfrak{A}_6.2^2$ and its normalizer, and so therefore the same holds for $M_{22}.2$. As we have already proved that the perverse equivalence exists, and have described it earlier, we simply give the simple modules, and then the decomposition matrix and π-values.

5.6.3.1. Simple modules. There are seven simple modules in the principal 3-block, of dimensions 1, 1, 55, 55, 98, 231 and 231. The ordering on the S_i is such that S_2 and S_4 are 231-dimensional modules, S_3 and S_6 are 55-dimensional modules, S_5 is 98-dimensional, and S_7 is the non-trivial 1-dimensional module. The choice of S_2 through to S_4 and S_6 is such that the Green correspondents are given below.

$$C_1 = 1, \qquad C_2 = \begin{array}{c}7\\34\\6\end{array}, \qquad C_3 = 3, \qquad C_4 = \begin{array}{c}6\\12\\7\end{array},$$

$$C_5 = \begin{array}{c}5\\67\\5\end{array}, \qquad C_6 = 2, \qquad C_7 = 4.$$

5.6.3.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_3	S_5	S_7	S_6	S_2	S_4
0	1_1	1						
0	55_1		1					
3	154_1	1	1	1				
3	1_2				1			
3	55_2					1		
4	385_1	1		1		1	1	
4	385_2		1	1	1			1
	154_2			1	1	1		
	560			1			1	1

5.6.4. The Higman–Sims group HS

Let $G = HS$. By [41, Example 4.8], there is a derived equivalence between $kB_0(G)$ and $kB_0(N)$. In this section we will produce a perverse equivalence between the two blocks, different to that of [41].

5.6.4.1. Simple modules. There are seven simple modules in the principal 3-block, of dimensions 1, 22, 154, 321, 748, 1176 and 1253. The ordering on the S_i is such that the dimensions of the S_i are (in order) 1, 1176, 154, 321, 1253, 748 and 22. The Green correspondents are given below.

$$C_1 = 1, \qquad C_2 = \begin{matrix} 7 \\ 34, \\ 6 \end{matrix} \qquad C_3 = 3, \qquad C_4 = \begin{matrix} 6 \\ 12, \\ 7 \end{matrix}$$

$$C_5 = \begin{matrix} 5 \\ 67, \\ 5 \end{matrix} \qquad C_6 = \begin{matrix} 25 \\ 67, \\ 25 \end{matrix} \qquad C_7 = 4.$$

5.6.4.2. The perverse equivalence. There is a perverse equivalence between $kB_0(G)$ and $kB_0(N)$ with local twist $\eta_Q = 1$ and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	S_3	S_7	S_5	S_4	S_2	S_6
0	1	1						
0	154		1					
3	22			1				
3	1408	1	1		1			
4	1750		1	1	1	1		
7	2750				1	1	1	
10	3200	1		1	1		1	1
	770			1				1
	1925	1					1	1

The explicit complexes are as follows.

$$\begin{aligned}
&X_7: && \mathcal{P}(7) \to \mathcal{P}(25) \to M_{2,1} \to C_7,\\
&X_5: && \mathcal{P}(5) \to \mathcal{P}(67) \to \mathcal{P}(5) \oplus M_{3,1} \to C_5,\\
&X_4: && \mathcal{P}(4) \to \mathcal{P}(2) \to \mathcal{P}(5) \to \mathcal{P}(6) \to C_4,\\
&X_2: && \mathcal{P}(2) \to \mathcal{P}(6) \to \mathcal{P}(6) \to \mathcal{P}(2) \to \mathcal{P}(4) \to \mathcal{P}(5) \to \mathcal{P}(7) \to C_2,\\
&X_6: && \mathcal{P}(6) \to \mathcal{P}(6) \to \mathcal{P}(6) \to \mathcal{P}(6)\\
& && \to \mathcal{P}(2) \to \mathcal{P}(2) \to \mathcal{P}(6) \to \mathcal{P}(6) \to \mathcal{P}(47) \to \mathcal{P}(25) \to C_6.
\end{aligned}$$

The cohomology of the complexes above is displayed in the following table.

