The Theory of Fusion Systems, Errata

David A. Craven, University of Birmingham

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I distinguish four types of corrections, in order of increasing seriousness:

(Extra) Additional information that was not available at the time of writing, or that I did not know about.

(Improve) Typographical issues, where what is written is still correct, but there is a nicer way of phrasing it, or I could choose a better symbol.

(Typo) Typographical errors, where I have spelled a word wrongly, used the wrong symbol, and so on.

(Error) Errors in proofs or statements.

When I give each correction, I will label it with one of these monikers.

(i) (Error) p5, Example 1.3. In fact H does act transitively on the subgroups of order 4, but not every map α induced on the 4-subgroups by conjugation in G is induced by conjugation by H, so that H controls weak fusion but not G-fusion.

Consider the following example in the same vein: let V be a vector space over $\mathbb{F}_2$ of dimension 5, whose automorphism group is $\mathrm{GL}_5(2)$. Let x be an element of order 31, and let y be an element of order 5 that normalizes x (this exists). Let G be a semidirect product of V and $\langle x, y\rangle$, a (soluble) group of order $2^5 \cdot 31 \cdot 5$, and let $H = \langle V, x\rangle$. Since there are 31 non-identity elements of V, x must permute them transitively, and so both H control weak fusion in P with respect to G.

Now consider the set S of subgroups of order 4 in V. There are $31 \cdot 5$ of these. Since x permutes the non-identity elements of V transitively, it must act freely on S, so has five orbits of length 31. However, as y acts on S non-trivially, we see that G must act transitively on S. However, H acts in five orbits, so that G and H have different actions on the subgroups of V of order 4. (Thanks to Justin Lynd and George Glauberman for pointing the error in 1.3 out.)

(ii) (Typo) p7, line 2 of Definition 1.7, the symbol “$\trianglelefteq (G)$" should be “$\mathcal{F}_P(G)$". (Thanks to Matt Towers for this.)

(iii) (Typo) p10, l7, '$|\mathrm{Aut}(Q)|$ is a 2-group' should be '$\mathrm{Aut}(Q)$ is a 2-group'. (Thanks to Benjamin Sambale for spotting this.)

(iv) (Improve) p15, line 15, the two $N_G(Q)$ and $N_G(R)$ should be $\mathrm{N}_G(Q)$ and $\mathrm{N}_G(R)$.

(v) (Improve) p19, line 4 of Theorem 1.33, the displayed equation should read

$$= \langle x^{-1}(x^g) \,:\, x \in Q \leqslant P,\ g \in \mathrm{O}^p(\mathrm{N}_G(Q))\rangle$$

This is simply replacing ϕ by g.

(vi) (Error) p34, paragraph 2 of the proof of Proposition 2.12. The fact that $\mathrm{Br}_P(\hat{X})$ is $\hat{X}$ if X is a singleton set or 0 otherwise has nothing to do with sizes of conjugacy classes: notice that by the definition of $\mathrm{C}_G(P)$, a P-conjugacy class X is a singleton set if and only if $X \subseteq \mathrm{C}_G(P)$. If X does not lie inside $\mathrm{C}_G(P)$ then $\mathrm{Br}_P(\hat{X}) = 0$ by definition, and if X does lie inside $\mathrm{C}_G(P)$ then it is a singleton.

(vii) (Error) p99, alternative definition of N_ϕ. This isn't quite right. I write $(Qc_g)\phi = (Q\phi)c_h$ in the displayed equation, but of course I want the maps $c_g\phi$ and ϕc_h to agree on Q, not just that they move Q to the same place (i.e., R). I mean that $qc_g\phi = q\phi c_h$ for all $q \in Q$. (Thanks to Benjamin Sambale for noticing this.)

(viii) (Error) p119, proof of Proposition 4.46. Everything written in the proof is OK, except I only show that $\mathrm{N}_Q(R) \leqslant R\mathrm{C}_P(R) = R$ in the proof. However, this can easily be fixed: if $Q \not\leqslant R$ then $QR > R$, so $\mathrm{N}_{QR}(R) > R$. The proof actually shows that $\mathrm{N}_{QR}(R) \leqslant R$, and we are done.