X_i	H^{-10}	H^{-9}	H^{-8}	H^{-7}	H^{-6}	H^{-5}	H^{-4}	H^{-3}	H^{-2}	H^{-1}	$[S_i]$
7								1/7	1		7
5								5		13	$5-1-3$
4							5/7/4		1		$4+1-5-7$
2				2	1/7/3	15/77/34	5		3		$2+3+7-4$
6	A_1	A_2	A_3	15/77/34	7/45		1/7/3	3		1	$6-2+4-5-7-7$

Here, $A_1 = 12/77/345/6$, $A_2 = 125/77/34$ and $A_3 = 12/77/345$.

5.6.4.3. A non-principal block of HS. We have $N_G(P) \simeq Z_2 \times (P \rtimes E)$. We denote by A (resp. B) the unique non-principal block of $\mathbf{F}_3G$ (resp. $\mathbf{F}_3N$) with defect group P. We have a canonical isomorphism $B \xrightarrow{\sim} \mathbf{F}_3P \rtimes E$ and we label simple modules for B as described in Section 5.6. By [29, Theorem 0.2], there is a derived equivalence between A and B, and we show it is perverse. Let us recall the construction.

The block A has seven simple modules. We denote by S_1 and S_3 the (dual) simple modules of dimension 49, by S_2 and S_4 the simple modules of dimension 154, by S_5 the simple module of dimension 77 and by S_6 and S_7 the (dual) simple modules of dimension 770. The choice is such that the Green correspondents are given below.

$$C_1 = \begin{matrix}1\\7\\3\end{matrix}, \qquad C_2 = 2, \qquad C_3 = \begin{matrix}3\\6\\1\end{matrix}, \qquad C_4 = 4, \qquad C_5 = 5, \qquad C_6 = 6, \qquad C_7 = 7.$$

There is a perverse equivalence between A and B with zero local twists and with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_5	S_2	S_4	S_6	S_7	S_1	S_3
0	77	1						
0	154_1		1					
0	154_2			1				
0	770_1				1			
0	770_2					1		
1	896_1	1			1		1	
1	896_2	1				1		1
	175	1					1	1
	1925	1	1	1	1	1		

The explicit complexes are as follows.

$$X_1\colon\ 0 \to \mathcal{P}(1) \to C_1 \to 0,$$
$$X_3\colon\ 0 \to \mathcal{P}(3) \to C_3 \to 0.$$

We have $H^{-1}(X_1) = 5/6/1$ and $H^{-1}(X_3) = 5/7/3$.

The outer automorphism of order 2 of HS swaps the simple modules S_1 and S_3. It follows that the perversity function is equivariant.

6. Prime 2

In this section, we assume that $\ell = 2$.

6.1. *Defect* 2×2

Let $G = \mathrm{PSL}_2(q)$ where $q \equiv 3, 5 \pmod 8$. We have a splendid Morita equivalence between $B_0(G)$ and $B_0(\mathfrak{A}_4)$ when $q \equiv 3 \pmod 8$ (resp. $B_0(\mathfrak{A}_5)$ when $q \equiv 5 \pmod 8$) [21]. It can be checked to be compatible with automorphisms.

There is a perverse equivalence between $kB_0(\mathfrak{A}_5)$ and $kB_0(\mathfrak{A}_4)$ [45], [48, §3]. We denote by T_2 the non-trivial simple $\mathbf{F}_2\mathfrak{A}_4$-module. There are three simple $B_0(\mathfrak{A}_5)$-modules. Over $\mathbf{F}_2$, the 2-dimensional modules $S_{2,1}$ and $S_{2,2}$ amalgamate into a 4-dimensional simple module S_2. The Green correspondents are

$$C_1 = 1, \qquad C_2 = \begin{matrix} 2 \\ 2 \end{matrix}.$$

There is a perverse equivalence between $B_0(G)$ and $B_0(N)$ with the π-values on the left, which makes the decomposition matrix look as follows.

π	Ord. Char.	S_1	$S_{2,1}$	$S_{2,2}$
0	1	1		
1	3_1	1	1	
1	3_2	1		1
	5_1	1	1	1

We have

$$X_2 = 0 \to \mathcal{P}(2) \to C_2 \to 0.$$

The cohomology is $H^{-1}(X_2) = \Omega^{-1}(C_2) = 11/2$.

Remark 6.1. Note that [51, §6.3] provides a direct proof of Broué's equivariant conjecture for blocks with Klein four defect groups, and it is easily seen that the equivalence constructed is perverse. The existence of derived equivalences for blocks with Klein four defect groups is due to Linckelmann [33, Corollary 1.5].

6.2. *Defect* $2 \times 2 \times 2$

6.2.1. *The group* $\mathrm{PSL}_2(8)$

Let $G = \mathrm{PSL}_2(8)$. A splendid Rickard equivalence has been constructed in [49, §2.3, Example 2]. It is not perverse, but we show below that there is a composition of two perverse equivalences in that case. The method used here is similar to that developed by Okuyama [41].

Let P be a Sylow 2-subgroup of G and $N = N_G(P)$. We have $P \simeq Z_2^3$ and $N/P \simeq Z_7$. There are two non-trivial simple $\mathbf{F}_2 N$-modules, T_2 and T_3. The labelling is chosen so that

$$\mathcal{P}(1) = \begin{matrix}1\\2\\3\\1\end{matrix}, \qquad \mathcal{P}(2) = \begin{matrix}2\\233\\11123\\2\end{matrix}, \qquad \mathcal{P}(3) = \begin{matrix}3\\11123\\223\\3\end{matrix}.$$

Let S_2 be the 6-dimensional irreducible $\mathbf{F}_2 G$-module and S_3 the 12-dimensional one. We have

$$\mathcal{P}(S_1) = \begin{matrix}S_1\\S_2\\S_1^3 S_3\\S_2^2\\S_1^3 S_3\\S_2\\S_1\end{matrix}, \qquad \mathcal{P}(S_2) = \begin{matrix}S_2\\S_1^3\\S_2^3\\S_1^6 S_3\\S_2^3\\S_1^3 S_3\\S_2\end{matrix}, \qquad \mathcal{P}(S_3) = \begin{matrix}S_3\\S_2\\S_1\\S_2\\S_1\\S_2\\S_3\end{matrix}.$$

The Green correspondents are

$$C_2 = \begin{matrix}3\\2\end{matrix}, \qquad C_3 = \begin{matrix}2\\23\\3\end{matrix}.$$

Consider A'' an $\mathbf{F}_2$-algebra with a perverse equivalence with $B_0(G)$ with perversity function $(0, 1, 0)$. The images of the simple A''-modules under the corresponding stable equivalence are

$$C_1'' = S_1, \qquad C_2'' = \Omega^{-1}\begin{pmatrix}S_1^3 \oplus S_3\\S_2\end{pmatrix}, \qquad C_3'' = S_3$$

while the (non-simple) image under the derived equivalence is

$$X_2'' = 0 \to \mathcal{P}(2) \to C_2'' \to 0.$$

Consider A' an $\mathbf{F}_2$-algebra with a perverse equivalence with kN with perversity function $(0, 2, 3)$. The images of the simple A'-modules under the corresponding stable equivalence are

$$C_1' = 1, \qquad C_2' = \mathrm{Res}_N^G C_2'', \qquad C_3' = C_3$$

while the images under the derived equivalence are

$$X_2' = 0 \to \mathcal{P}(3) \to \mathcal{P}(3) \to \mathcal{P}(2) \to C_3 \to 0,$$
$$X_3' = 0 \to \mathcal{P}(2) \to \mathcal{P}(23) \to C_2'.$$