The complete argument would look as follows:

Let R be a centric, radical subgroup of P. We claim that $\mathrm{Aut}_{QR}(R)$ is a normal subgroup of $\mathrm{Aut}_\mathcal{F}(R)$: if this is true, then since R is $\mathcal{F}$-radical, $\mathrm{Aut}_{QR}(R) \leqslant \mathrm{Inn}(R)$, so that $\mathrm{N}_{QR}(R) \leqslant R\mathrm{C}_P(R) = R$. If $Q \not\leqslant R$ then $QR > R$, so $\mathrm{N}_{QR}(R) > R$, which is a contradiction, so that $Q \leqslant R$, as needed.

We now prove the claim. Let $c_g \in \mathrm{Aut}_{QR}(R)$, let $\phi \in \mathrm{Aut}_\mathcal{F}(R)$, and let $\psi \in \mathrm{Aut}_\mathcal{F}(QR)$ be an extension of ϕ. Since both Q and R are ψ-invariant, $\mathrm{N}_{QR}(R)$ is ψ-invariant. For $x \in QR$, x normalizes R if and only if $c_x \in \mathrm{Aut}(R)$, so it suffices to show that, if $c_g \in \mathrm{Aut}_{QR}(R)$ then $(c_g)^\psi \in \mathrm{Aut}_{QR}(R)$. However, $(c_g)^\psi = c_{g\psi}$, and $g\psi \in \mathrm{N}_{QR}(R)$ as $g \in \mathrm{N}_{QR}(R)$ and $\mathrm{N}_{QR}(R)$ is ψ-invariant, proving the claim.

(Thanks to Robert Leek for pointing this out.)

(ix) (Error), p135-6, in the description of ψ^ϕ, the point is that this is not necessarily well defined in general, and Proposition 5.3 should be that the kernel T is strongly $\mathcal{F}$-closed if and only if, whenever ψ is a map in $\mathcal{F}$, the map ψ^ϕ is well defined. If T is strongly $\mathcal{F}$-closed then ψ^ϕ is an injection, by the original Proposition 5.3.

A complete proof for the modified Proposition 5.3 is as follows:

Suppose that T is strongly $\mathcal{F}$-closed, and let $\psi : A \to B$ be a morphism in $\mathcal{F}$. If a and a' are representatives of the same right coset of T in P then $a'a^{-1} \in A \cap T$; as T is strongly $\mathcal{F}$-closed, $(a'\psi)(a^{-1}\psi) \in B \cap T$, so that $(a\psi)\phi = (a'\psi)\phi$, and ψ^ϕ is well defined.

Conversely, suppose that ψ^ϕ is well defined for all ψ in $\mathcal{F}$. If $\psi : A \to B$ is a homomorphism in $\mathcal{F}$, and $a \in A \cap T$, then both a an 1 label the same right coset of T, so that $(a\psi)\phi = (1\psi)\phi = 1$, and $a\psi \in T$. This proves that T is strongly $\mathcal{F}$-closed.

In general, for an arbitrary map $\psi : A \to B$ and a homomorphism $\phi : P \to Q$, ψ^ϕ is well defined if and only if $(A \cap \ker\phi)\psi \leqslant \ker\phi$. If ψ is bijective and $(\psi^{-1})^\phi$ is also well defined, then $(A \cap \ker\phi)\psi = B \cap \ker\phi$.

(Thanks to Kasper Andersen for pointing this out.)

(x) (Typo) p138, Definition 5.9, last line. Instead of $Q\phi = Q$ we need $Q\psi = Q$.

(Thanks to Benjamin Sambale for noting this.)

(xi) (Typo) p189, proof of 6.2, l2. I wrote $\mathcal{F}$-isomorphism, and I meant simply ‘morphism’, as of course $|\mathrm{N}_P(R)|$ need not be as large as $|\mathrm{N}_P(Q)|$ in general.

(Thanks to Kasper Andersen for pointing this out.)

(xii) (Typo) p189, proof of 6.2, l5, R should be $\mathrm{N}_P(R)$.

(Thanks to Kasper Andersen for pointing this out.)

(xiii) (Typo) p192, proof of 6.8, l13, Q is fully normalized, but the extension is because Q is also receptive.

(Thanks to Kasper Andersen for pointing this out.)

(xiv) (Typo) p196, l19. $1 \leqslant i < t$ should be $1 \leqslant i < t-1$, as one cannot compose ϕ_i and ϕ_{i+1} if $i = t-1$.

(Thanks to Kasper Andersen for pointing this out.)