The composite equivalence

$$A''\text{-stab} \xrightarrow[\sim]{(0,1,0)} B_0(\mathbf{F}_2 G)\text{-stab} \xrightarrow[\sim]{\mathrm{Res}} \mathbf{F}_2 N\text{-stab} \xrightarrow[\sim]{(0,2,3)^{-1}} A'\text{-stab}$$

sends simple modules to simple modules, hence comes from a Morita equivalence between A'' and A' [34, Theorem 2.1]. Hence, we may assume that $A'' = A'$, and the composition

$$D^b\big(B_0(\mathbf{F}_2G)\big) \xrightarrow[\sim]{(0,1,0)^{-1}} D^b\big(A'\big) \xrightarrow[\sim]{(0,2,3)} D^b(\mathbf{F}_2N)$$

lifts the stable equivalence induced by restriction. Thus there is a composition of two perverse equivalences

$$B_0(\mathbf{F}_2G) \xleftarrow{(0,1,0)} A' \xrightarrow{(0,2,3)} \mathbf{F}_2N,$$

where the algebra A' is acted on by Z_3, and the equivalences are compatible with the action of $Z_3 = \mathrm{Out}(\mathrm{PSL}_2(8))$.

6.2.2. The Ree groups

Let $G = {}^2G_2(q)$ where $q = 3^{2n+1}$. Note that ${}^2G_2(3) \simeq \mathrm{PSL}_2(8) \rtimes Z_3$. There is a splendid Morita equivalence between $B_0(G)$ and $B_0({}^2G_2(3))$ [41, Example 3.3]. Therefore we obtain from Section 6.2.1 a derived equivalence as a composition of two perverse equivalences.

6.2.3. The Janko group J_1

Let $G = J_1$. In this case, a splendid Rickard equivalence has been constructed by Gollan and Okuyama [25] and we recall their construction, with a more direct proof.

6.2.3.1. Simple modules. There are four simple modules over $\mathbf{F}_2$ in the principal block of G, of dimensions 1, 20, 76 and 112. We label the simple modules $S_1, \dots, S_4$ by increasing dimension.

Let P be a Sylow 2-subgroup of G, and write $N = N_G(P)$. We have $N \simeq Z_2^3 \rtimes (Z_7 \rtimes Z_3)$. There are three non-trivial $\mathbf{F}_2N$-modules: T_2 of dimension 2, and T_3 and T_4 of dimension 3. The labelling is chosen so that

$$\mathcal{P}(1) = \begin{matrix}1\\3\\4\\1\end{matrix}, \qquad \mathcal{P}(2) = \begin{matrix}2\\33\\44\\2\end{matrix}, \qquad \mathcal{P}(3) = \begin{matrix}3\\344\\1234\\3\end{matrix}, \qquad \mathcal{P}(4) = \begin{matrix}4\\1234\\334\\4\end{matrix}.$$

Let $H = \mathrm{PSL}_2(8).3$. There are four simple modules over $\mathbf{F}_2$ in the principal block of H, of dimensions 1, 2, 6 and 12. We denote by S_2' the 12-dimensional one, by S_3' the 6-dimensional one and by S_4' the 2-dimensional one.

6.2.3.2. The equivalence. The construction of Section 3.3.1 extends to the case of $P \simeq Z_2^3$ (see [51, §6.4]). Let Q be subgroup of P of order 2 (there is a unique N-conjugacy class of such subgroups). We have $C_G(Q)/Q \simeq \mathfrak{A}_5$, while $C_N(Q)/Q \simeq \mathfrak{A}_4$. We define $\mathcal{E} = \{V_{4,1}, V_{4,2}\}$ to be the set of non-trivial simple modules of the principal block of $kC_G(Q)/Q$, and we construct as in Section 3.3.1 (see [51, §6.4]) a complex that induces a splendid standard stable equivalence $F : \mathbf{F}_2B_0(G)\text{-stab} \xrightarrow{\sim} \mathbf{F}_2N\text{-stab}$. Let $C_i = F(S_i)$. Let P_2 be the projective cover of the non-trivial simple $\mathbf{F}_2(C_H(Q)/Q)$-module and let $M = \mathrm{Ind}_{C_H(Q)}^{H} \mathrm{Res}_{C_H(Q)}^{C_H(Q)/Q} P_2$.

We have

$$C_2 \simeq \mathrm{Res}_N^H S_2' \quad \text{and} \quad C_4 \simeq 0 \to M \to V \to 0$$

where $V = \begin{matrix}244\\233\end{matrix}$.

Let A be an $\mathbf{F}_2$-algebra equipped with a standard perverse equivalence $D^b(A) \xrightarrow{\sim} D^b(\mathbf{F}_2 B_0(G))$, corresponding to the perversity function $(0,0,1,0)$. Let L be the largest submodule of $\mathcal{P}(S_3)$ containing S_3 as a submodule and such that L/S_3 has no composition factor S_3. Then, $\operatorname{Res}_N^G(\Omega^{-1}L) \simeq \operatorname{Res}_N^H S_3'$ in $\mathbf{F}_2 N$-stab. Furthermore, $\operatorname{Res}_Q S_3'$ is projective, and hence $F(\Omega^{-1}L) \simeq S_3'$.

Let A' be an $\mathbf{F}_2$-algebra equipped with a standard perverse equivalence $D^b(A') \xrightarrow{\sim} D^b(\mathbf{F}_2 B_0(H))$, corresponding to the perversity function $(0,0,0,1)$. Let L' be the largest submodule of $\mathcal{P}(S_4')$ containing S_4' as a submodule and such that L'/S_4' has no composition factor S_4'. Then, $L'' = \operatorname{Res}_N^H \Omega^{-1}(L')$ is an indecomposable module of dimension 72. There is an exact sequence

$$0 \to M \to \mathcal{P}(23344) \oplus V \to L'' \to 0$$

showing that $C_4 \simeq L''$ in $\mathbf{F}_2 N$-stab.

We have a diagram of standard stable equivalences

$$\begin{tikzcd}
A\text{-stab} \arrow[r, "(0,0,1,0)", "\sim"'] & \mathbf{F}_2 B_0(G)\text{-stab} \arrow[dr, "F", "\sim"'] & & \mathbf{F}_2 B_0(H)\text{-stab} \arrow[dl, "\mathrm{Res}"', "\sim"] & A'\text{-stab} \arrow[l, "(0,0,0,1)"', "\sim"] \\
& & kN\text{-stab} & &
\end{tikzcd}$$

The set of images of simple A-modules in kN-stab coincide with that of simple A'-modules. It follows that the composite equivalence A-stab $\to A'$-stab comes from a Morita equivalence [34, Theorem 2.1]. So, we have obtained a composition of two perverse equivalences

$$kB_0(G) \xleftarrow{(0,0,1,0)} A \xrightarrow{(0,0,0,1)} kB_0(H).$$

6.3. $\mathrm{PSL}_2(\ell^n)$

Let $G = \mathrm{PSL}_2(\ell^n)$ for some integer $n \geqslant 1$ and ℓ a prime. Let $\tilde{G} = \operatorname{Aut}(G)$. Okuyama [42] has constructed a sequence of derived equivalences as in Section 4.2.6. The sets I_r used by Okuyama are invariant under $\operatorname{Out}(G)$. It follows that there is a complex C of $kB_0(N_\Delta(G,\tilde{G}))$-modules whose restriction to $G \times N_G(P)^{\mathrm{opp}}$ is a two-sided tilting complex. It actually induces a splendid Rickard equivalence. This derived equivalence between principal blocks of G and $N_G(P)$ extends to a derived equivalence between principal blocks of G' and $N_{G'}(P)$, for any $G \leqslant G' \leqslant \tilde{G}$, and that extended equivalence is a splendid Rickard equivalence if $\ell \nmid [G' : G]$.

Note that the equivalences defined by Okuyama are not perverse in general. It is not known if they are compositions of perverse equivalences.

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