(xv) (Typo) p196, l20. $B_i\phi$ should be $B_i\phi_i$.

(Thanks to Kasper Andersen for pointing this out.)

(xvi) (Improve) p200. The proof of 6.11 in the book uses induction on the number of conjugacy classes in the set $\mathcal{H}'$ to establish the result for all conjugacy classes. Of course, it should be assumed that all of the members of $\mathcal{H}'$ are saturated, and then prove that some class not in the set is saturated, thus showing that every set of classes consists of saturated members.

The inductive setup is a bit rushed, and so the intention might not have been clear. One can also assume that there is a non-saturated class, take a minimal such one, then derive a contradiction.

(xvii) (Improve) p201, proof of 6.16. In the second paragraph, I prove that $\mathcal{Q}$ is saturated in $\mathcal{E}$, not in $\mathcal{F}$. Since $\mathcal{Q}$ consists of subgroups that are not centric, and $\mathcal{F}$ is generated by automorphisms of centric subgroups, every $\mathcal{F}$-map between members of $\mathcal{Q}$ is an $\mathcal{E}$-map, and so the two statements are the same.

One also should use the *proof* of 6.9, rather than the statement. In the situation of 6.16 we are inductively saturated, not saturated, so 6.9 technically does not apply. Generally, inductively saturated and saturated fusion systems behave the same, but of course the distinction should be made.

(Thanks to Kasper Andersen for pointing this out.)

(xviii) (Typo) p224, Theorem 7.21, then $W(P) \leqslant \mathcal{F}_P(G)$ should be $W(P) \trianglelefteq \mathcal{F}_P(G)$.

(Thanks to Benjamin Sambale.)

(xix) (Typo) p238, in Proposition 7.48 R should be a normal subgroup of P.

(Thanks to George Glauberman for pointing this out.)

(xx) (Typo) p253, Example 7.63, because of the conventions chosen for bisets here, the twisted diagonal subgroup should actually be $\{(x^{-1}, x\phi) : x \in K\}$. (This is because we have to place an inverse somewhere to turn left actions into right actions, and I chose to put it there.)

(Thanks to George Glauberman for pointing this out.)

(xxi) (Typo) p256, line 2, ‘left P-action’ should be ‘left G-action’.

(Thanks to George Glauberman for pointing this out.)

(xxii) (Improve) p265, last line of the proof of Lemma 7.79. What is written is wrong: what is meant is, as Q ranges over all such subgroups, we generate $\mathfrak{foc}(\mathcal{F})$, and so W controls transfer.

(Thanks to George Glauberman for pointing this out.)

(xxiii) (Error) p266, just before Corollary 7.82, I write that K_∞ controls transfer for all odd primes, whereas it’s only known for $p \geqslant 5$. This is stated correctly earlier on the page, and in Corollary 7.82 itself. (As far as I know, it is currently unknown whether K^∞ and K_∞ control transfer for $p = 3$, even using the classification of the finite simple groups.)

(xxiv) (Error) p272, Lemma 8.4(iv), this should be that $\mathcal{E}$ is normal in $\mathcal{F}$ if it is normal in $\bar{\mathcal{E}}$, not if and only if. Indeed, Example 8.18 states that the 'only if' side does not hold, and Corollary 8.19 gives one situation in which the 'if and only if' does hold.

(Thanks to Bob Oliver for pointing this out.)

(xxv) (Error) p280, Example 8.20. Instead of $V \rtimes (H_1 \times H_2)$, which doesn't exist, one should take two copies of $V \rtimes H$, $G = (V_1 \rtimes H_1) \times (V_2 \rtimes H_2)$, set $G_i = \langle V_1, V_2, H_i \rangle \trianglelefteq G$, and note that again $\mathcal{E}_i = \mathcal{F}_{Q_i}(G_i)$ have intersection $\mathcal{E}$ lying on $V_1 \times V_2$, with automorphism group $S_5 \times S_5$. The two maximal p'-subgroups we take are now $C_3 \times C_3$ and $C_5 \times C_5$.

(Thanks to Gernot Stroth for pointing out the error in the example.)

(xxvi) (Extra) p349/350, Conjecture 9.44 has been solved by Andy Chermak, and the translation of his proof into obstruction theory by Bob Oliver also solves Conjecture 9.45. Conjecture 9.49 is still open, however.